Lecture 14: Row Echelon Form. Elementary Row Operations.

**Aim Lecture** Key to solving lin eqns involves 2 concepts.

i) Elementary Row Operations (EROs): which

& ii) Row Echelon Form: which are systems of lin eqns, sufficiently

**Analyse Easy Example** Solve

\[ x - 2y = 3 \quad \ldots (1) \]
\[ x + y = -3 \quad \ldots (2) \]

Eliminate \( x \): (2) - (1)
In fact solns to (*) same as solns to

\[ x - 2y = 3 \quad \cdots (1) \]

\[ = \quad \cdots (2) \]

**Row Echelon Form** Consider system of lin eqns \( A \mathbf{x} = \mathbf{b} \) with \( A \) an \( m \times n \)-matrix with entries in field \( \mathbb{F} \), and \( \mathbf{b} \in \mathbb{F}^m \). Need solve for \( \mathbf{x} \in \mathbb{Q} \)

Q When can you solve for variables, 1 at a time as in (**) above?

A When \( A \) is in

First some auxiliary defns.

Leading entry means first (= leftmost) non-
A leading row (or column) is one

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
\pi & -1 & 1 & 1 \\
\end{pmatrix}
\]

**Defn** A matrix $A$ is in row echelon form if

i) all non-leading rows are

ii) as you go down rows, the leading
e.g. Following are not in row echelon form
\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

e.g. Following is in row echelon form

**Back-Substitution** Can solve \((A|b)\) easily using “back-substitution” method below if
e.g. Solve

$$
\begin{pmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & 1 & -1 \\
0 & 0 & 2 & 4
\end{pmatrix}
$$

i.e. $x_1 + x_2 + 2x_3 = 1 \ldots (1)$

(3) $\implies x_3$

(2) $\implies x_2+$

(1) $\implies x_1$

Hence, soln is $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$

If there are non-leading columns in $A$, you need to introduce a
e.g. Solve (over the reals)

\[
\begin{pmatrix}
1 & -1 & 1 & -1 & 0 \\
0 & 0 & 1 & 2 & 1
\end{pmatrix}
\]

N.B. Expect soln given by a plane

Columns 2 & 4 not leading so intro

2nd row \implies

1st row \implies

Hence, \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \)

Elementary Row Operations

Q How can you alter (simplify) a system $(A|b)$ of lin eqns without

A You can apply any of the following

ERO1 Switch 2 rows of

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{pmatrix}
\]

Same set of solns as this corresp to

ERO2 Multiply a row by
Let’s rewrite original example in new notn

\[ x - 2y = 3 \]
\[ x + y = -3 \]