MATH 1231 MATHEMATICS 1B 2010.

For use in Dr Chris Tisdell’s lectures

Calculus Section 3B: - Second order ODEs.

Created and compiled by Chris Tisdell

S1: The homogeneous case

S2: The nonhomogeneous case

S3: MAPLE.

S1: Second order ODEs

The second order linear ODE with constant coefficients, has the form

\( (NH) \quad ay'' + by' + cy = f(x), \quad a \neq 0, b, c = \text{consts} \)

where \( y \) is a function of \( x \) which we have to determine. If \( f(x) = 0 \), then we say that the equation

\[ ay'' + by' + cy = 0 \quad (H) \]

is \textit{homogeneous}.

\textit{How do we solve the above ODE and what is the form of the solution?}
Ex: Solve

\[ y'' - y' - 6y = 0. \quad (D) \]
In the previous example, the coefficient of $y'$ was “split” into two parts to form

$$-1y' = -(3 - 2)y'$$

Also note that the coefficient of $y$ in (D) is obtained by multiplying our 3 and -2. Furthermore, 3 and -2 occur in our final answer for $y$. This is no accident and both 3 and -2 will be solve the “characteristic equation” associated with (D), namely

$$\lambda^2 - \lambda - 6 = 0.$$ 

We normally go straight to the characteristic equation, solve it and simply write down the general solution.

In general the characteristic equation for

$$ay'' + by' + cy = 0$$

is $a\lambda^2 + b\lambda + c = 0$. 
Real and distinct roots.

If the characteristic equation has real and distinct roots $\lambda_1$ and $\lambda_2$ then the general solution to our ODE is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}, \quad A, B = \text{constants}.$$ 

**Ex:** Solve $y'' - 4y' - 5y = 0$, $y(0) = 1$, $y'(0) = 0$. 


Repeated roots.

Ex: Solve

\[ y'' - 4y' + 4y = 0. \]

Note that we can rearrange our ODE to form

\[ [y'' - 2y'] - 2[y' - 2y] = 0 \]

or, equivalently,

\[ \frac{d}{dx} [y' - 2y] - 2[y' - 2y] = 0. \]

We can now reduce our eqn to a first order ODE via the substitution \( u = y' - 2y \), obtaining

\[ u' - 2u = 0 \]

which may be solved to obtain \( u = Ke^{2x} \). Backsubstitution for \( u \) yields

\[ Ke^{2x} = y' - 2y \]

which can be solved for \( y \) to form

\[ y = Ae^{2x} + Bxe^{2x}. \]

Note that the coefficient of \( x \) in the exponents our solution (ie, 2) is just the repeated root of the characteristic eqn

\[ \lambda^2 - 4\lambda - 4 = 0. \]
The technique we used above works in general and so if the characteristic equation has a single repeated root, \( \lambda \), then the general solution is

\[
y = Ae^{\lambda x} + Bxe^{\lambda x}.
\]
Ex: Solve

\[ y'' + 6y' + 9y = 0. \]
**Complex roots.**

Finally, the third possibility is that the roots are complex. They occur in conjugate pairs and so are of the form

\[ \lambda = \alpha \pm i\beta. \]

The general solution will be

\[ y(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x). \]

This solution can be justified from our earlier method with some work and the use of Euler’s formula for \( e^{ix} \).
Ex: Solve

\[ y'' - 2y' + 5y = 0. \]
To summarise, we solve the characteristic equation of
\[ ay'' + by' + cy = 0 \]
which is
\[ a\lambda^2 + b\lambda + c = 0. \]

If

- roots are real and distinct, \( \lambda_1, \lambda_2 \), then the general solution is
  \[ y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}; \]
- roots are equal, \( \lambda \), then the general solution is
  \[ y(x) = Ae^{\lambda_1 x} + Bxe^{\lambda_2 x}; \]
- roots are complex, \( \lambda = \alpha \pm i\beta \), then the general solution is
  \[ y(x) = e^{\alpha x} \left[ A \cos \beta x + B \sin \beta x \right]. \]
S2: The nonhomogeneous case.

If $y_1(x)$ solves (H) and $y_2(x)$ solves (NH) then the linear combo $Y := y_1 + y_2$ also satisfies (NH) as

$$
Y'' + aY' + bY
= a[y_1'' + y_2''] + b[y_1' + y_2'] + c[y_1 + y_2]
= [ay_1'' + by_1' + cy_1] + [ay_2'' + by_2' + cy_2]
= [0] + f(x)
= f(x).
$$

The idea above is will be used frequently to solve (NH).

We first solve (H) and then construct a particular solution to (NH). We then add the two solutions to obtain the general solution to (NH).
Ex: Solve

\[ y'' - 5y' + 6y = 2x + 3. \]
Ex: Solve

\[ y'' - 5y' + 6y = 12e^{5x}. \]
We can construct a table of what to try as a particular solution for given $f(x)$.

<table>
<thead>
<tr>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$, a $n$th deg. polyn.</td>
</tr>
<tr>
<td>$P(x)e^{ax}$</td>
</tr>
<tr>
<td>$P(x)\cos ax$</td>
</tr>
<tr>
<td>$P(x)\sin ax$</td>
</tr>
<tr>
<td>$P(x)e^{ax}\sin bx$ or $P(x)e^{ax}\cos bx$</td>
</tr>
</tbody>
</table>
Respective construction for $y_p$.

<table>
<thead>
<tr>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(x)$, $n$th deg. polyn.</td>
</tr>
<tr>
<td>$Q(x)e^{ax}$</td>
</tr>
<tr>
<td>$Q_1(x)\cos ax + Q_2(x)\sin ax$</td>
</tr>
<tr>
<td>$Q_1(x)\cos ax + Q_2(x)\sin ax$</td>
</tr>
<tr>
<td>$Q_1(x)e^{ax}\cos bx + Q_2(x)e^{ax}\sin bx$</td>
</tr>
</tbody>
</table>

***Care must be taken however, when using the above table as the following examples will show. ***
**Ex:** Solve $y'' - 5y' + 6y = 12e^{2x}$.

An even more unpleasant example is:

**Ex:** Solve $y'' - 4y' + 4y = 2e^{2x}$. 
Thus, as a general rule, if the right hand side of the equation has a function which is already in the kernel (i.e. one of the homogeneous solutions), we multiply by $x$ until the resulting function is no longer a solution to the homogeneous equation.

**Ex:** Solve $y'' + y = \cos x$, $y(0) = 3$, $y'(0) = 0$. 
3. Appendix: MAPLE

The following command is used in MAPLE to solve ODE's (if possible).

```maple
>dsolve(deqn, y(x));
```

For example

```maple
>dsolve(diff(y(x),x$2)-y(x) = 1, y(x));
```

\[ y(x) = -1 + C_1 \exp(x) + C_2 \exp(-x) \]

```maple
>dsolve({diff(v(t),t) + 2*t = 0, v(1) = 5}, v(t));
```

\[ v(t) = -t^2 + 6 \]