Lecture 10: Linear Maps or Transformations

Recall, a (homogeneous) linear function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is one of the form

\[
    f(x_1, \ldots, x_n)^T = C
\]

More generally, a lin fn \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one of the form

**Aim Lecture 10** Generalise the notion of linear functions to linear maps \( T : V \rightarrow W \) between
Defn Let $V, W = \text{vector space } / \mathbb{F}$ and $T : V \rightarrow W$ a function. We say $T$ is a linear map or

a. (Addition Condn) For any $v, w \in V$

& b. (Scalar Multn Condn) For any $\lambda \in \mathbb{F}$

E.g. 1 Let $A \in M_{mn}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ by

Proof: (Addn condition) for $v, w \in V$

\[ T_A(v + w) = \]
so addn condn holds
(Scalar Multn Condn) if also $\lambda \in \mathbb{F}$
$T_A(\lambda \mathbf{v}) = \quad$
giving scalar
Since the addn & scalar multn condn

**E.g. 2** Saruman needs
$x_1$
$x_2$
Each orc
Each
If $y_1, y_2$ are
$\mathbf{y} = \quad$
so $\mathbf{y}$ is a lin fn of
E.g. 3 $\mathcal{C} =$

$\mathcal{C}^1 =$ subspace of continuously

Define $T : \mathcal{C}^1 \rightarrow \mathcal{C}$ by

$T$ is linear.

Why? Addn: for $f, g \in \mathcal{C}^1$

$T(f + g) =$

so addn condn holds

Scalar Multn: for $\lambda \in \mathbb{R}$

$T(\lambda f) =$
Hence, scalar multn condn also

**E.g. 4** Let \( T : M_{23}(\mathbb{F}) \longrightarrow M_{32}(\mathbb{F}) \) be defined by

\( T \) is linear.

Why? Addn: For \( A, B \in M_{23} \)

\( T(A + B) = \)

Scalar Multn: for \( \lambda \in \mathbb{F} \)

\( T(\lambda A) = \)

Hence, scalar multn condn also

Linear maps are very special. One way to view them is
**Fact** A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear iff its graph is

We’ll omit the proof and definition of graph but hopefully the following example shows what’s going on.

**Non- E.g. 5**

$T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) = x^2$.

Graph of $T$

$T(2) = \quad$ but $2T(1) =$

Scalar multn condn fails so

**Propn 1** Let $T : V \rightarrow W$ be linear. Then

a. $T0 =$ \hspace{1cm} b. $T(\mathbf{v}) =$

6
Proof:

Thm 1 (Preservation of Lin Comb) Let $T : V \rightarrow W$ be linear. Then for scalars vectors we have

$T(\lambda_1 v_1 + \ldots + \lambda_n v_n) =$

Proof: By induction on $n$. Just do $n = 2$
case here
\[ T(\lambda_1 v_1 + \lambda_2 v_2) = T(\lambda_1 v_1) + \]

**Remark** Thm \( \Rightarrow \) if you know \( T : V \rightarrow W \) is linear and the values of \( T \) on a spanning set for \( V \) then \( T \) is determined.

To illustrate this,

**E.g. 6** Suppose \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) linear & 
\( T(1, 1)^T = (1, 1, 1)^T, T(1, 0)^T = (2, 2, 1)^T. \) 
Note \( (1, 1)^T, (1, 0)^T \) span 
Then

**E.g. 7** Suppose \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) linear and
\[ T(1, 0)^T = (1, 2)^T, T(0, 1)^T = (3, 4)^T. \]

Then \[ T(\lambda, \mu)^T = T(\lambda(1, 0)^T + \lambda \ T(1, 0)^T + \lambda \]

In fact, any linear \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is multn by

**Thm 2** (Matrix Representation Thm)

Let \( T : \mathbb{F}^n \rightarrow \mathbb{F}^m \) be linear.

\( e_1, \ldots, e_n \in \mathbb{F}^n \)

Define \( a_1 = T e_1, \ldots \)

\( A = \)

Then \( T = T_A \) i.e.
Proof: $T(x_1, \ldots, x_n)^T = T(x_1 \mathbf{e}_1 +$

$= T(x_1 \mathbf{e}_1) +$

$= x_1 T$

$= x_1$

**Visualising Linear Transformations**

Consider $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ linear with

$T \mathbf{e}_1 = \mathbf{a}_1, T \mathbf{e}_2 = \mathbf{a}_2$

$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$
Note grid lines go to

**E.g. 8** Scaling axes

\[
A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}
\]

Let \( T = T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \)