MATH3711: Higher Algebra (2006, S1)

Problem Sheet 2

1. Find the subgroup of \( \mathbb{Z} \) generated by 4 and 6.

2. Consider \( \sigma \in S_6 \) defined using 2 line notation by
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 5 & 6 & 4 & 2 & 1
\end{pmatrix},
\]
Write out \( \sigma \) explicitly as a product of transpositions and hence determine whether it is odd or even. Verify your answer by computing \( \sigma\Delta \) where \( \Delta \) is the difference product.

3. Show that \( \Delta^2 \) is a symmetric function.

4. Let \( f(x_1, \ldots, x_n) \) be a complex polynomial. Show that the following two conditions on \( f \) are equivalent: i) for any transposition \( \tau \) we have \( \tau f = -f \) and ii) for any \( \sigma \in S_n \) we have \( \sigma f = f \) when \( \sigma \) is even and \( \sigma f = -f \) when \( \sigma \) is odd. Such a polynomial is said to be anti-symmetric. Find some examples.

5. There is a right-handed version of all the results in lectures 6. For \( H \leq G \), we define a right coset of \( H \) in \( G \) to be a set of the form \( Hg := \{hg|h \in H\} \) for some \( g \in G \). Show that \( G \) is also a disjoint union of right cosets. (The sophisticated approach is via \( G^{op} \).) The set of right cosets is denoted \( H \backslash G \). Let \( \iota : G \rightarrow G : g \mapsto g^{-1} \) be the inverse map. It is clearly a bijection. Show that \( \iota(Hg) = g^{-1}H \) so \( \iota \) gives a 1-1 correspondence between \( H \backslash G \) and \( G/H \). That is why there is no left or right index.

6. Let \( G \) be the symmetric group on 4 symbols \( S_4 \) and \( H \) be the subset \( \{\sigma|\sigma(4) = 4\} \). Show that \( H \) is a subgroup. Compute all the left and right cosets of \( H \) in \( G \). Verify Lagrange’s theorem and the 1-1 correspondence between left and right cosets given in question 5.

7. Let \( H, K \) be subgroups of \( G \) of order 3 and 5 respectively. Use Lagrange’s theorem to show that \( H \cap K = 1 \).

8. Let \( G \) be a group with prime order. Use Lagrange’s theorem to find all subgroups of \( G \). Show that \( G \) is cyclic.

9. Using the previous exercise or otherwise, find all subgroups of \( S_3 \).

10. Show associativity of the subset product claimed in lecture 7 i.e. for subsets \( K_1, K_2, K_3 \) of a group \( G \) we have \( (K_1K_2)K_3 = K_1(K_2K_3) \).

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11. Let $G = \mathbb{C}^*$ and $H$ be the subset of complex numbers of modulus 1. Show that $H$ is a normal subgroup of $G$ and describe the cosets of $H$. Show that $G/H$ is isomorphic to a subgroup of $\mathbb{R}^*$.

12. Show that $A_n \leq S_n$ is generated by 3-cycles.

13. Let $G = GL_2$ and let $H$ be the subgroup of elements of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, c \in \mathbb{R}^*$ and $b \in \mathbb{R}$. Compute all the left and right cosets of $H$ in $G$. If you know some projective geometry you may wish to show that $G/H$ can be naturally identified with the real projective line.

14. Compute explicitly all cosets of $SL_n := \{M \in GL_n | \det M = 1 \}$ in $GL_n$.

15. Compute all cosets of $O_2$ in $GL_2$.

16. Let $G$ be a group and $H$ be a subgroup of index two. Show that $H$ is normal.

17. Why is $H = A_n$ normal in $G = S_n$? Find a group isomorphic to $G/H$.

18. Let $z \in \mathbb{C}^*$ and $\phi$ be multiplication by $z$. Is $\phi$ a group homomorphism from a) $\mathbb{C} \rightarrow \mathbb{C}$, b) $\mathbb{C}^* \rightarrow \mathbb{C}^*$?

19. Find all isomorphisms $\phi : \mathbb{Z}/p\mathbb{Z} \sim \rightarrow \mathbb{Z}/p\mathbb{Z}$ where $p$ is prime.

20. Isomorphic groups should be identical as far as their group structure is concerned. To illustrate this, consider an isomorphism $\phi : G \rightarrow G'$. Show

   (a) $G$ is abelian if and only if $G'$ is.
   (b) $G, G'$ have the same order.
   (c) There is a natural bijection between the subgroups of $G$ and the subgroups of $G'$. It preserves orders, inclusions and normality.
   (d) If $g \in G$ has order $n$, so does $\phi(g)$.

21. Show that $S_3$ and $\mathbb{Z}/6\mathbb{Z}$ both have order 6 (so are isomorphic sets) but are not isomorphic as groups.

22. Fix an integer $n \geq 2$. Suppose that $G$ is a group with distinct elements $\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \sigma\tau, \ldots, \sigma^{n-1}\tau\}$ where $\sigma^n = 1 = \tau^2$ and $\tau\sigma = \sigma^{-1}\tau$. Show that $G$ is isomorphic to $D_n$.

23. Find all normal subgroups $H$ of $D_n$. Show that $G/H$ is dihedral or cyclic. (N.B. This means isomorphic to a dihedral group or cyclic group).

24. For $\sigma \in S_n$ we let $\Phi(\sigma)$ be the linear transformation $\Phi(\sigma) : (x_1, \ldots, x_n)^t \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})^t$. Show that $\Phi : S_n \rightarrow GL_n$ is a group homomorphism. Determine its image.

25. Show that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the group $\mu_n$ introduced in problem sheet 1.