Lecture 9: Existence & Construction of Bases

**Aim** We show the existence of bases in the finite dimensional case and illustrate methods of finding them.

**Thm 1** Let $V = \text{vector space}/\text{field } \mathbb{F}$

Let $S \subset V$ be a spanning set of $n$ elts. Any subspace $W$

Proof: Consider all lin indep subsets $B = \{w_1, \ldots, w_m\}$ of $W$.

Pick one such $B$ with

This is possible as thm 1 lecture 8 $\implies$
We seek to show that $B$ is a basis.

Why? Since $B$ is lin indep we need only

Let $w \in W$. Since $B$ was chosen so that $m$

was maximal,

Consider non-trivial

If $\lambda = 0$

Solving for $w$

Since $w$ was an arbitrary element of $W$, $B$
Found basis as max lin indep subset. Method was non-constructive i.e. there’s no algorithm with which to construct a basis.

**Classification of Subspaces of** $\mathbb{R}^3$

Let $W = \dim \mathbb{R}^n = n$ so thm 1 $\implies$

$W =$

If $n = 3$ there are

$m=0:$

$m=1:$

$m=2:$
m=3:
\[ \therefore \text{the 3 lin indep} \]

**Constructing Bases**

**Thm 2** (Reducing spanning sets to bases)

Let \( A = (v_1 \ v_2 \ldots \ v_n) \). Recall \( \text{col}(A) = \)

Let \( U \) be a

Then we have the following basis of \( \text{col}(A) \)

\[ B = \{v_i\} \]

Proof: will hopefully be clear from following

e.g. (else see notes §6.7.3 p.48)

**E.g. 1**

\[
A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & -3 & 2 & 1 \end{pmatrix} = (v_1 \ v_2 \ v_3 \ v_4)
\]
Find a basis for $\text{col}(A)$.

Ans:

1st & 2nd

Thm 2 $\implies$

**Why did it work?** i.e.

Check lin indep: Omitting 3rd & 4th column
from above calculation, we see

Check span: 3rd & 4th columns correspond to parameters in soln

Pick soln $\mathbf{x}$ with

Back substn $\implies$

$$\mathbf{0} = A \mathbf{x} =$$

Sim, setting

get soln $\mathbf{x} =$
so $v_4 \in$

$\therefore$ Span($v_1, v_2$) =

Hence, $\{v_1, v_2\}$ is lin indep & spans col($A$)
so is a basis.

**Thm 3** (Completing a lin indep set to a basis)

Suppose $W$ is a subspace of $\mathbb{F}^m$ and $S = \{v_1, \ldots, v_n\} \subset W$

There exists a basis $B$ of

Proof: In fact we have

**Algorithm for finding $B$**

Let $\{w_1, \ldots, w_r\}$ be

Consider the matrix
\[ A = \]

Note \( \text{col}(A) \supseteq \)

so

Hence we can apply the method of thm 2 to

Note: \( \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \) lin indep \( \implies \)

\[ \because \text{the basis } B \text{ produced by this method} \]

Hopefully, the reason why this works will be clear from the following e.g.

**E.g. 2** Let \( S = \{\mathbf{v}_1 = (1, 2, -1)^T, \mathbf{v}_2 = (3, 2, -1)^T\} \). Extend \( S \) to a basis of \( \mathbb{R}^3 \).
Ans: Note vectors not parallel  \( \implies \)
Let \( \mathbf{w}_1 \),

1st, 2nd & 4th

i.e.

**Why did it work?** i.e.
If either the 1st or 2nd column was not lead-
ing then, ignoring

we see we can solve

This contradicts

**Remark** What happens to this method if $S$ is lin dependent so that $S$ cannot form part of a basis?

Ans: Method produces a basis with as many members of $S$ as possible.

**More Results on Dim**

**Propn** Let $W$ be a subspace of
If \( \dim V = \dim W \) then

**E.g. 3** The only 10-dim subspace of

Proof Propn: Let \( B \) be a basis for \( W \).

\( B \) is

Cor 1 lecture 8 \( \iff \)

\( \therefore \)