Lecture 19: Scaling. Normal Distribution

**Aim** Introduce the most important distribution, the normal distribution.

**Probability Density of \( g(X) \)**

\( X \) cont random var with

Let \( g(x) \) be increasing diff’ble

**Lemma** Let \( Y = g(X) \) have prob density \( f_1(y) \) where \( y = g(x) \). Then

a) \( F_1(y) = P(Y = g(X) < P(X) \)

b) \( f(x) = g'(x)f_1(g(x)) \)

Proof: a) holds as \( g \)

b) follows from
Important special case is

**Scaling**

We can scale $X$ to $Y =$

where $a >$

Lemma $\implies f(x)$

or in other words

$f_1(y) =$

Recall from lecture 18

$E(Y) =$

Suppose graph of $f(x)$ is
**Note** Graph changes by

1) Mean
2) horizontal scale “stretches”
3) vertical scale

How does \( \text{Var}(X) \) change with scaling?

**Propn 1** \( \text{Var}(aX + b) = \)

\[ \sigma(aX + b) = \]

Proof: \( \text{Var}(aX + b) = E((aX + b)^2) - E(aX + b)^2 \)
cont’d

\[ \sigma(aX + b) = \]

**Cor** If \( X \) has mean \( \mu \) standard deviation \( \sigma \)
then \( Y = \frac{X - \mu}{\sigma} \)

**A Useful Integral**

**Lemma**

\[
I := \int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi}
\]

Proof: Consider solid of revolution on revolving \( w = e^{-x^2/2} \) about \( w \)-axis.
This is graph of

\[ r = \]

We’ll compute volume under graph by

Slices: Fix \( x \)

Cross-sectional area \( A(x) = \)
Hence, volume

Cylindrical Shells:

\[
\text{Volume} = \int_0^\infty 2\pi rwdr
\]

\[\therefore I^2 = \]

**Standard Normal Distribn**

**Defn** A cont random var $Z$ has standard
normal distrbution if its prob density fn is

$$\phi(z) =$$

Note: 1. Lemma ⇒

2. The graph is symmetrical

**Remark** It is hard to say at the moment why the normal distribution is so important. We’ll see later. Note for now that it has a similar bell shape to the binomial distribution except that it’s continuous.

E.g. $$B(n = 8, p = .5, k) =$$
Propn Let $Z$ be

a) $E(Z) = \quad$ b) $\sigma(Z) =$

Proof: a) $\phi(z)$ even $\implies$
b) $\text{Var}(Z) = E(Z^2) =$

Normal Distribn

We scale the standard normal variable $Z$ to get

**Defn** A cont random var $X$ has normal distribn if

**Notn** Write $X \sim N(\mu, \sigma^2)$.

Results on scaling show
Facts If \( X \sim \)

a) \( E(X) = \)  

b) \( \text{Var}(X) = \)