Section 3: Functions of several variables.

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S1: **Motivation.** Phenomena of a complex nature usually depend on more than one variable.

**Applications matter!** The amount of power $P$ (in watts) available to a wind turbine can be summarised by the equation

$$P = \frac{1}{2} \left(\frac{49}{40}\right) (\pi r^2) v^3$$

where

$r = \text{diameter of turbine blades exposed to the wind (m)}$

$v = \text{wind speed in m/sec}$

$49/40$ is the density of dry air at 15 deg C at sea level (kg/m$^3$).

See that the power $P$ depends on two variables, $r$ and $v$, that is, $P = f(r, v)$.
You have already studied functions of 1 variable at school. You developed curve–sketching skills and a knowledge of calculus for functions of the type

\[ y = f(x). \]

In this section we extend these ideas to functions of many variables. In particular, we will learn the idea of a derivative for these more complicated functions.

Such ideas give us the power to more accurately model and understand complex phenomena like that of the previous example.
S2: Functions of two variables.

We will consider functions of the type

\[ z = f(x, y), \quad (x, y) \in U \]

where \( U \subseteq \mathbb{R}^2 \) is the domain of \( f \) and \( f \) is real–valued. We write \( f : U \subseteq \mathbb{R}^2 \to \mathbb{R} \).

**Examples** of \( f \) and \( U \):

\[
\begin{align*}
  f(x, y) &= x \cos y + xy \sin x, \quad U = \mathbb{R}^2 \\
  f(x, y) &= \frac{1}{2x - y}, \quad U = \{(x, y) \in \mathbb{R}^2 : y \neq 2x\} \\
  f(x, y) &= \sqrt{x + y}, \quad U = \{(x, y) \in \mathbb{R}^2 : x + y \geq 0\}.
\end{align*}
\]

More simply:
S3: Visualising & sketching.

As a first step to understanding functions of two variables, we now develop some methods for visualising and sketching their graphs.

**DEFINITIONS  Level Curve, Graph, Surface**

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a level curve of $f$. The set of all points $(x, y, f(x, y))$ in space, for $(x, y)$ in the domain of $f$, is called the graph of $f$. The graph of $f$ is also called the surface $z = f(x, y)$.
Graphs for functions of 2 variables will be surfaces in $\mathbb{R}^3$. 

Surface $z = f(x, y)$
For a given function $f$, we can determine the nature of its graph by examining how the surface intersects with various planes and then build the surface from these curves.

**Horizontal planes: Contour curves & level curves.**

A **contour curve** of $f$ is the curve of intersection between the surface $z = f(x, y)$ and a horizontal plane $z = c$, $c = \text{constant}$. For simple cases, a contour curve can be easily drawn in $\mathbb{R}^3$ and observe that this curve also lies in the plane $z = c$. If we sketch our contour curve in the $XY$-plane, then we obtain what is known as a **level curve** of $f$. 
The contour curve \( f(x, y) = 100 - x^2 - y^2 = 75 \) is the circle \( x^2 + y^2 = 25 \) in the plane \( z = 75 \).

The level curve \( f(x, y) = 100 - x^2 - y^2 = 75 \) is the circle \( x^2 + y^2 = 25 \) in the \( xy \)-plane.
(a) $z = e^{-(x^2 + y^2)^{1/8}}(\sin x^2 + \cos y^2)$

(b) $z = \sin x + 2 \sin y$
(c) $z = (4x^2 + y^2)e^{-x^2-y^2}$

(d) $z = xy e^{-y^2}$
Ex: Sketch the surface

\[ z = x^2 + \frac{y^2}{9}. \]
Ex: Sketch the level curves associated with $f(x, y) = y^2 - x^2$.

There are many other important surfaces, which we list a little later.
Applications matter!

The idea of “contour curves” is similar to that used to prepare contour maps where lines are drawn to represent constant altitudes. Walking along a line would mean walking on a level path.
**Level surface**

**DEFINITION**  **Level Surface**

The set of points \((x, y, z)\) in space where a function of three independent variables has a constant value \(f(x, y, z) = c\) is called a **level surface** of \(f\).

\[
\sqrt{x^2 + y^2 + z^2} = 1 \\
\sqrt{x^2 + y^2 + z^2} = 2 \\
\sqrt{x^2 + y^2 + z^2} = 3
\]

**FIGURE**  
The level surfaces of \(f(x, y, z) = \sqrt{x^2 + y^2 + z^2}\) are concentric spheres.
Common surfaces + their properties.

Ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

The ellipsoid

\( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

has elliptical cross-sections in each of the three coordinate planes.
Paraboloid \( z/c = x^2/a^2 + y^2/b^2 \)

The elliptical paraboloid \( (x^2/a^2) + (y^2/b^2) = z/c \) shown for \( c > 0 \). The cross-sections perpendicular to the \( z \)-axis above the \( xy \)-plane are ellipses. The cross-sections in the planes that contain the \( z \)-axis are parabolas.
Hyperboloid (1 sheet) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Part of the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ in the $xz$-plane

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ in the plane $z = c$

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the $xy$-plane

Part of the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ in the $yz$-plane

The hyperboloid $(\frac{x^2}{a^2}) + (\frac{y^2}{b^2}) - (\frac{z^2}{c^2}) = 1$

Planes perpendicular to the $z$-axis cut it in ellipses. Vertical planes containing the $z$-axis cut it in hyperbolas.
Hyperboloid (2 sheets) \[ \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

The hyperbola \[ \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1 \]
in the plane \( z = c\sqrt{2} \)

The hyperbola \[ \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1 \]
in the \( yz \)-plane

The hyperboloid \((z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1\)

Planes perpendicular to the \( z \)-axis above and below the vertices cut it in ellipses. Vertical planes containing the \( z \)-axis cut it in hyperbolas.
Elliptic Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

The line $z = -\frac{c}{b} y$ in the $yz$-plane

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the plane $z = c$

The line $z = \frac{c}{a} x$ in the $xz$-plane

The elliptical cone $(x^2/a^2) + (y^2/b^2) = (z^2/c^2)$

Planes perpendicular to the $z$-axis cut the cone in ellipses above and below the $xy$-plane. Vertical planes that contain the $z$-axis cut it in pairs of intersecting lines.
Hyperbolic Paraboloid \( \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c} \)

The hyperbolic paraboloid \( \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0 \). The cross-sections in planes perpendicular to the \( z \)-axis above and below the \( xy \)-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.
Review your understanding:

1) Generally speaking, what is the form of the graph of $f(x, y)$?

2) T/F: If the level curves of a function $f(x, y)$ are concentric circles, then the graph is a cone.

3) T/F: For $z = f(x, y)$ the $z$ value measures how far each point on the surface lies above or below the point $(x, y)$ in the $XY$–plane.
S4: Limits and continuity.

For functions of two variables, the idea of a limit is more profound due to the more general domains of these functions.

If $R$ is the domain of $f$ then we can approach $(x_0, y_0)$ from many different directions (not just from 2 directions as in first-year studies).
DEFINITION  Limit of a Function of Two Variables

We say that a function \( f(x, y) \) approaches the limit \( L \) as \((x, y)\) approaches \((x_0, y_0)\), and write

\[
\lim_{{(x, y) \to (x_0, y_0)}} f(x, y) = L
\]

if, for every number \( \epsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that for all \((x, y)\) in the domain of \( f \),

\[
|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.
\]

Roughly speaking our definition says that the distance between \( f(x, y) \) and \( L \) becomes (arbitrarily) small when the distance between \((x, y)\) and \((x_0, y_0)\) is sufficiently small (but not zero).

Above we always assume that \((x, y)\) is in the domain of \( f \) so that limits of boundary points may be included.
THEOREM  Properties of Limits of Functions of Two Variables

The following rules hold if \( L, M, \) and \( k \) are real numbers and

\[
\lim_{{(x, y) \to (x_0, y_0)}} f(x, y) = L \quad \text{and} \quad \lim_{{(x, y) \to (x_0, y_0)}} g(x, y) = M.
\]

1. **Sum Rule:**
\[
\lim_{{(x, y) \to (x_0, y_0)}} (f(x, y) + g(x, y)) = L + M
\]

2. **Difference Rule:**
\[
\lim_{{(x, y) \to (x_0, y_0)}} (f(x, y) - g(x, y)) = L - M
\]

3. **Product Rule:**
\[
\lim_{{(x, y) \to (x_0, y_0)}} (f(x, y) \cdot g(x, y)) = L \cdot M
\]

4. **Constant Multiple Rule:**
\[
\lim_{{(x, y) \to (x_0, y_0)}} (kf(x, y)) = kL \quad \text{(any number \( k \))}
\]

5. **Quotient Rule:**
\[
\lim_{{(x, y) \to (x_0, y_0)}} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \quad M \neq 0
\]

6. **Power Rule:** If \( r \) and \( s \) are integers with no common factors, and \( s \neq 0 \), then
\[
\lim_{{(x, y) \to (x_0, y_0)}} (f(x, y))^{r/s} = L^{r/s}
\]
provided \( L^{r/s} \) is a real number. (If \( s \) is even, we assume that \( L > 0 \).)
Ex: If

\[ f(x, y) := \frac{x^2 + y^2 + 1}{x + y} \]

then calculate

\[ \lim_{(x, y) \to (1, 2)} f(x, y). \]
Ex: If

\[ f(x, y) := x^2 + y^2 + 3 \]

then formally prove \( f(x, y) \to 3 \) as \((x, y) \to (0, 0)\).
Ex: If

\[ f(x, y) := \frac{y}{x^4 + 1} \]

then formally prove \( f(x, y) \to 0 \) as \( (x, y) \to (0, 0) \).
Two-Path Test for Nonexistence of a Limit
If a function $f(x, y)$ has different limits along two different paths as $(x, y)$ approaches $(x_0, y_0)$, then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

Ex: Show that

$$f(x, y) := \frac{3x^3 y}{x^4 + y^4}$$

has no limit as $(x, y) \to (0, 0)$. 
**DEFINITION**  Continuous Function of Two Variables

A function \( f(x, y) \) is **continuous at the point** \((x_0, y_0)\) if

1. \( f \) is defined at \((x_0, y_0)\),
2. \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) \) exists,
3. \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0,y_0) \).

A function is **continuous** if it is continuous at every point of its domain.

**Ex:** If \( f(x,y) := \frac{2xy}{2 + \sin x} \) then show \( f \) is continuous at \((0,0)\). Hint: Use Young’s inequality \( 2ab \leq a^2 + b^2 \).
Ex: Show that

\[ f(x, y) = \begin{cases} 
\frac{2x^2}{x^2+y^2}, & (x, y) \neq (0, 0); \\
0, & (x, y) = (0, 0) 
\end{cases} \]

is not continuous at \((0, 0)\).
Ex: By switching to polar co–ordinates $x = r \cos \theta$, $y = r \sin \theta$ and using the fact that if $f$ has a limit $L$ then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r \to 0} f(r \cos \theta, r \sin \theta) = L$$

show that

$$f(x, y) = \begin{cases} 
\frac{x^3}{x^2+y^2}, & (x, y) \neq (0, 0); \\
0, & (x, y) = (0, 0) 
\end{cases}$$

is continuous at $(0, 0)$. 
S5: Partial differentiation.

We know from elementary calculus that the idea of a derivative is very helpful in the mathematical analysis of applied problems.

We now extend this concept to functions of two variables.

For a function of two variables $f = f(x, y)$, the basic idea is to determine the rate of change in $f$ with respect to one variable, while the other variable is held fixed.
DEFINITION     Partial Derivative with Respect to x

The **partial derivative of** \( f(x, y) \) **with respect to** \( x \) **at the point** \( (x_0, y_0) \) **is**

\[
\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},
\]

provided the limit exists.

Essentially \( \partial f / \partial x \) is just the derivative of \( f \) with respect to \( x \), keeping the \( y \) variable fixed.
The intersection of the plane \( y = y_0 \) with the surface \( z = f(x, y) \), viewed from above the first quadrant of the \( xy \)-plane.
DEFINITION  Partial Derivative with Respect to $y$

The partial derivative of $f(x, y)$ with respect to $y$ at the point $(x_0, y_0)$ is

$$\frac{\partial f}{\partial y} \bigg|_{(x_0, y_0)} = \left. \frac{d}{dy} f(x_0, y) \right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

Essentially $\partial f / \partial y$ is just the derivative of $f$ with respect to $y$, keeping the $x$ variable fixed.
The intersection of the plane \( x = x_0 \) with the surface \( z = f(x, y) \), viewed from above the first quadrant of the \( xy \)-plane.
Ex: If $f(x, y) := x^3y + y^2$ then calculate $\partial f/\partial x$ and $\partial f/\partial y$. 
**EXAMPLE** Finding Partial Derivatives at a Point

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$ 

**Solution** To find $\partial f/\partial x$, we treat $y$ as a constant and differentiate with respect to $x$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$ 

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat $x$ as a constant and differentiate with respect to $y$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$ 

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. 

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Ex: If $f(x, y) := \sin(xy)$ then calculate: $f_x(0, \pi); f_y(2, 0)$. 
Ex: For $f(x, y) := x^{2/3}y^{1/3}$ show $\frac{\partial f}{\partial x} = 0$ at $(0, 0)$. 
Product and quotient rules for partial differentiation are defined in the natural way:

\[
\frac{\partial}{\partial x} (uv) = u_x v + v_x u \\
\frac{\partial}{\partial x} \left( \frac{u}{v} \right) = \frac{u_x v - v_x u}{v^2} \\
\frac{\partial}{\partial y} (uv) = u_y v + v_y u \\
\frac{\partial}{\partial y} \left( \frac{u}{v} \right) = \frac{u_y v - v_y u}{v^2}.
\]
EXAMPLE Finding a Partial Derivative as a Function

Find \( \partial f/\partial y \) if \( f(x, y) = y \sin xy \).

Solution We treat \( x \) as a constant and \( f \) as a product of \( y \) and \( \sin xy \):

\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y} (y)
\]

\[
= (y \cos xy) \frac{\partial}{\partial y} (xy) + \sin xy = xy \cos xy + \sin xy.
\]
Ex: The plane \( x = 1 \) intersects the paraboloid \( z = x^2 + y^2 \) in a parabola as shown in the following diagram. Calculate the slope of that tangent line to the parabola at the point \((1,2,5)\).
Partial derivatives + continuity

\[ z = \begin{cases} 
0, & xy \neq 0 \\
1, & xy = 0 
\end{cases} \]

The graph of \( f(x, y) = \begin{cases} 
0, & xy \neq 0 \\
1, & xy = 0 
\end{cases} \)

consists of the lines \( L_1 \) and \( L_2 \) and the four open quadrants of the \( xy \)-plane. The function has partial derivatives at the origin but is not continuous there.

**Continuity of partial derivatives** of \( f \) implies continuity of \( f \) (and also implies what is known as differentiability of \( f \)).
Higher derivatives:

We define and denote the second–order partial derivatives of \( f \) as follows:

\[
\begin{align*}
    f_{xx} &= (f_x)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
    f_{xy} &= (f_x)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
    f_{yx} &= (f_y)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
    f_{yy} &= (f_y)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}
\end{align*}
\]
See that there are four partial second-order derivatives and two of them are “mixed”.

The order of differentiation is, in general, important (but see below for an important exception).
Ex: If \( f(x, y) = 1 + x^5 + y^3 \) then compute all four 2nd–order partial derivatives. What do you notice about the mixed derivatives?

**THEOREM**  
**The Mixed Derivative Theorem**

If \( f(x, y) \) and its partial derivatives \( f_x, f_y, f_{xy}, \text{ and } f_{yx} \) are defined throughout an open region containing a point \((a, b)\) and are all continuous at \((a, b)\), then

\[
 f_{xy}(a, b) = f_{yx}(a, b).
\]
Ex: Consider the function \( f(x, y) = xy + x + y \).

(a) Calculate \( f_{xx} \) and \( f_{yy} \).

(b) Use (a) to show

\[
f_{xx} - xf_{yy} = 0 \tag{1}
\]
Applications matter!

Equation (1) is known as a *partial differential equation* (PDE).

A PDE involves: partial derivatives of an unknown function and an equals sign.

The PDE (1) is a special equation known as the Euler–Tricomi equation, which is used to describe “transonic” fluid (air) flow over aircraft.

Part (b) from the previous example says that $f(x, y) = xy + x + y$ is a solution to the Euler–Tricomi equation.
S6: Differentiability & chain rules.

The goal of this section is to suitably define the concept of differentiability of functions $f = f(x, y)$ and explore some of the interesting consequences and applications including the chain rule.

In first-year you learnt that a function $f = f(x)$ is differentiable at a point $x = x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

exists (2)

and we denote the value of this limit by $f'(x_0)$. 
In particular, when we say that a function \( f = f(x) \) is differentiable at \( x_0 \) (an interior point of \( \text{dom } f \)), we mean that there is a (unique) affine function \( A \) that suitably approximates \( f \) near \( x_0 \).

In the case \( f = f(x) \), the affine function is of the form \( A(x) = ax + b \), where \( a \) and \( b \) are particular constants. It turns out that the graph of \( A \) is just the tangent line to \( f \) at \( x_0 \) and so

\[
A(x) = f(x_0) + f'(x_0)(x - x_0) = f(x_0) + L(x - x_0)
\]

so that: \( a = f'(x_0) \); and \( b = f(x_0) - f'(x_0)x_0 \). Above, \( L \) is the linear function that represents multiplication by \( a = f'(x_0) \).
What do we mean by “A suitably approximates $f$ near $x_0$”? We mean:

(i) $f(x_0) = A(x_0)$; and

(ii) $f(x) - A(x)$ approaches 0 faster than $x$ approaches $x_0$, that is,

$$\lim_{x \to x_0} \frac{f(x) - A(x)}{x - x_0} = 0,$$

which may be equivalently written as: there is a function $\varepsilon(x)$ such that

$$f(x) = f(x_0) + L(x - x_0) + (x - x_0)\varepsilon(x - x_0),$$

and

$$\lim_{x \to 0} \varepsilon(x) = 0.$$

The above kind of approximation is known as “linear” or “first degree” approximation.
The concept of differentiability is more subtle in the case \( f = f(x, y) \) but we can build a useful definition very naturally from the previous discussion.

When we say that a function \( f = f(x, y) \) is differentiable at \((x_0, y_0)\) (an interior point of \( \text{dom} \ f \)), we mean that there is a (unique) affine function \( A = A(x, y) \) that suitably approximates \( f \) near \((x_0, y_0)\) in the sense that \( f(x, y) - A(x, y) \) goes to zero faster than \((x, y)\) goes to \((x_0, y_0)\). That is, there is a linear function \( L \) such that

\[
\lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - L(x-x_0,y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.
\]

The above may be equivalently written as: there is a function \( \varepsilon = \varepsilon(x, y) \) such that

\[
f(x, y) = f(x_0, y_0) + L(x-x_0, y-y_0)
+ \varepsilon(x-x_0, y-y_0)\sqrt{(x-x_0)^2 + (y-y_0)^2},
\]

and

\[
\lim_{(x,y) \to (0,0)} \varepsilon(x, y) = 0.
\]
**Independent learning ex:** What do you think is the particular form of $A(x, y)$ or $L(x, y)$ and what might the graph of $A$ represent?

**Indep. learning ex:** Show that for $f = f(x)$ the definition of differentiability (3) is equivalent to our first-year definition of differentiability (2) (with $L$ representing multiplication by $a = f'(x_0)$).

Many important functions of two (or more) variables satisfy the following.

**COROLLARY OF THEOREM**  
**Continuity of Partial Derivatives Implies Differentiability**

If the partial derivatives $f_x$ and $f_y$ of a function $f(x, y)$ are continuous throughout an open region $R$, then $f$ is differentiable at every point of $R$.

Thus, functions such as polynomials are always differentiable.
Chain rules.

We have discussed various rules for partial differentiation, like product and quotient rules. What other concepts can we apply to help us to find partial derivatives?

Remember the chain rule for functions of one variable? For functions of more than one variable, the chain rule takes a more profound form.

The easiest way of remembering various chain rules is through simple diagrams.
Case I: \( w = f(x) \) with \( x = g(r, s) \)

If \( w = f(x) \) and \( x = g(r, s) \), then

\[
\frac{\partial w}{\partial r} = \frac{dw}{dx} \frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{\partial x}{\partial s}.
\]
Ex: If \( w = f(r^2 + s^2) \), with \( f \) differentiable, then show that \( f \)
satisfies the partial differential equation (PDE)

\[
s f_r - r f_s = 0.
\]

Indep. learning ex: If \( a \) is a constant and \( f \) and \( g \) are diff’able
then show \( z = f(x + at) + g(x - at) \) satisfies the “wave” equation
\( z_{tt} = a^2 z_{xx} \).
Case II: $w = f(x, y)$ with $x = x(t), y = y(t)$

**THEOREM** Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives $f_x$ and $f_y$ and if $x = x(t), y = y(t)$ are differentiable functions of $t$, then the composite $w = f(x(t), y(t))$ is a differentiable function of $t$ and

$$
\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),
$$

or

$$
\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
$$
Ex: If $f(x, y) = xy^2$ with $x = \cos t$ and $y = \sin t$ then use the chain rule to find $df/dt$. 
Applications matter! Ex: The pressure $P$ (in kilopascals); volume $V$ (litres) and temperature $T$ (degrees $K$) of a mole of an ideal gas are related by the equation $P = 8.31T/V$. Find the rate at which the pressure is changing wrt time when: $T = 300$; $dT/dt = 0.1$; $V = 100$; $dV/dt = 0.2$.

We calculate $dP/dt$ and evaluate it at the above instant.
Case III: \( w = f(x, y) \) with \( x = g(r, s) \), \( y = h(r, s) \)

If \( w = f(x, y) \), \( x = g(r, s) \), and \( y = h(r, s) \), then

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}.
\]
Ex: Let $f$ have continuous partial derivatives. Show that

$$z = f(u - v, v - u)$$

satisfies the PDE

$$zu + zv = 0.$$
Case IV: \( w = f(x, y, z) \) with \( x = x(t), y = y(t), z = z(t) \)

**THEOREM**  **Chain Rule for Functions of Three Independent Variables**

If \( w = f(x, y, z) \) is differentiable and \( x, y, \) and \( z \) are differentiable functions of \( t \), then \( w \) is a differentiable function of \( t \) and

\[
\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.
\]
EXAMPLE Changes in a Function’s Values Along a Helix

Find $dw/dt$ if

$$w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t.$$  

In this example the values of $w$ are changing along the path of a helix. What is the derivative’s value at $t = 0$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t.$$  \hspace{1cm} \text{Substitute for the intermediate variables.}

$$\left( \frac{dw}{dt} \right)_{t=0} = 1 + \cos (0) = 2.$$  \hspace{1cm} \blacksquare
The upper half of the helix

\[ \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \]
Case V: \( w = f(x, y, z) \) with \( x = g(r, s), \ y = h(r, s), \ z = k(r, s) \)

**THEOREM**  
Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that \( w = f(x, y, z), \ x = g(r, s), \ y = h(r, s), \) and \( z = k(r, s) \). If all four functions are differentiable, then \( w \) has partial derivatives with respect to \( r \) and \( s \), given by the formulas

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}
\]

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}.
\]
EXAMPLE Partial Derivatives

Express \( \partial w/\partial r \) and \( \partial w/\partial s \) in terms of \( r \) and \( s \) if

\[
w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.
\]

Solution

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}
\]

\[
= (1) \left( \frac{1}{s} \right) + (2)(2r) + (2z)(2)
\]

\[
= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \quad \text{Substitute for intermediate variable} \ z.
\]

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
\]

\[
= (1) \left( -\frac{r}{s^2} \right) + (2) \left( \frac{1}{s} \right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}
\]
Applications matter! Advection, in mechanical and chemical engineering, is a transport mechanism of a substance or a conserved property with a moving fluid.

The *advection* PDE is

\[
\frac{a}{\partial x} + \frac{\partial u}{\partial t} = 0
\]

(4)

where \( a \) is a constant and \( u(x, t) \) is the unknown function.

Use the chain rule to show that a solution to (4) is of the form \( u(x, t) = f(x - at) \) where \( f \) is a differentiable function.

*Perhaps the best image to have in mind is the transport of salt dumped in a river. If the river is originally fresh water and is flowing quickly, the predominant form of transport of the salt in the water will be advective, as the water flow itself would transport the salt. Above, \( u(x, t) \) would represent the concentration of salt at position \( x \) at time \( t \).
S7: Gradient & directional derivative.

We now generalise our ability to determine the rate of change of \( f \) to \textit{any} direction. The ideas are extensions of partial derivatives.

\textbf{DEFINITION} \hspace{1em} \textit{Directional Derivative}

The \textbf{derivative of} \( f \) \textbf{at} \( P_0(x_0, y_0) \) \textbf{in the direction of} \( \text{the unit vector} \ u = u_1i + u_2j \) is the number

\[
\left( \frac{df}{ds} \right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s},
\]

provided the limit exists.
FIGURE The rate of change of $f$ in the direction of $\mathbf{u}$ at a point $P_0$ is the rate at which $f$ changes along this line at $P_0$. 

Line $x = x_0 + su_1$, $y = y_0 + su_2$ 

$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$ 

Direction of increasing $s$ 

Point $P_0(x_0, y_0)$
The equation \( z = f(x, y) \) represents a surface \( S \) in space (see following diagram).

If \( z_0 = f(x_0, y_0) \) then the point \( P(x_0, y_0, z_0) \) lies on \( S \).

The vertical plane that passes through \( P \) and \( P_0(x_0, y_0) \) that is parallel to \( \hat{u} \) intersects \( S \) in a curve \( C \).

The rate of change of \( f \) in the direction of \( \hat{u} \) is the slope of the tangent line to \( C \) at \( P \).
The slope of curve $C$ at $P_0$ is $\lim_{Q \to P} \text{slope} (PQ)$; this is the directional derivative

$$\left( \frac{df}{ds} \right)_{u,P_0} = (D_u f)_{P_0}.$$
There is a more efficient formula for the directional derivative than the one we have seen. The new formula involves $\nabla f$, the gradient of $f$, which we now explore more deeply.

**DEFINITION**  **Gradient Vector**  
The gradient vector (gradient) of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

obtained by evaluating the partial derivatives of $f$ at $P_0$.

**Ex:** If $f(x, y) := x^3 + y$ then calculate $\nabla f$ and $\nabla f(1, 2)$.

**THEOREM**  **The Directional Derivative Is a Dot Product**  
If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{u,P_0} = (\nabla f)_{P_0} \cdot \mathbf{u},$$

the dot product of the gradient $f$ at $P_0$ and $\mathbf{u}$. 
EXAMPLE Finding the Directional Derivative Using the Gradient

Find the derivative of \( f(x, y) = xe^y + \cos(xy) \) at the point \((2, 0)\) in the direction of \( \mathbf{v} = 3\mathbf{i} - 4\mathbf{j} \).

**Solution** The direction of \( \mathbf{v} \) is the unit vector obtained by dividing \( \mathbf{v} \) by its length:

\[
\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{\sqrt{5}} \mathbf{i} - \frac{4}{\sqrt{5}} \mathbf{j}.
\]

The partial derivatives of \( f \) are everywhere continuous and at \((2, 0)\) are given by

\[
f_x(2, 0) = (e^y - y \sin(xy))_{(2,0)} = e^0 - 0 = 1
\]

\[
f_y(2, 0) = (xe^y - x \sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.
\]

The gradient of \( f \) at \((2, 0)\) is

\[
\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}
\]

(Figure 14.26). The derivative of \( f \) at \((2, 0)\) in the direction of \( \mathbf{v} \) is therefore

\[
(D_u f)|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u}
\]

\[
= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5} \mathbf{i} - \frac{4}{5} \mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.
\]

![Diagram](Image)

**FIGURE** Picture \( \nabla f \) as a vector in the domain of \( f \). In the case of \( f(x, y) = xe^y + \cos(xy) \), the domain is the entire plane. The rate at which \( f \) changes at \((2, 0)\) in the direction \( \mathbf{u} = (3/5)\mathbf{i} - (4/5)\mathbf{j} \) is \( \nabla f \cdot \mathbf{u} = -1 \)
If \( \theta \) is the angle between the vectors \( \hat{u} \) and \( \nabla f \) then the formula

\[
D_{\hat{u}} f = \nabla f \cdot \hat{u} = |\nabla f||\hat{u}| \cos \theta = |\nabla f| \cos \theta
\]

reveals the following properties.

**Properties of the Directional Derivative**

\( D_{u} f = \nabla f \cdot u = |\nabla f| \cos \theta \)

1. The function \( f \) increases most rapidly when \( \cos \theta = 1 \) or when \( u \) is the direction of \( \nabla f \). That is, at each point \( P \) in its domain, \( f \) increases most rapidly in the direction of the gradient vector \( \nabla f \) at \( P \). The derivative in this direction is

\[
D_{u} f = |\nabla f| \cos (0) = |\nabla f|.
\]

2. Similarly, \( f \) decreases most rapidly in the direction of \(-\nabla f\). The derivative in this direction is \( D_{u} f = |\nabla f| \cos (\pi) = -|\nabla f|\).

3. Any direction \( u \) orthogonal to a gradient \( \nabla f \neq 0 \) is a direction of zero change in \( f \) because \( \theta \) then equals \( \pi/2 \) and

\[
D_{u} f = |\nabla f| \cos (\pi/2) = |\nabla f| \cdot 0 = 0.
\]

**Why do we use a unit vector** \( \hat{u} \)** in our definition of directional derivative?**

In this case, \( D_{\hat{u}} f \) is the rate of change of \( f \) *per unit change* in the direction of \( \hat{u} \).
At every point \((x_0, y_0)\) in the domain of a differentiable function \(f(x, y)\), the gradient of \(f\) is normal to the level curve through \((x_0, y_0)\).

**FIGURE** The gradient of a differentiable function of two variables at a point is always normal to the function’s level curve through that point.
**Ex:** At the point $(1, 1)$, determine the directions in which $f(x, y) := x^2/2 + y^2/2$: increases most rapidly; decreases most rapidly; has zero change.

![Diagram showing the gradient vector](image)

**FIGURE** The direction in which $f(x, y) = (x^2/2) + (y^2/2)$ increases most rapidly at $(1, 1)$ is the direction of $
abla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at $(1, 1, 1)$. 

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Applications matter! Experiments show that if a piece of material is heated on one side and cooled on another, then heat flows in the direction of maximum decrease of temperature. That is, heat flows from hot regions toward cold regions. If the temperature \( T = T(x, y) \) is given by

\[
T = x^3 - 3xy^2
\]

then determine the direction of maximum decrease of temperature at the point \( P(1, 2) \).
Similarly, for $f(x, y, z)$ we define

$$\nabla f := \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

and verbally refer to $\nabla f$ as “grad $f$”. Note that $\nabla f$ is vector–valued (but $f$ is not)!

**DEFINITION** Gradient Field

The **gradient field** of a differentiable function $f(x, y, z)$ is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$ 

**Ex:** If $f(x, y, z) := x^3 + y + z^2$ then calculate $\nabla f$ and $\nabla f(1, 2, 3)$. 
Ex: Calculate the derivative of $f(x, y, z) := x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the dir’n of $v = 2i - 3j + 6k$. 
Algebra Rules for Gradients

1. **Constant Multiple Rule:** \( \nabla(kf) = k\nabla f \) (any number \( k \))

2. **Sum Rule:** \( \nabla(f + g) = \nabla f + \nabla g \)

3. **Difference Rule:** \( \nabla(f - g) = \nabla f - \nabla g \)

4. **Product Rule:** \( \nabla(fg) = f\nabla g + g\nabla f \)

5. **Quotient Rule:** \( \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \)

Chain rule involving the gradient:

\[
\frac{d}{dt}[f(r(t))] = \nabla f(r(t)) \cdot r'(t).
\]
Applications matter! In the following contour map of the West Point area in New York, see that the tributary streams to the Hudson flow perpendicular to the contours. Explain and justify!
Tangent plane, normal line and other applications of $\nabla f$.

$\nabla f$  

$\mathbf{v}_1$  

$\mathbf{v}_2$  

$P_0$  

$f(x, y, z) = c$

**FIGURE** The gradient $\nabla f$ is orthogonal to the velocity vector of every smooth curve in the surface through $P_0$. The velocity vectors at $P_0$ therefore lie in a common plane, which we call the tangent plane at $P_0$. 
DEFINITIONS    Tangent Plane, Normal Line

The **tangent plane** at the point \( P_0(x_0, y_0, z_0) \) on the level surface \( f(x, y, z) = c \) of a differentiable function \( f \) is the plane through \( P_0 \) normal to \( \nabla f \big|_{P_0} \).

The **normal line** of the surface at \( P_0 \) is the line through \( P_0 \) parallel to \( \nabla f \big|_{P_0} \).

**Tangent Plane to** \( f(x, y, z) = c \) **at** \( P_0(x_0, y_0, z_0) \)

\[
f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0
\]

**Normal Line to** \( f(x, y, z) = c \) **at** \( P_0(x_0, y_0, z_0) \)

\[
x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t
\]
Ex: Calculate the tangent plane and the normal line to the surface
\[ x^2 + y^2 + z = 9 \] at the point \( P_0(1, 2, 4) \).
The surface
\[ x^2 + y^2 + z - 9 = 0 \]

**FIGURE** The tangent plane and normal line to the surface
\[ x^2 + y^2 + z - 9 = 0 \] at \( P_0(1, 2, 4) \)
Let the graph of $z = f(x, y)$ represent the surface of a mountain lying above the $XY$-plane.

The angle of inclination $\psi$, which measures the steepness of the terrain in the direction $\hat{u}$ is

$$\tan \psi = \text{slope of tangent in dir'n } \hat{u} = D_{\hat{u}}f.$$
Check your understanding.

**Ex:** T/F: The expression \( \nabla (\nabla f) \) is well-defined.

**Ex:** T/F: The expression \( f \nabla \) is well-defined.

**Ex:** T/F: There is a \( f(x, y) \) such that \( \nabla f = 5 \).

**Ex:** Express \( f_x \) in terms of \( D_{\hat{u}}f \) for some \( \hat{u} \).

**Ex:** If \( \nabla f \) is, loosely speaking, some sort of derivative of \( f \), then what is the opposite operation that cancels the \( \nabla \)?

**Ex:** T/F: \( D_{\hat{u}}f \) is the scalar component of \( \nabla f \) in the direction of \( \hat{u} \).

**Ex:** T/F: \( \nabla (f^n) = nf^{n-1} \nabla f \).
S8: Linear approximation.

**DEFINITIONS**  **Linearization, Standard Linear Approximation**

The *linearization* of a function $f(x, y)$ at a point $(x_0, y_0)$ where $f$ is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the *standard linear approximation* of $f$ at $(x_0, y_0)$.

Geometrically, $z = L(x, y)$ is the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0)$.

If $f$ is smooth enough then the tangent plane will provide a good approximation to $f$ for points near to $(x_0, y_0)$. 
By “smooth”, we mean the surface has no corners, sharp peaks or folds.
Graph of the error between $e^x \sin y$ and its tangent plane at $(0,0)$. 
The Error in the Standard Linear Approximation

If $f$ has continuous first and second partial derivatives throughout an open set containing a rectangle $R$ centered at $(x_0, y_0)$ and if $M$ is any upper bound for the values of $|f_{xx}|, |f_{yy}|$, and $|f_{xy}|$ on $R$, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on $R$ by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M(1 \cdot |x - x_0| + |y - y_0|)^2.$$

The above concept roughly says that the polynomial $L(x, y)$ gives a "first-order" approximation to $f(x, y)$ near the point $(x_0, y_0)$ in the sense that: $L(x_0, y_0) = f(x_0, y_0)$ (ie, the two surfaces touch at the point $(x_0, y_0)$)

$$\lim_{(x,y) \to (x_0,y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

(ie, the error is negligible when compared to $\sqrt{(x - x_0)^2 + (y - y_0)^2}$.)
S9: Error estimation.

Using the tangent plane as an approximation to $f$ near the point $(x_0, y_0)$ we obtain

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for small $\Delta x$ and $\Delta y$.

The above concept has important consequences in error estimation.

When taking measurements (say, some physical dimensions), errors in the measurements are a fact of life.

We now look at the effects of small changes in quantities and error estimation.
We define
\[ \Delta f := f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \]
as the \textit{increment} in \( f \).

Rearranging (5) we obtain
\[ \Delta f \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y. \]  
(6)

If we take absolute values in (6) and use the triangle inequality then we obtain
\[ |\Delta f| \leq |f_x(x_0, y_0)| |\Delta x| + |f_y(x_0, y_0)| |\Delta y|. \]  
(7)

As a general guide:

\begin{itemize}
  \item (6) is useful for approximating errors;
  \item while (7) is useful for estimating maximum errors.
\end{itemize}
Ex: The frequency \( f \) on a LC circuit is given by

\[
f(x, y) = \frac{x^{-1/2}y^{-1/2}}{2\pi}
\]

where \( x \) is the inductance and \( y \) is the capacitance. If \( x \) is decreased by 1.5% and \( y \) is decreased by 0.5% then find the approximate percentage change in \( f \).

We use (6). For our problem:

\[
\Delta x = -1.5\% \text{ of } x = -0.015x \text{ and }
\]
\[
\Delta y = -0.5\% \text{ of } y = -0.005y.
\]

Also

\[
\frac{\partial f}{\partial x} = \frac{-x^{-3/2}y^{-1/2}}{4\pi}, \quad \frac{\partial f}{\partial y} = \frac{-x^{-1/2}y^{-3/2}}{4\pi}.
\]

Thus (6) gives
\[ \Delta f \approx -x^{-\frac{3}{2}}y^{-\frac{1}{2}}\frac{1}{4\pi}(-0.015x) + -x^{-\frac{1}{2}}y^{-\frac{3}{2}}\frac{1}{4\pi}(-0.005y) \]

\[ = x^{-\frac{1}{2}}y^{-\frac{1}{2}}\frac{1}{2\pi} \left( \frac{0.015}{2} + \frac{0.005}{2} \right) \]

\[ = \left[ \frac{0.015}{2} + \frac{0.005}{2} \right] f \]

\[ = 0.01f. \]

The approximate % change in \( f \) is:

\[ \frac{\Delta f}{f} \times 100 \approx 0.01 \times 100 = 1\%. \]
Ex: Consider a cylinder with base radius $r$ and height $h$ measured (resp.) to be 5 cm and 12 cm, both calculated to nearest mm. What is the expectation for the maximum % error in calculating the volume?
The differentials $dx$ and $dy$ are independent variables and so they can be assigned any values. Sometimes we take

$$dx = \Delta x = x - x_0, \quad dy = \Delta y = y - y_0.$$ 

We then have the following definition of the total differential of $f$:

**DEFINITION**  **Total Differential**

If we move from $(x_0, y_0)$ to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0) \, dx + f_y(x_0, y_0) \, dy$$

in the linearization of $f$ is called the **total differential of $f$**.
Applications matter! A manufacturer produces cylindrical storage tanks with height 25m and radius 5m. As a quality control engineer hired by the company, perform an analysis on how sensitive the tanks’ volumes are to small variations in height and radius. Which measurement (height or radius) would you advise the company to pay particular attention to?

Independent learning ex: What happens if the dimensions of the tanks are switched? Is your advice to the company the same?
S10: Taylor polynomials and Taylor series.

Taylor polynomials and series for $f(x)$.

In first–year you discovered Taylor polynomials and Taylor series.

In particular, the aim was to develop a method for representing a (differentiable) function $f(x)$ as an (infinite) sum of powers of $x$. The main thought–process behind the method is that powers of $x$ are easy to evaluate, differentiate and integrate, so by rewriting complicated functions as sums of powers of $x$ we can greatly simplify our analysis.
DEFINITIONS Taylor Series, Maclaurin Series

Let $f$ be a function with derivatives of all orders throughout some interval containing $a$ as an interior point. Then the Taylor series generated by $f$ at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n + \cdots.$$

The Maclaurin series generated by $f$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots,$$

the Taylor series generated by $f$ at $x = 0$. 

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Colin Maclaurin was a professor of mathematics at Edinburgh university. Newton was so impressed by Maclaurin’s work that he offered to pay part of Maclaurin’s salary.
<table>
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<tr>
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<th>Maclaurin Series</th>
<th>Converges to ( f(x) ) for</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td>[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ]</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ]</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ]</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \frac{1}{1-x} )</td>
<td>[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots ]</td>
<td>(</td>
</tr>
<tr>
<td>( \frac{1}{1+x} )</td>
<td>[ \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots ]</td>
<td>(</td>
</tr>
<tr>
<td>( \ln(1 + x) )</td>
<td>[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots ]</td>
<td>(</td>
</tr>
<tr>
<td>( \tan^{-1} x )</td>
<td>[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots ]</td>
<td>(</td>
</tr>
<tr>
<td>( (1 + x)^a )</td>
<td>[ \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \cdots ]</td>
<td>(</td>
</tr>
</tbody>
</table>

Common Maclaurin series are above. If we take a finite number of terms in our series, then we obtain Taylor and Maclaurin polynomials, which are useful for approximation of functions.
Taylor’s Theorem.

**THEOREM**  
**Taylor’s Theorem**

If $f$ and its first $n$ derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between $a$ and $b$, and $f^{(n)}$ is differentiable on the open interval between $a$ and $b$, then there exists a number $c$ between $a$ and $b$ such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!} (b - a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!} (b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!} (b - a)^{n+1}.$$  

Taylor’s theorem is an extension of the mean value theorem.
When applying Taylor’s theorem, we frequently wish to hold $a$ fixed and consider $b$ as an independent variable. If we change $b$ to $x$ in Taylor’s formula, then it is easier to use in these cases. We obtain:

**Taylor’s Formula**

If $f$ has derivatives of all orders in an open interval $I$ containing $a$, then for each positive integer $n$ and for each $x$ in $I$,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

The above version of Taylor’s theorem says that for all $x \in I$ we have

$$f(x) = P_n(x) + R_n(x).$$
Taylor polynomials + series for $f(x, y)$

The Taylor series expansion $T(x, y)$ of a function $f(x, y)$ of two independent variables about a point $(a, b)$ is

$$T(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) +$$

$$\frac{1}{2!} [f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2]$$

$$+ \cdots \text{(higher order terms)}$$
We will be interested in Taylor polynomials of “first” and “second” order, ie

\[ T_1(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \]

\[ T_2(x, y) := T_1(x, y) + \frac{1}{2!} \left[ f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2 \right]. \]

In particular, \( T_1 \) and \( T_2 \) will provide (respectively) first– and second–degree approximations to \( f(x, y) \) near \( (a, b) \).
Ex: Calculate the Taylor polynomial (up to and including quadratic terms) about \((a, b) = (0, 0)\) for \(f(x, y) = e^x \sin y\).
We can graph the difference between the $f$ and its Taylor polynomial.

```latex
> \text{plot3d}(\sin(y)\exp(x)-(y + x*y), x = -1 .. 1, y = -1 .. 1);
```

See that near $(a, b) = (0, 0)$ the difference is small, but as we wander away from $(0, 0)$ the difference grows.
Ex: Calculate the Taylor polynomial (up to and including quadratic terms) about \((a, b) = (0, 0)\) for

\[
f(x, y) = \frac{1}{1 - x - y}.
\]
Ex: Use the formula to calculate the Taylor polynomial (up to and including quadratic terms) about \((a, b) = (1, 0)\) for

\[ f(x, y) = \ln(x^2 + y^2). \]
Taylor’s formula / theorem for \( f(x, y) \)

**Taylor’s Formula for \( f(x, y) \) at the Point \((a, b)\)**

Suppose \( f(x, y) \) and its partial derivatives through order \( n + 1 \) are continuous throughout an open rectangular region \( R \) centered at a point \((a, b)\). Then, throughout \( R \),

\[
f(a + h, b + k) = f(a, b) + (hf_x + kf_y)\bigg|_{(a,b)} + \frac{1}{2!} (h^2f_{xx} + 2hkf_{xy} + k^2f_{yy})\bigg|_{(a,b)}
\]

\[
+ \frac{1}{3!} (h^3f_{xxx} + 3h^2kf_{xxy} + 3hk^2f_{xyy} + k^3f_{yyy})\bigg|_{(a,b)} + \cdots + \frac{1}{n!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f\bigg|_{(a,b)}
\]

\[
+ \frac{1}{(n + 1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f\bigg|_{(a+b,c+b+c)}.
\]

**Taylor’s Formula for \( f(x, y) \) at the Origin**

\[
f(x, y) = f(0, 0) + xf_x + yf_y + \frac{1}{2!} (x^2f_{xx} + 2xyf_{xy} + y^2f_{yy})
\]

\[
+ \frac{1}{3!} (x^3f_{xxx} + 3x^2yf_{xxy} + 3xy^2f_{xyy} + y^3f_{yyy}) + \cdots + \frac{1}{n!} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^n f
\]

\[
+ \frac{1}{(n + 1)!} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^{n+1} f\bigg|_{(0,0)}
\]
Under the conditions of Taylor’s theorem, the $n$–th order Taylor polynomial $T_n$ for $f$ about $(0, 0)$ closely approximates $f$ to the $n$–th degree near $(0, 0)$ in the sense that:

$$T_n(0, 0) = f(0, 0),$$

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - T_n(x, y)}{\left(\sqrt{x^2 + y^2}\right)^n} = 0.$$

Furthermore, $T_n$ is the only polynomial of $n$–th degree that satisfies the above.
Ex. Calculate the first–order Taylor polynomial to \( f(x, y) := e^x + y \) about \((0, 0)\) and prove that it is a first–degree approximation to \( f \) near \((0, 0)\).
Where does the Taylor polynomial for \( f(x, y) \) come from?

Let \( F(t) := f(tu + a, tv + b) \) where \( u \) and \( v \) are held fixed. Let’s calculate the 2nd–order Taylor poly of \( F \) about \( t = 0 \). From the chain rule we have:

\[
F'(t) = u f_x(tu + a, tv + b) + v f_y(tu + a, tv + b)
\]

and so

\[
F'(0) = u f_x(a, b) + v f_y(a, b).
\]

Similarly, the chain rule yields:

\[
F''(t) = u^2 f_{xx}(tu + a, tv + b) \\
+ 2uv f_{xy}(tu + a, tv + b) + v^2 f_{yy}(tu + a, tv + b)
\]

and so

\[
F''(0) = u^2 f_{xx}(a, b) + 2uv f_{xy}(a, b) + v^2 f_{yy}(a, b).
\]
If we replace: $u$ with $(x - a)$; and $v$ with $(y - b)$ then our (2nd-order) Taylor polynomial for $F(t)$ about $t = 0$ is

$$T_2(0) := F(0) + F'(0) t + F''(0) t^2 / 2!$$

which, for $t = 1$, becomes

$$T_2(a, b) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2!} \left[ f_{xx}(a, b)(x - a)^2 + 2 f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2 \right].$$
Maple:

Maple’s plot3d command is used for drawing graphs of surfaces in three-dimensional space. The syntax and usage of the command is very similar to the plot command (which plots curves in the two-dimensional plane). As with the plot command, the basic syntax is plot3d(what,how); but both the "what" and the "how" can get quite complicated.

The commands for plotting the following paraboloid are:

\[
\begin{align*}
> & z := 3x^2 + y^2;

> & plot3d(z, x=-2..2, y=-2..2);
\end{align*}
\]