
**Aim Lecture** Examine how transposes allow you to swap results from left

**Propn 1** \(((A^T)^T) = \)

**Proof** Clear from any e.g.

**Propn 2** For matrices \(A, B \& \lambda \in \mathbb{F}\), the following hold if they make sense.

1) \((A + B)^T = \)

2) \((\lambda A)^T = \)

3) \((AB)^T = \)

**Proof** Just compute both sides using defns.
We’ll only check 3)

$[(AB)^T]_{ij} = [AB]_{ji} = \sum$

$(AB)^T =$

**Applicn** We’ll show matrix distrib law $L(M + N) = LM + LN$ gives other distrib law $(A + B)C =

(A + B)C = (((A + B)C)^T)^T

= (C^T(A + B)^T)^T = (C^T(A^T + B^T))^T$

**Defn** A matrix $A$ is symmetric if
e.g.
N.B. Any symmetric matrix is square.

**EROs & Elementary matrices**

e.g.

\[
\begin{pmatrix}
a & b \\
c & d \\
e & f
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix} =
\]

The right multn by the $2 \times 2$-matrix above performs an “elem column operation” col 2

\rightarrow

We generalise as follows.

For $i \neq j$ & $c \in \mathbb{F}$ define $n \times n$-matrix by

$E_c(i, j) = $
i.e. $E_c(i, j)$ looks like $I_n$ except the

**Fact** Let $A = (a_1 \ldots a_n) \in M_{mn}$. Then $AE_c(i, j)$ is the same as $A$ except the

i.e. right multn by $E_c(i, j)$ performs the elem col opern

**Proof** Let’s check $j$-th col of $AE_c(i, j)$ which is

$A$
Can check other col sim.

Let’s apply transpose to fact with $A = B^T$.

$$(B^T E_c(i, j))^T = (E_c(i, j))^T B = E_c(j),$$

Since $T$ swaps columns &

**Fact** $^T$ Left multn by $E_c(j, i)$ performs the ERO

**Proof** Let’s check $j$-th row, others easier.

$j$-th row $E_c(j, i)B = j$-th col $B^T E_c(i, j)$

$= j$-th col $B^T +$

$= j$-th row $B +$

**Factorising** Suppose we apply the ERO $A$

We can get $A'$ from $A$ by applying the “in-
verse” ERO

\[ \text{Fact}^T \implies A = E_{-c}() \]

**Row Swaps** can also be implemented by matrix multn. For \( i \neq j \), let

\[ E(i, j) := \]

i.e. \( E(i, j) \) is like \( I_n \) except that

(i, i)-th & (j, j)-th entries

(i, j)-th &

**Fact** Left multn by \( E(i, j) \) performs the ERO
No **proof** It’s clear from any e.g.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d \\
e & f \\
\end{pmatrix}
= 
\begin{pmatrix}
a & b \\
c & d \\
e & f \\
\end{pmatrix}
\]

**Defn** A square matrix $A$ is said to be upper (resp. lower

\[
[A]_{ij} =
\]

**e.g.**

\[
\begin{pmatrix}
1 & 2 & 3 \\
& 4 & 5 \\
& & 6 \\
\end{pmatrix},
\begin{pmatrix}
1 \\
2 & 3 \\
4 & 5 & 6 \\
\end{pmatrix}
\]

N.B. The transpose of an upper triang ma-
trix is

Matrices of the form $E(i, j)$, $E_c(i, j)$ are called

**Propn** Any square matrix $A$ can be factorised as

$$A = E_1 E_2 \ldots E_k A'$$

where $A'$ is upper

**Proof** Clear from facts above &
e.g.

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 2 \\
0 & 1 & 3
\end{pmatrix}
\]