Lecture 18: Continuous Random Variables

A continuous random var $X : \Omega \rightarrow \mathbb{R}$ can take any value in $\mathbb{R}$.

E.g. $X =$ humidity in Sydney.

Aim Lecture Set up probability theory for these random var.

Approach

Consider e.g. above. We’ll use

Range of values is
We’ll divide this into 4
of length $\Delta$
Let $p_i =$
Draw normalised graph of $p'_k :=$

We want to shrink $\Delta \to 0$ & will see later that this normalisation removes dependence on $\Delta$.

**Key Point**

$P(X$

Typical graph of $p'_k$ for $\Delta =$
Let’s subdivide

Note: Each original rectangle breaks up into

**What Happens as $\Delta \to 0$?**

1. $p_k = \text{area}$

   $\therefore P(X = x)$

2. The graph of $p'_k$ tends towards a presumably cont fn $f(x)$. Prob is represented by

3. Since $\sum p_k = 1, p_k \geq 0$ we expect
Above heuristics suggest

**Defn (Continuous Random Var)**  
A random var $X : \Omega \rightarrow \mathbb{R}$ is continuous if there’s an

which gives the probability

$P(a <$

such that

1.  
2.

We call $f(x)$ the probability density fn or
Any fn $f(x)$ satisfying 1 & 2 is called a

**N.B.** $P(X = a) =$

so $P(a \leq X < b) =$

**E.g. 1** $X =$ time (in hrs) Homer Simpson

Suppose prob density fn $f(x)$ is proportional to $x$. What’s $f(x)$?

Ans: $f(x) =$

Need only determine
Negative Exponential Distribution

E.g. 2  Let $\alpha > 0$ be a parameter.

Define $f(x)$

$f(x)$ is a prob density fn. Why?

1.

2. $\int_{-\infty}^{\infty} f(x) \, dx =$

It’s graph
Note 1 & 2 means we usually have \( f(x) \rightarrow \)

**Cumulative Distrbn Fn**

Given a probability density fn \( f(x) \) for random var \( X \), it’s sometimes easier to work with actual probabilities.

**Defn** The cumulative distrbn fn of \( f(x) \) is \( F(x) := \)

**E.g. 2 again** Negative exp distrbn

For \( x \leq 0, F(x) = \)

If \( x \geq 0, F(x) = \)
Can compute the median value from this i.e. the value of $x$ such that

$$\frac{1}{2} = P(X)$$

$$e^{-\alpha x} =$$

Median $x =$

**Note** 1. If $f$ is cont, $F'(x) =$

2. $f(x) \geq 0 \implies$

3. $\int_{-\infty}^{\infty} f(x)dx = 1 \implies$
4. \( P(a < X < b) = \)

\textbf{Mean (Cont case)}

Again use discrete approx to suggest defn.

Recall

\[ p'_k = \frac{p_k}{\Delta} \rightarrow \]

Usual limiting argument shows the mean in discrete approx

\[ \sum x_k p_k = \]

Suggests

\textbf{Defn (Expected Value)}

Let \( f(x) \) be prob density fn for cont random var \( X \). The
\[ E(X) := \]

**E.g. 2 again** \( f(x) \) negative exp distrbn with param \( \alpha \).

\[ E(X) = \]

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**Formulae Involving Expected Values**

As in discrete case, given a continuous fn \( g : \mathbb{R} \rightarrow \mathbb{R} \), we can consider new random
var $g(X)$.

**Propn 1** If $X$ is a cont random var with prob density $f(x)$ then

$$E(g(X)) =$$

Proof: See ex 28, ch. 9 of the notes.

**Propn 2** a) $E(g(X) + h(X)) =$

b) For $a, b \in \mathbb{R}$, $E(aX + b) =$

Proof: Use propn 1 in both cases.

a)
b) Sim (see discrete case for hints).

**Variance (Cont case)**

Let $f(x)$ be prob density fn for random var $X$. The

$$\text{Var}(X) :=$$

Its square root $\sigma(X)$ is

**Propn 3** \( \text{Var}(X) = \)

Proof: Same as in discrete case. Use propn 2.

**E.g. 2 again** Negative exp distrbn fn $f(x)$ with param $\alpha$.

Recall $f(x) =$

\& $E(X) =$
cont’d $Var(X) =$

E.g. 3

$X =$ diameter of Spider

Assume exp distribn with mean 2mm. Find expected value of cross-

Ans: $E($