Lecture 14: Invertible Linear Maps

**Aim Lecture** Coordinates allow you to identify fin dim vector spaces with
More gen, invertible

**Defn** (One-to-one) Let $X, Y$ be sets & $f : X \rightarrow Y$ be a function. We say that $f$ is one-to-one (1-1) or injective if the soln

i.e. $f(x) = f(x')$

**E.g. 1** $f(x) = x^2$ is not

Notion of 1-1 simplifies in the linear case.

**Propn 2** Let $T : V \rightarrow W$ be linear.
Then $T$ is 1-1 iff
Proof: Recall from lecture 12 propn 2 that given \( w \in W \) & particular

\[
\therefore \text{soln is unique}
\]

E.g. 2 Show the soln to the IVP

\[
\frac{dy}{dx} - y = \frac{\pi^x}{\sqrt{e^{\cos x} + \cosh x^\pi}}, y(0) = 0
\]

is unique?

Ans: Let \( T \) be defined by

Note \( T \) is linear being

By propn, suffice show ker \( T = 0 \)
i.e. (*)
has unique soln \( y = 0 \).

**Defn (Onto)** A fn \( f : X \rightarrow Y \) is

i.e. \( \text{im } f \)

**Inverse Functions**
Let \( X, Y \) be sets and \( f : X \rightarrow Y \) any fn.

**Propn-Defn** The following condns on \( f \) are equivalent.
a) $f$ is 1-1 &

b) the eqn $f(x) = y$ always has a
denoted $x = $

c) there’s a fn $g :$

Proof: Clear a) & b) equiv
If b) holds, then c) holds on

Conversely, if $g : Y \rightarrow X$ is as in c), then

$x = g(y)$ is

Check is a soln: $f(g(y))$

Unique: if also $f(x') = y =

x' = g($

**Propn 3** If $T : V \rightarrow W$ is linear and
invertible then

Proof: in case $T = T_A$ for $A \in M_{mn}(\mathbb{F})$.
Since $T$ is invertible, $T \mathbf{x} = A \mathbf{x} = \mathbf{b}$ always

Hence, $T^{-1} =$

**E.g. 3** Let $V = \text{vector space/ field } \mathbb{F}$
$B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$
Define $S_B : V \longrightarrow \mathbb{F}^n$ by
Recall $S_B(x_1 \mathbf{v}_1 + \ldots + x_n \mathbf{v}_n) =$
Thm lecture 7 $\Rightarrow$
$S_B$ is invertible since $S_B(\mathbf{v}) = (x_1, \ldots, x_n)^T$
has unique soln
Part a) of following propn generalises fact that a subspace has smaller dimension than the ambient space.

**Propn 4** Let $T : V \rightarrow W$ be linear.

Let $B = \{v_1, \ldots, v_n\} \subset$

a) $T$ 1-1 & $B$ lin indep $\Rightarrow$

Hence if $W$ is finite dim then so is

b) $T$ onto & $B$ a spanning set $\Rightarrow$

Proof: a) We’ll show $T(B)$ is

Suppose $0 =$

Need all $\lambda_i = 0$.

$0 =$
$T$ 1-1 $\implies$

$B$ lin indep

Hence $T(B)$ is

If $B$ is a basis, then lecture 8 cor 1 $\implies$

b) Sim, see proof lemma 2 lecture 9.

**Cor** If $T : V \to W$ is an invertible

Then bases (resp. spanning sets, resp. lin indep sets) of $V$ & $W$ correspond via $T$ & $T^{-1}$.

**E.g. 4** Let $B := \{v, w\}$ be a basis for $V$.

Then $\{v + w, v - w\}$

Why? Consider invertible lin map $S_B$:
Hence \( \{ \mathbf{v} + \mathbf{w} \} \)

**Propn 5** Let \( T : V \rightarrow W \) be a lin map

Consider invertible \( S_r : V' \rightarrow V, \ S_l \)

\( S_lT S_r : \)

a) \( \ker(S_lT S_r) = \)

b) \( \text{im} \ (S_lT S_r) = \)

Proof: a) \( \mathbf{x} \in \ker(S_lT S_r) \) iff

iff

b) \( S_lT S_r(V') = \)

We can now prove propn 3, lecture 12 &
propn 1, lecture 13.
**Cor** Let $T : V \longrightarrow W$ be of dimensions $n \& m$.

& $B_V, B_W$ be finite ordered

Let $A$ be the matrix

a) The corresp map on coord vectors is $T_A =$

b) $[\ker T]_{B_V} =$

c) $[\text{im } T]_{B_W} =$

Proof: a) is an easy ex. In fact, you can prove the gen matrix reprn thm by applying matrix reprn thm to

b) $\ker A =$

c) is almost identical.
A nice application of rank-nullity is the following.

**Propn 6** Let $T : V \rightarrow W$ be linear and suppose $\dim V = \dim W$ is finite.

Then $T$ is invertible iff either

Proof: If $T$ is 1-1: $\text{null } T = 0 \implies \text{rank } T =$

If $T$ is onto: