MATH3710: Higher Algebra I, 
Problem Sheet 3

1. Consider the subgroup \( \mathbb{R} \) of \( \mathbb{C} \) (you need not show it is a subgroup). Describe geometrically, all the cosets of \( \mathbb{R} \) in \( \mathbb{C} \). Identify the group \( \mathbb{C}/\mathbb{R} \) i.e. show it is isomorphic to a well-known group we have already studied in class.

2. Recall that \( \mathbb{R} \) is a group when endowed with addition and \( H := \{ z \in \mathbb{C} | |z| = 1 \} \) is a subgroup of \( \mathbb{C}^* \). Using the exponential function and the first isomorphism theorem, show that \( H \) is isomorphic to a quotient group of \( \mathbb{R} \). State explicitly what this quotient group is. Show using similar methods or otherwise that \( \mathbb{Q}/\mathbb{Z} \) is isomorphic to a subgroup of \( \mathbb{C}^* \).

3. Let \( \phi : \mathbb{C}^* \rightarrow \mathbb{C}^* : z \mapsto z^n \) for some positive integer \( n \). Show that \( \phi \) is a group homomorphism. Find \( \ker \phi, \text{im } \phi \). What isomorphism does the the first isomorphism theorem give? Verify that the fibres of \( \phi \) are indeed the cosets of \( \ker \phi \).

4. Weak version of Chinese remainder theorem. Let \( m, n \) be relatively prime positive integers. Consider the homomorphism \( \phi : \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) defined by \( \phi(a) = (a + m\mathbb{Z}, a + n\mathbb{Z}) \). Find \( \ker \phi \). Compare the orders of \( \mathbb{Z}/ \ker \phi \) and \( \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) to determine the image of \( \phi \). Use the first isomorphism theorem to find which cyclic group \( \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) is isomorphic to.

5. Let \( T, U \) be sets and \( S \) be their disjoint union. Consider the subset \( G \) of \( \text{Perm} \, S \) consisting of permutations \( \sigma \) such that \( \sigma(T) = T, \sigma(U) = U \). (Note that \( G \) is a subgroup). Use the universal property of products to construct a group isomorphism \( \sim \rightarrow \text{Perm} \, T \times \text{Perm} \, U \).

6. Let \( G \) be the dihedral group of order \( 2n \) and \( N \) the (unique) cyclic subgroup of order \( n \) (\( N =< \sigma > \) in the lecture notes). Let \( H \) be the group generated by any \( \tau \notin N \). Verify the third isomorphism theorem in this case and compute explicitly the isomorphism.
7. Suppose $N \trianglelefteq G, N' \trianglelefteq G'$. Show that $N \times N'$ is naturally a normal subgroup of $G \times G'$ and show $(G \times G')/(N \times N') \simeq (G/N) \times (G'/N')$.

8.