1. Sigma notation
2. What is a series?
3. The big question
4. What you should already know
5. Telescoping series
6. Convergence
7. Integral test
8. Comparison tests
9. Ratio test
10. Leibniz’s test
11. Absolute & conditional convergence
12. Maple.

Lecture notes created by Chris Tisdell. Images are from “Thomas’ Calculus” by Wier, Hass and Giordano, Pearson, 2008; and “Calculus” by Rogowski, W H Freeman & Co., 2008.
1. Sigma notation.

Think of writing down all the natural numbers from 1 to 100, added together. There are a lot of terms to write out!

To save on notation involving summation, we use so-called sigma notation, for example

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 99 + 100.$$  

The $i$ in the above is known as the dummy variable in the sense that $i$ does not appear when we write out the terms of our sum. Thus,

$$\sum_{j=1}^{100} j = 1 + 2 + 3 + \cdots + 99 + 100.$$  

In general,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n,$$

with a special case being

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$  

Ex: What is the sum $1 + 2 + \cdots + 100$?
2. What is a series?

An *infinite* series is the sum of an *infinite* sequence of numbers:

\[ a_1 + a_2 + a_3 + \cdots + a_n + \cdots \]

How to add together infinitely many numbers is not so clear. In this section we examine such ideas.

Infinite series *sometimes* have a finite sum. For example, consider

\[ 1/2 + 1/4 + 1/8 + 1/16 + \cdots = 1 \]

which may be verified by adding up the areas of the repeatedly halved unit square.

Other series do not have a finite sum. Consider

\[ 1 + 2 + 3 + 4 + \cdots. \]

It is not obvious whether the following infinite series has a finite sum or not

\[ 1 + 1/2 + 1/3 + 1/4 + \cdots. \]
In general, an infinite series is denoted by sigma notation as:

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \cdots \]

\[ = \lim_{N \to \infty} \sum_{n=1}^{N} a_n. \]

\[ = \lim_{N \to \infty} (a_1 + a_2 + \cdots + a_N) \]

Since there are infinitely many terms to add together, we cannot just keep adding to see what comes out!

In some cases, we analyse what we obtain by summing the first \( N \) terms of a sequence (and stopping).

The sum of the first \( N \) terms

\[ s_N = a_1 + a_2 + \cdots + a_N \]

is called the \( N \)-th partial sum. It is an ordinary finite sum and can be calculated by normal addition. Does the sequence \( s_N \) approach a limit?
DEFINITIONS Infinite Series, $n$th Term, Partial Sum, Converges, Sum

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an infinite series. The number $a_n$ is the $n$th term of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

is the sequence of partial sums of the series, the number $s_n$ being the $n$th partial sum. If the sequence of partial sums converges to a limit $L$, we say that the series converges and that its sum is $L$. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.
3. The big question.

We are interested in the big question: 

“Does a given series converge?”

(and not ‘what does it converge to?’)

If we know that a series converges (ie. that it is a finite number) then we know it will obey the rules of algebra. We can then use this important property to our advantage.

In general, it is very difficult to obtain the value that (convergent) series converge to. We will be able to find out for only a few limited cases.
4. What you should already know.

At school you learnt about the infinite geometric series.

The geometric progression:

\[ 1, x, x^2, x^3, \cdots \]

gives rise to:

\[ 1, 1 + x, 1 + x + x^2, \cdots \]

and to the geometric series \( \sum_{k=0}^\infty x^k \). Furthermore, remember that

\[ s_N = 1 + x + x^2 + \cdots + x^N = \frac{1 - x^N}{1 - x}, \quad x \neq 1. \]

**THEOREM 1** Sum of a Geometric Series  
A geometric series with common ratio \( r \) converges if \( |r| < 1 \) and diverges if \( |r| \geq 1 \). Furthermore,

\[
\sum_{n=0}^\infty cr^n = \frac{c}{1 - r}, \quad |r| < 1
\]

\[
\sum_{n=M}^\infty cr^n = \frac{cr^M}{1 - r}, \quad |r| < 1
\]

We see that the series \( 1 + 1/2 + 1/4 + 1/8 + \cdots \) must converge (to 2).
5. Telescoping series.

A special type of infinite series which we can deal with is the so–called telescoping series.

Ex. Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

Note that there was a simple formula for the partial sums in this example. However, this is the exception rather than the rule. There is usually no general formula for $s_N$. We must develop techniques for studying other series that do not rely on formulae for $s_N$. 
6. \( n \)-th term test.

We begin with our journey with a simple theorem.

**THEOREM**

\[
\sum_{n=1}^{\infty} a_n \text{ converges, then } a_n \to 0.
\]

The above theorem is simple, but is not extremely useful. The following equivalent theorem is in a more usable form.

<table>
<thead>
<tr>
<th>The ( n )th-Term Test for Divergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{n=1}^{\infty} a_n ) diverges if ( \lim_{n \to \infty} a_n ) fails to exist or is different from zero.</td>
</tr>
</tbody>
</table>

Be careful with this simple test. You cannot use it to show that a series converges. The test is only useful for showing that a series diverges.

Ex: The series \( \sum_{n=1}^{\infty} \frac{n}{2n+3} \) cannot converge.
What is **wrong** with the following?

Let

\[ S = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots. \]

We then have

\[
- S = -1 + (1 - 1 + 1 - 1 + \cdots) \\
= -1 + S.
\]

Thus a rearrangement gives \( S = 1/2 \).

---

**Linearity of convergent series.**

**THEOREM**

If \( \sum a_n = A \) and \( \sum b_n = B \) are convergent series, then

1. **Sum Rule:** \( \sum (a_n + b_n) = \sum a_n + \sum b_n = A + B \)
2. **Difference Rule:** \( \sum (a_n - b_n) = \sum a_n - \sum b_n = A - B \)
3. **Constant Multiple Rule:** \( \sum ka_n = k\sum a_n = kA \) \hspace{1em} (Any number \( k \)).
7. Integral test.

**Corollary**
A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

A key property of positive series is that the partial sums $S_N$ form an increasing sequence. Each partial sum is obtained from the previous one by adding a positive number.

![Graph showing partial sums](image)

The partial sum $S_N$ is the sum of the areas of the $N$ shaded rectangles.

The above corollary is used in the proof of the following very important test.

**THEOREM**  The Integral Test
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function of $x$ for all $x \geq N$ ($N$ a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_{N}^{\infty} f(x) \, dx$ both converge or both diverge.
Ex: Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Ex. Show that $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges.
$p-$series:

**THEOREM** : Convergence of $p$-Series  
The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

**Proof:**

**THEOREM**: The Comparison Test

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer $N$.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer $N$.

The proof uses the corollary at the start of the previous section.

When making comparisons, we generally compare with the $p$–series mentioned above.

Ex. $\sum_{k=1}^{\infty} \frac{1}{k^4 + 3}$
Ex. Consider $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$. What happens if I compare this series with the series $\sum_{n=2}^{\infty} \frac{1}{n^3}$?
Limit form of the comparison test.

The following test is particularly useful for series in which \( a_n \) is a rational function of \( n \).

**THEOREM . Limit Comparison Test**

Suppose that \( a_n > 0 \) and \( b_n > 0 \) for all \( n \geq N \) (\( N \) an integer).

1. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c > 0 \), then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge.

2. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) and \( \sum b_n \) converges, then \( \sum a_n \) converges.

3. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \) and \( \sum b_n \) diverges, then \( \sum a_n \) diverges.

The proof uses the first comparison test.
Ex. \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-4}} \)

Ex. Does \( \sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}} \) converge?

**Theorem**: The Ratio Test
Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho.$$ 

Then

(a) the series converges if $\rho < 1$,
(b) the series diverges if $\rho > 1$ or $\rho$ is infinite,
(c) the test is inconclusive if $\rho = 1$.

**Proof of Test:**

When $\rho > 1$ we do not have $a_n \to 0$ as $n \to \infty$ so the series diverges.

For $\rho < 1$ we have $a_{n+1}/a_n \leq \rho$. We can show by induction that $a_n \leq \rho^n a_0$. Now $\sum_{n=0}^{\infty} \rho^n$ converges, since it is a geometric series with $\rho < 1$ and so by the comparison test, the original series converges.

We use the ratio test generally when exponentials and factorials are involved.
Jean d’Alembert was a French mathematician who was a pioneer in the study of differential equations and their use in physics. He studied the equilibrium and motion of fluids.
Ex: Discuss the convergence/divergence of:

(a) $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

(b) $\sum_{n=0}^{\infty} \frac{(2n)!}{n!n!}$

Ex: What about applying the ratio test to $\sum_{n=0}^{\infty} \frac{4^n n!n!}{(2n)!}$?
10. Leibniz’s Test

So far we have concentrated on series with non-negative terms. How, then, do we treat a series like

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n} = 1 - 1/2 + 1/3 - 1/4 + \cdots? \]

Such a series is called an alternating series.

The following test will be useful for alternating series:

**THEOREM** \[ \text{The Alternating Series Test (Leibniz’s Theorem)} \]

The series

\[ \sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots \]

converges if all three of the following conditions are satisfied:

1. The \( u_n \)'s are all positive.
2. \( u_n \geq u_{n+1} \) for all \( n \geq N \), for some integer \( N \).
3. \( u_n \to 0 \).

Gottfried Leibniz was a German mathematician who developed the present day notation for the differential and integral calculus though he never thought of the derivative as a limit.
Ex. Show $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Ex. Show $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ converges.
11. Absolute convergence.

**DEFINITION** Absolutely Convergent

A series $\sum a_n$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Consider the series: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$. We see that

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2^n} \right| = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

which is a GP and converges. Thus, we say $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ converges absolutely.

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. We see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

which does not converge. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge absolutely.

**THEOREM** The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
Ex: Show that \(\sum_{n=1}^{\infty} \frac{\sin n}{n^2}\) converges absolutely.

Ex: Does \(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\) converge absolutely?
Conditional convergence.

**DEFINITION**  Conditionally Convergent
A series that converges but does not converge absolutely converges conditionally.

We have seen that \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges, but does not converge absolutely. Thus the series converges conditionally.
12. MAPLE notes.

The following commands are relevant to the material of this chapter.

\[ \text{sum}(f, k=m..n) \]

is used to compute the sum of \( f(k) \) as \( k \) goes from \( m \) to \( n \).

\[ \text{sum}(k^2, k=1..4) \]

\[ \text{sum}(k^2, k=1..n) \]

\[ \frac{1}{3} (n+1)^3 - \frac{1}{2} (n+1)^2 + \frac{1}{6} n + \frac{1}{6} \]

\[ \text{sum}(1/k^2, k=1..\text{infinity}) \]

\[ \frac{\pi^2}{6} \]