1. Given the following equation in a group $x^{-1}yxz^2 = 1$, solve for $y$.

2. Let $GL_n(\mathbb{Z})$ be the set of $n \times n$ matrices $M$ with integer entries such that $M^{-1}$ exists and also has integer entries. Show that $GL_n(\mathbb{Z})$ forms a group when endowed with matrix multiplication.

3. In this question, we identify $1 \times 1$ matrices with their unique entry so that $GL_1(\mathbb{C})$ gets identified with $\mathbb{C}^*$, the non-zero elements in $\mathbb{C}$. Let $\mu$ be the subset of roots of unity of $\mathbb{C}^*$. (Recall that a root of unity is a complex number $\zeta$ such that $\zeta^n = 1$ for some integer $n$). Show that, $\mu$ is a subgroup of $\mathbb{C}^*$. Show that the subset $\mu_n$ of $n$-th (not necessarily primitive) roots of unity is in turn a subgroup of $\mu$.

4. (Mildly non-trivial according to my students in previous years.) Show that any finitely generated subgroup of $\mu$ is cyclic. Show that $\mu$ is not finitely generated and find a non-trivial subgroup of $\mu$ which is not finitely generated.

5. Consider a permutation $\sigma \in S_n$ and a $k$-cycle $(a_1 a_2 \ldots a_k)$. Express the product $\sigma(a_1 \ldots a_k)\sigma^{-1}$ in cycle notation. What is its order?

6. Write out the multiplication table for $S_3$.

7. In the symmetric group $S_6$, describe all the elements of the subgroup $H$ generated by the 3 generators $(12), (34), (56)$. In particular, what is the order of $H$?

8. Describe explicitly, the subgroup $H$ of $GL_2(\mathbb{C})$ generated by the matrices

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\zeta & 0 \\
0 & \zeta^{-1}
\end{pmatrix}
$$

where $\zeta$ is a primitive $n$-th root of unity. This is the binary dihedral group.
9. Determine explicitly the elements of the cyclic group generated by
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

10. Consider the matrix
\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

Let \(Sp_2(\mathbb{R}) := \{A | A \text{ is a real 2 by 2 matrix, } A^tJA = J\}\). Show that \(Sp_2(\mathbb{R})\) is a subgroup of \(GL_2(\mathbb{R})\). It is called the symplectic group.

11. Find the order of the permutation \((12)(345)\). More generally, given disjoint cycles \(\sigma, \tau\), find the order of \(\sigma\tau\).

12. Find the orders of all elements in the dihedral group \(D_n\).

13. Show that the subset \(SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})\) of matrices of determinant 1 is a subgroup.

14. For a \(k\)-cycle \((a_1a_2\ldots a_k)\) find \((a_1a_2\ldots a_k)^n\) for any integer \(n\).

15. Let \(H < S_5\) be the subgroup of the symmetric group generated by \((23), (34)\). Describe \(H\) in such a way that it allows you to compute the order of \(H\). What is the order? (Hint: there is a simple reason why \(H\) has at most 6 elements.)

16. Let \(T\) be a subset of \(S\). Show that \(\{\sigma \in \text{Perm } S | \sigma(t) = t \text{ for every } t \in T\}\) is a subgroup of \(\text{Perm } S\).