Section 1: Functions of severable variables.

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S1: Motivation.

Phenomena of a complex nature usually depend on more than one variable.

**Applications matter!** The amount of power $P$ (in watts) available to a wind turbine can be summarised by the equation

$$P = \frac{1}{2} \left( \frac{49}{40} \right) (\pi r^2)v^3$$

where

$r = $ diameter of turbine blades exposed to the wind (m)

$v = $ wind speed in m/sec

$49/40$ is the density of dry air at 15 deg C at sea level (kg/m$^3$).

See that the power $P$ depends on two variables, $r$ and $v$, that is, $P = f(r, v)$. 
You have already studied functions of 1 variable at school. You developed curve–sketching skills and a knowledge of calculus for functions of the type

\[ y = f(x). \]

In this section we extend these ideas to functions of many variables. In particular, we will learn the idea of a derivative for these more complicated functions.

Such ideas give us the power to more accurately model and understand complex phenomena like that of the previous example.
S2: Functions of two variables.

We will consider functions of the type

\[ z = f(x, y), \quad (x, y) \in U \]

where \( U \subseteq \mathbb{R}^2 \) is the domain of \( f \) and \( f \) is real–valued. We write \( f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \).

Examples of \( f \) and \( U \):

\[ f(x, y) = 3x + 2y^4, \quad U = \mathbb{R}^2 \]
\[ f(x, y) = x \cos y + xy \sin x, \quad U = \mathbb{R}^2 \]
\[ f(x, y) = \frac{1}{x - y}, \quad U = \{ (x, y) \in \mathbb{R}^2 : x \neq y \} \]

More simply:
S3: Visualising & sketching.

As a first step to understanding functions of two variables, we now develop some methods for visualising and sketching their graphs.

**DEFINITIONS**  **Level Curve, Graph, Surface**

The set of points in the plane where a function \( f(x, y) \) has a constant value \( f(x, y) = c \) is called a level curve of \( f \). The set of all points \( (x, y, f(x, y)) \) in space, for \( (x, y) \) in the domain of \( f \), is called the graph of \( f \). The graph of \( f \) is also called the surface \( z = f(x, y) \).

Graphs for functions of 2 variables will be surfaces in \( \mathbb{R}^3 \).
For a given function \( f \), we can determine the nature of its graph by examining how the surface intersects with various planes and then build the surface from these curves.

**Horizontal planes: Contour curves & level curves.**

A **contour curve** of \( f \) is the curve of intersection between the surface \( z = f(x, y) \) and a horizontal plane \( z = c \), \( c = \) constant. For simple cases, a contour curve can be easily drawn in \( \mathbb{R}^3 \) and observe that this curve also lies in the plane \( z = c \). If we sketch our contour curve in the \( XY \)-plane, then we obtain what is known as a **level curve** of \( f \).

The contour curve \( f(x, y) = 100 - x^2 - y^2 = 75 \) is the circle \( x^2 + y^2 = 25 \) in the plane \( z = 75 \).

The level curve \( f(x, y) = 100 - x^2 - y^2 = 75 \) is the circle \( x^2 + y^2 = 25 \) in the \( xy \)-plane.
(a) $z = e^{-(x^2 + y^2)/8} \sin x^2 + \cos y^2$

(b) $z = \sin x + 2 \sin y$

(c) $z = (4x^2 + y^2)e^{-x^2-y^2}$

(d) $z = xye^{-y^2}$
**Ex:** Sketch some contour curves in $\mathbb{R}^3$ of

$$f(x, y) = x^2 + y^2/4.$$ 

Also sketch the corresponding level curves in the plane.
Use of Vertical planes.

It is sometimes helpful to study the intersection of a surface with vertical planes to obtain curves that give us a different perspective from contour lines.

**Ex:** Sketch the curves of intersections between the surface

\[ z = x^2 + \frac{y^2}{4} \]

and: the $YZ$–plane; the $XZ$–plane.
Putting it all together.

**Ex:** Combine the information from the previous examples to sketch the surface

\[ z = x^2 + \frac{y^2}{4}. \]

There are many other important surfaces, which we list in the appendix.

**Independent learning exercise:** Sketch the graph of \( f(x, y) = x^2 - y^2 \). Hint: sketch the level curves in the plane first.
Applications matter!

The idea of "contour curves" is similar to that used to prepare contour maps where lines are drawn to represent *constant* altitudes. Walking along a line would mean walking on a level path.
S4: Partial differentiation.

We know from elementary calculus that the idea of a derivative is very helpful in the mathematical analysis of applied problems.

We now extend this concept to functions of two variables.

For a function of two variables \( f = f(x, y) \), the basic idea is to determine the rate of change in \( f \) with respect to one variable, while the other variable is held fixed.
**DEFINITION**  Partial Derivative with Respect to $x$

The **partial derivative of $f(x, y)$ with respect to $x$** at the point $(x_0, y_0)$ is

$$
\frac{\partial f}{\partial x} \bigg|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},
$$

provided the limit exists.

Essentially $\frac{\partial f}{\partial x}$ is just the derivative of $f$ with respect to $x$, keeping the $y$ variable fixed.
Define Partial Derivative with Respect to y

The **partial derivative of \( f(x, y) \) with respect to \( y \) at the point \((x_0, y_0)\)** is

\[
\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = \frac{d}{dy} f(x_0, y) \bigg|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},
\]

provided the limit exists.

The intersection of the plane \( x = x_0 \) with the surface \( z = f(x, y) \), viewed from above the first quadrant of the \( xy \)-plane.

Essentially \( \partial f / \partial y \) is just the derivative of \( f \) with respect to \( y \), keeping the \( x \) variable fixed.
**Example**  Finding Partial Derivatives at a Point

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$  

**Solution**  To find $\partial f/\partial x$, we treat $y$ as a constant and differentiate with respect to $x$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$ 

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat $x$ as a constant and differentiate with respect to $y$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$ 

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.  

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Ex: If \( f(x, y) = e^{xy} \) then calculate: \( f_x(1, 1); f_y(2, 0) \).

Independent learning exercise: For \( f(x, y) = x^{2/3}y^{1/3} \) show \( \frac{\partial f}{\partial x} = 0 \) at \((0, 0)\).
Applications matter! The amount of power $P$ (in watts) available to a wind turbine can be summarised by the equation

$$P = \frac{1}{2} \left( \frac{49}{40} \right) (\pi r^2) v^3$$

where

$r = \text{diameter of turbine blades exposed to the wind (m)}$

$v = \text{wind speed in m/sec}$

$49/40$ is the density of dry air at 15 deg C at sea level (kg/m$^3$). Calculate the rate of change in the power wrt the wind speed at the instant when: $r = 1/7$; $v = 1$. 
Product and quotient rules for partial differentiation are defined in the natural way:

\[
\frac{\partial}{\partial x} (uv) = u_x v + v_x u
\]

\[
\frac{\partial}{\partial x} \left( \frac{u}{v} \right) = \frac{u_x v - v_x u}{v^2}
\]

\[
\frac{\partial}{\partial y} (uv) = u_y v + v_y u
\]

\[
\frac{\partial}{\partial y} \left( \frac{u}{v} \right) = \frac{u_y v - v_y u}{v^2}.
\]

**EXAMPLE** Finding a Partial Derivative as a Function

Find \( \frac{\partial f}{\partial y} \) if \( f(x, y) = y \sin xy \).

**Solution** We treat \( x \) as a constant and \( f \) as a product of \( y \) and \( \sin xy \):

\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y} (y)
\]

\[
= (y \cos xy) \frac{\partial}{\partial y} (xy) + \sin xy = xy \cos xy + \sin xy.
\]
Ex: The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola as shown in the following diagram. Calculate the slope of that tangent line to the parabola at the point (1,2,5).
Higher derivatives:

We define and denote the second–order partial derivatives of $f$ as follows:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

See that there are four partial second–order derivatives and two of them are "mixed".

The order of differentiation is, in general, important (but see below for an important exception).
Ex: If \( f(x, y) = 1 + x^3 + y^4 \) then compute all four 2nd–order partial derivatives. What do you notice about the mixed derivatives?

**THEOREM**     The Mixed Derivative Theorem

If \( f(x, y) \) and its partial derivatives \( f_x, f_y, f_{xy}, \) and \( f_{yx} \) are defined throughout an open region containing a point \( (a, b) \) and are all continuous at \( (a, b) \), then

\[
f_{xy}(a, b) = f_{yx}(a, b).
\]
**Ex:** Consider the function \( f(x, t) = x^2 + \frac{t^2}{2} \).

(a) Calculate \( f_{xx} \) and \( f_{tt} \).

(b) Use (a) to show

\[ f_{xx} - 2f_{tt} = 0 \]  \hspace{1cm} (1)
Applications matter!

Equation (1) is known as a partial differential equation (PDE).

A PDE involves: partial derivatives of an unknown function and an equals sign.

The PDE (1) is a special equation known as the wave equation, which is used to describe the flow of waves in water and air.

Part (b) from the previous example says that \( f(x, t) = x^2 + \frac{t^2}{2} \) is a solution to the wave equation.
S5: Normal vector + tangent plane

See that the two tangent lines \( (l_1 \text{ and } l_2) \) to the surface at \( P \) appear to lie in a plane that is tangent to the surface at \( P \).

A normal vector to the tangent plane to the surface \( z = f(x, y) \) at \( P(x_0, y_0, z_0) \) is given by

\[
n = \pm \begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{pmatrix}.
\]
To justify our expression for \( \mathbf{n} \), we construct two vectors (\( \mathbf{u} \) and \( \mathbf{v} \)) that are parallel to \( l_1 \) and \( l_2 \) (resp.) and then cross them together, ie \( \mathbf{n} = \mathbf{u} \times \mathbf{v} \).

See that \( l_1 \) sits in the vertical plane \( y = y_0 \) and has slope \( f_x(x_0, y_0) \) within this vertical plane. Similarly, see that \( l_2 \) sits in the vertical plane \( x = x_0 \) and has slope \( f_y(x_0, y_0) \) within this vertical plane. Thus, our two parallel vectors will be

\[
\mathbf{u} = (1, 0, f_x(x_0, y_0)), \quad \mathbf{v} = (0, 1, f_y(x_0, y_0))
\]

and so our normal vector is

\[
\mathbf{n} = \mathbf{u} \times \mathbf{v} = (-f_x(x_0, y_0), -f_y(x_0, y_0), 1).
\]

We can also take any (non–zero) multiple of our \( \mathbf{n} \), so that a normal vector will also be

\[
(f_x(x_0, y_0), f_y(x_0, y_0), -1).
\]
We compute what the equation of a tangent plane to the graph of \( f(x, y) \) at a point \( P(x_0, y_0, z_0) \) ought to be if \( f \) is “smooth” enough.

In \( \mathbb{R}^3 \) a non–vertical plane has equation

\[
z = ax + by + c. \tag{2}\]

Now, we know that the equation of a plane containing a point \( P \) and with normal vector \( \mathbf{n} \) is

\[
\mathbf{n} \cdot (\mathbf{x} - \mathbf{P}) = 0
\]

and so the eqn of tangent plane will be

\[
(f_x(x_0, y_0), f_y(x_0, y_0), -1) \cdot ((x, y, z) - (x_0, y_0, f(x_0, y_0))) = 0
\]

**Plane Tangent to a Surface** \( z = f(x, y) \) at \( (x_0, y_0, f(x_0, y_0)) \)

The plane tangent to the surface \( z = f(x, y) \) of a differentiable function \( f \) at the point \( P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0)) \) is

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.
\]
Ex: Compute the equation of the tangent plane and upward-pointing normal vector to the graph of $f(x, y) = 9 - (x^2 + y^2)$ at the point $(1, 2)$. 

The tangent plane and normal line to the surface $x^2 + y^2 + z - 9 = 0$ at $P_0(1, 2, 4)$.
Applications Matter! Why is a normal vector important? A (unit) normal to a surface plays a crucial role in the study of fluid flow. Let $S$ be a smooth surface and $\mathbf{v}(x, y, z)$ be a velocity vector field of a fluid (in length / unit time).

To compute the flow rate (or “flux”) of the fluid over $S$ (in volume / unit time) we produce a (unit) normal vector $\mathbf{n}$ to $S$ and integrate $\mathbf{v} \cdot \mathbf{n}$ over $S$.

The flow of fluid in a long cylindrical pipe. The vectors $\mathbf{v} = (a^2 - r^2)\mathbf{k}$ inside the cylinder that have their bases in the $xy$-plane have their tips on the paraboloid $z = a^2 - r^2$. 
S6: Linear Approximation.

**DEFINITIONS  Linearization, Standard Linear Approximation**

The **linearization** of a function \( f(x, y) \) at a point \((x_0, y_0)\) where \( f \) is differentiable is the function

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

The approximation

\[
f(x, y) \approx L(x, y)
\]

is the **standard linear approximation** of \( f \) at \((x_0, y_0)\).

Geometrically, \( z = L(x, y) \) is the tangent plane to the surface \( z = f(x, y) \) at the point \((x_0, y_0)\).

If \( f \) is smooth enough then the tangent plane will provide a good approximation to \( f \) for points near to \((x_0, y_0)\).

By “smooth”, we mean the surface has no corners, sharp peaks or folds.
Graph of the error between $e^x \sin y$ and its tangent plane at (0,0).
S7: Error estimation.

Using the tangent plane as an approximation to $f$ near the point $(x_0, y_0)$ we obtain

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for small $\Delta x$ and $\Delta y$.

The above concept has important consequences in error estimation.

When taking measurements (say, some physical dimensions), errors in the measurements are a fact of life.

We now look at the effects of small changes in quantities and error estimation.
We define
\[ \Delta f := f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \]
as the *increment* in \( f \).

Rearranging (3) we obtain
\[ \Delta f \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y. \tag{4} \]
If we take absolute values in (4) and use the triangle inequality then we obtain
\[ |\Delta f| \leq |f_x(x_0, y_0)| |\Delta x| + |f_y(x_0, y_0)| |\Delta y|. \tag{5} \]

As a general guide:

- (4) is useful for approximating errors;
- while (5) is useful for estimating maximum errors.
Ex: The frequency $f$ on a LC circuit is given by

$$f(x, y) = \frac{x^{-1/2}y^{-1/2}}{2\pi}$$

where $x$ is the inductance and $y$ is the capacitance. If $x$ is decreased by 1.5% and $y$ is decreased by 0.5% then find the approximate percentage change in $f$.

We use (4). For our problem:

$$\Delta x = -1.5\% \text{ of } x = -0.015x \text{ and } \Delta y = -0.5\% \text{ of } y = -0.005y.$$ 

Also

$$\frac{\partial f}{\partial x} = -\frac{x^{-3/2}y^{-1/2}}{4\pi}, \quad \frac{\partial f}{\partial y} = -\frac{x^{-1/2}y^{-3/2}}{4\pi}.$$ 

Thus (4) gives
\[ \Delta f \approx \frac{-x^{\frac{3}{2}}y^{\frac{1}{2}}}{4\pi}(-0.015x) + \frac{-x^{\frac{1}{2}}y^{\frac{3}{2}}}{4\pi}(-0.005y) \]

\[ = \frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{2\pi} \left(0.015\right) + \frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{2\pi} \left(0.005\right) \]

\[ = \frac{2}{2\pi} \left[ \frac{0.015}{2} + \frac{0.005}{2} \right] f \]

\[ = 0.01f. \]

The approximate % change in \( f \) is:

\[ \frac{\Delta f}{f} \times 100 = 0.01 \times 100 = 1\%. \]
Ex: Consider a cylinder with base radius $r$ and height $h$ measured (resp.) to be 5 cm and 12 cm, both calculated to nearest mm. What is the expectation for the maximum % error in calculating the volume?
Applications matter! A manufacturer produces cylindrical storage tanks with height 25m and radius 5m. As a quality control engineer hired by the company, perform an analysis on how sensitive the tanks’ volumes are to small variations in height and radius. Which measurement (height or radius) would you advise the company to pay particular attention to?

![Diagram of a cylindrical tank with height h = 25 and radius r = 5.](a)

![Diagram of a circular base with radius r = 25 and height h = 5.](b)

Independent learning ex: What happens if the dimensions of the tanks are switched? Is your advice to the company the same?
S8: Chain rules.

We have discussed various rules for partial differentiation, like product and quotient rules. What other concepts can we apply to help us to find partial derivatives?

Remember the chain rule for functions of one variable? For functions of more than one variable, the chain rule takes a more profound form.

The easiest way of remembering various chain rules is through simple diagrams.
Case I: \( w = f(x) \) with \( x = g(r, s) \)

If \( w = f(x) \) and \( x = g(r, s) \), then

\[
\frac{\partial w}{\partial r} = \frac{dw}{dx} \frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{\partial x}{\partial s}.
\]
Ex: If \( w = \tan^{-1}(s/r) \) then calculate \( \partial w / \partial r \) via the chain rule.
Case II: $w = f(x, y)$ with $x = x(t)$, $y = y(t)$

**THEOREM**  Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives $f_x$ and $f_y$ and if $x = x(t)$, $y = y(t)$ are differentiable functions of $t$, then the composite $w = f(x(t), y(t))$ is a differentiable function of $t$ and

$$\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$
Ex: If $f(x, y) = x^2y^2$ with $x = \cos t$ and $y = \sin t$ then use the chain rule to find $df/dt$. 
Applications matter! Ex: The pressure $P$ (in kilopascals); volume $V$ (litres) and temperature $T$ (degrees $K$) of a mole of an ideal gas are related by the equation $P = 8.31T/V$. Find the rate at which the pressure is changing wrt time when: $T = 300; \frac{dT}{dt} = 0.1; V = 100; \frac{dV}{dt} = 0.2$.

We calculate $dP/dt$ and evaluate it at the above instant.
Case III: \( w = f(x, y) \) with \( x = g(r, s), \ y = h(r, s) \)

If \( w = f(x, y), \ x = g(r, s), \) and \( y = h(r, s), \) then

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}.
\]

**Ex:** Let \( f \) have continuous partial derivatives. Show that

\[
z = f(u - v, v - u)
\]

satisfies the PDE

\[
z_u + z_v = 0.
\]
Case IV: \( w = f(x, y, z) \) with \( x = x(t), y = y(t), z = z(t) \)

**THEOREM**  
Chain Rule for Functions of Three Independent Variables
If \( w = f(x, y, z) \) is differentiable and \( x, y, \) and \( z \) are differentiable functions of \( t \), then \( w \) is a differentiable function of \( t \) and

\[
\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.
\]
EXAMPLE Changes in a Function’s Values Along a Helix

Find \( dw/dt \) if

\[
w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t.
\]

In this example the values of \( w \) are changing along the path of a helix. What is the derivative’s value at \( t = 0 \)?

Solution

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

\[
= (y)(-\sin t) + (x)(\cos t) + (1)(1)
\]

\[
= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1
\]

\[
= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t.
\]

\[
\left( \frac{dw}{dt} \right)_{t=0} = 1 + \cos (0) = 2.
\]

The upper half of the helix

\( r(t) = (\cos t)i + (\sin t)j + tk \)
Case V: \( w = f(x, y, z) \) with \( x = g(r, s), \ y = h(r, s), \ z = k(r, s) \)

**THEOREM**  **Chain Rule for Two Independent Variables and Three Intermediate Variables**

Suppose that \( w = f(x, y, z), \ x = g(r, s), \ y = h(r, s), \) and \( z = k(r, s) \). If all four functions are differentiable, then \( w \) has partial derivatives with respect to \( r \) and \( s \), given by the formulas

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}
\]

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
\]
EXAMPLE  Partial Derivatives

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of $r$ and $s$ if

$$w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.$$  

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \quad \text{Substitute for intermediate variable } z.$$  

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}$$
Applications matter! Advection, in mechanical and chemical engineering, is a transport mechanism of a substance or a conserved property with a moving fluid. *

The advection PDE is

$$a \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

(6)

where $a$ is a constant and $u(x, t)$ is the unknown function.

Use the chain rule to show that a solution to (6) is of the form $u(x, t) = f(x - at)$ where $f$ is a differentiable function.

*Perhaps the best image to have in mind is the transport of salt dumped in a river. If the river is originally fresh water and is flowing quickly, the predominant form of transport of the salt in the water will be advective, as the water flow itself would transport the salt. Above, $u(x, t)$ would represent the concentration of salt at position $x$ at time $t$. 

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Ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

Paraboloid \( \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \)
Hyperboloid (1 sheet) \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

The hyperboloid \((x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1\) is a surface in three-dimensional space. Planes perpendicular to the z-axis cut it in ellipses. Vertical planes containing the z-axis cut it in hyperbolas.

Hyperboloid (2 sheets) \[ \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

The hyperboloid \((z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1\) is a surface in three-dimensional space. Planes perpendicular to the z-axis above and below the vertices cut it in ellipses. Vertical planes containing the z-axis cut it in hyperbolas.
Elliptic Cone \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = z \)

The elliptical cone \((x^2/a^2) + (y^2/b^2) = (z^2/c^2)\)

Planes perpendicular to the z-axis cut the cone in ellipses above and below the xy-plane. Vertical planes that contain the z-axis cut it in pairs of intersecting lines.

Hyperbolic Paraboloid \( \frac{y^2}{b^2} - \frac{x^2}{z^2} = \frac{z}{c} \)

The hyperbolic paraboloid \((y^2/b^2) - (x^2/a^2) = z/c, c > 0\). The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.
Maple:

Maple’s plot3d command is used for drawing graphs of surfaces in three-dimensional space. The syntax and usage of the command is very similar to the plot command (which plots curves in the two-dimensional plane). As with the plot command, the basic syntax is plot3d(what,how); but both the ”what” and the ”how” can get quite complicated.

The commands for plotting the following paraboloid are:
> z:=3*x^2+y^2;

\[ z := 3x^2 + y^2 \]

> plot3d(z, x=-2..2,y=-2..2);