Aim Lecture For $T : V \rightarrow W$ linear, understand when you can solve

Image

Defn-Propn (Image) Let $T : V \rightarrow W$ be linear. The image of $T$ is

Also rank $T :=$

If $T = T_A$ also write

Note $\text{im } A = \text{col } A$.

Proof: Follows easily from subspace thm-defn. We’ll just check closure under addn. For $v, w \in V$,
so \( \text{im } T \) is closed

**E.g. 1** \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) orthog projn onto

**Finding Bases for Image**

For \( \mathbb{F}^m \) case, we computed bases for \( \text{col}(A) \) in thm 2 lecture 9. We reduce to this case using

**Propn 1** Let \( T : V \rightarrow W \) be

& \( B_V, B_W \) be finite ordered

Let \( A \) be the matrix

Then \([\text{im } T]_{B_W} = \)


**E.g. 2** Let \( M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \).
Define $T : M_{22}(\mathbb{F}) \rightarrow M_{22}(\mathbb{F})$ by $TC := MC$. Note that $T$ is linear by

Find a basis for $\text{im } T$ and rank $T$.

Ans: First find matrix $A \in M_{44}(\mathbb{F})$ representing $T$ wrt

$e_{11} =$

$$
T \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
$$
Hence $A = (T$

$= \ldots$

We compute a basis for $\text{col}(A)$ as in lecture 9.

$A \rightarrow \ldots$

Hence a basis is

$v_1 = \ldots$

They correspond to

$C_1 = \ldots$
By propn 1 & lemma 2 of lecture 9, 
\( \{C_1, C_2\} \) is

**Rank-Nullity Thm**

**Thm** Let \( T : V \longrightarrow W \) be a linear map, 
\( V, W \) finite dim.

Let \( A \) be a matrix 

wrt 

Let \( U \) be a row 

a) null \( T = \) no. non-leading columns 

b) rank \( T = \) 

c) (Rank-Nullity) \( \dim V = \) no.

Proof: Do easy case only where \( T = T_A : \)
\( \mathbb{F}^n \longrightarrow \mathbb{F}^m \) where \( A = (v_1 \ldots v_n) \)
a) Each non-leading column in $U$ gives a param in

$\therefore$ gives a basis

$\therefore$ null $A =$

& a) holds.

b) Thm 2, lecture 9 gives basis

$\{v_i | i$

so b) holds.

c) Add a) & b)

Gen case reduces to this one using methods of lecture 14.

E.g. 3 A geometric picture illustrating rank-nullity thm
Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an orthog projn onto 1-dim subspace $W$. 
E.g. 4 We’ll use the rank-nullity thm to show that the intersection of \( m \) hyperplanes through \( 0 \) in \( \mathbb{R}^n \) has

For \( \mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n - \mathbf{0} \) consider the hyperplanes

\[ H_i := \]

Now \( \mathbf{v} \cdot \mathbf{v}_i = \mathbf{v}_i^T \mathbf{v} \implies H_i = \]

Hence, \( \cap H_i = \cap \ker \mathbf{v}_i^T = \)

Now \( \text{im } A \) is a sub
so rank $A =$  
Hence $\dim \cap H_i = \dim \ker A =$  

**E.g. 5** Let $A \in M_{nn}$. Define fn $T : M_{nn} \rightarrow M_{nn}$ by  
$TB := AB$  
Show $\text{im } T \neq$  

**Ans** Check 1st $T$ is linear. If $B, C \in$  
$T(B + C) =$  

Also $T(\lambda B) =$  
So $T$ is  
If $A = 0$ then $\text{im } T = 0$ so suppose
so $A \in$

$\dim \text{im } T =$