1. Rotations

‘Einstein’ Summation Convention

Basis vectors:

\[ i = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad j = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad k = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]

Abbreviation: if an index occurs exactly twice in a term then sum that index from 1 to 3. E.g., if

\[ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \]

then

\[ v = v_i e_i = \sum_{i=1}^{3} v_i e_i = v_1 e_1 + v_2 e_2 + v_3 e_3. \]
Position vector: write
\[ r = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \quad \text{or} \quad x = x_i e_i = x_1 e_1 + x_2 e_2 + x_3 e_3. \]

Define the \textit{Kronecker delta}
\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \]
and \textit{Levi-Civita symbol}
\[ \epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3) \\ & \text{or } (2, 3, 1) \text{ or } (3, 1, 2), \\ -1 & \text{if } (i, j, k) \text{ is } (2, 1, 3) \\ & \text{or } (1, 3, 2) \text{ or } (3, 2, 1), \\ 0 & \text{otherwise}, \end{cases} \]
so that
\[ e_i \cdot e_j = \delta_{ij} \quad \text{and} \quad e_i \times e_j = \epsilon_{ijk} e_k. \]
Notice $\delta_{ij}$ is symmetric in $i$ and $j$,
\[ \delta_{ij} = \delta_{ji}, \]
whereas $\epsilon_{ijk}$ is *skew*-symmetric with respect to odd permutations,
\[ \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}. \]

Use $\delta_{ij}$ and $\epsilon_{ijk}$ in calculations involving dot products,
\[ \mathbf{v} \cdot \mathbf{w} = (v_i e_i) \cdot (w_j e_j) = v_i w_j e_i \cdot e_j = v_i w_j \delta_{ij} = v_i w_i, \]
and cross products,
\[ \mathbf{v} \times \mathbf{w} = (v_i e_i) \times (w_j e_j) = v_i w_j e_i \times e_j = v_i w_j \epsilon_{ijk} e_k. \]

Remember that
\[ \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \quad \text{whereas} \quad \mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}, \]
and that
\[ \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \quad \text{whereas} \quad \mathbf{v} \times \mathbf{v} = 0. \]
Recall that if the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta$ ($0 \leq \theta \leq \pi$), then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta \quad (1)$$

and

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin \theta \mathbf{e} \quad (2)$$

where $\mathbf{e}$ is a unit vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$ with direction given by the right-hand rule.

In particular, for non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ we have $\mathbf{u} \cdot \mathbf{v} = 0$ iff $\theta = \pi/2$, i.e., iff $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
We call \( u \cdot (v \times w) \) the **scalar triple product** of the vectors \( u, v, w \).

1.1 **Exercise.** Show that

\[
    u \cdot (v \times w) = \varepsilon_{ijk} u_i v_j w_k = \det \begin{bmatrix}
        u_1 & v_1 & w_1 \\
        u_2 & v_2 & w_2 \\
        u_3 & v_3 & w_3
    \end{bmatrix} =: \det[u, v, w].
\]

Thus,

\[
    u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u)
\]

and

\[
    u \cdot (v \times w) = (u \times v) \cdot w.
\]

1.2 **Exercise.** Show that \( |u \cdot (v \times w)| \) is the volume of the parallelepiped generated by \( u, v, w \).

Given a matrix \( A = [a_{ij}] \in \mathbb{R}^{3 \times 3} \),

\[
    w = Av \iff w_i = a_{ij} v_j.
\]
1.3 Exercise. Show that $(Av) \cdot w = v \cdot (A^T w)$.

1.4 Exercise. Show that $Au \cdot (Av \times Aw) = (\det A) u \cdot (v \times w)$.

The above exercise implies that if $\det A = 1$ then the volumes of the parallelepipeds generated by $\{u, v, w\}$ and $\{Au, Av, Aw\}$ respectively are the same.

In other words, the linear map $x \mapsto Ax$ is volume-preserving if $\det A = 1$. 
We call $u \times (v \times w)$ the vector triple product of $u$, $v$, $w$.

1.5 Lemma. For $i, j, m, n \in \{1, 2, 3\}$,

$$
\sum_{k=1}^{3} \epsilon_{ijk} \epsilon_{mnk} = \begin{cases} 
+1 & \text{if } i = m \text{ and } j = n \text{ and } i \neq j, \\
-1 & \text{if } i = n \text{ and } j = m \text{ and } i \neq j, \\
0 & \text{otherwise},
\end{cases}
$$

or, more succinctly,

$$
\epsilon_{ijk} \epsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}.
$$

1.6 Theorem. $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$. 
Orthogonal Matrices

A matrix $Q \in \mathbb{R}^{n \times n}$ is orthogonal if

$$Q^T = Q^{-1},$$

or in other words if $Q^T Q = I = QQ^T$. The set of all orthogonal matrices,

$$O(n) = \{Q \in \mathbb{R}^{n \times n} : Q^T = Q^{-1}\},$$

is called the orthogonal group of order $n$.

1.7 Theorem. For $Q \in \mathbb{R}^{n \times n}$, the following are equivalent:

1. $Q \in O(n)$;
2. $(Qv) \cdot (Qw) = v \cdot w$ for all $v, w \in \mathbb{R}^n$;
3. $|Qv| = |v|$ for all $v \in \mathbb{R}^n$.

Thus, orthogonal transformations preserve angles and distances.
1.8 Example. It is readily verified that

\[
Q = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \in O(2).
\]

In fact, the action of \( Q \) on a vector \( v \) represents the rotation of \( v \) about the origin by the angle \( \theta \) since

\[
|w| = |v|, \quad w \cdot v = |w||v| \cos \theta,
\]

where \( w = Qv \).

1.9 Exercise. Show that if \( Q \in O(3) \) then for all \( v, w \in \mathbb{R}^3 \),

\[
Qv \times Qw = (\det Q) Q(v \times w).
\]

1.10 Exercise. Show that if \( Q \in O(n) \) then \( \det Q = \pm 1 \).
The set

\[ SO(n) = \{ Q \in O(n) : \det Q = +1 \} \]

is called the \textit{special} orthogonal group of order \( n \). When \( Q \in SO(n) \) we say that \( v \mapsto Qv \) is a \textit{proper} orthogonal transformation.

The preceding results show that for \( Q \in SO(3) \),

\[ Qv \cdot Qw = v \cdot w \quad \text{and} \quad Qv \times Qw = Q(v \times w). \quad (3) \]

It turns out that a transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) is proper orthogonal, i.e., belongs to \( SO(3) \), iff it represents a \textit{rotation} about an axis through the origin. (We will prove the “if” part in the next theorem. The “only if” part is one of the tutorial problems.) Given this fact, we see that (3) also follows from (1) and (2).
The identities (3) help to explain why many geometric and physical quantities are expressed in terms of dot and cross products: such quantities should not depend on the choice of a right-handed, Euclidean coordinate system for physical space.

1.11 Theorem. Let $a, v \in \mathbb{R}^3$ with

$$|a| = 1 \quad \text{and} \quad v \neq 0.$$  

The vector $w$ obtained by rotating $v$ through an angle $\theta$ about $a$ is

$$w = (1 - \cos \theta)(a \cdot v)a + (\cos \theta)v + (\sin \theta)a \times v.$$  

Moreover, $w = Rv$ where $R = [r_{ij}] \in SO(3)$ has entries

$$r_{ij} = (1 - \cos \theta)a_ia_j + (\cos \theta)\delta_{ij} - (\sin \theta)\epsilon_{ijk}a_k.$$  

1.12 Corollary. Given any two vectors $v, w \in \mathbb{R}^3$ with $|v| = |w|$, there exists a rotation matrix $Q \in SO(3)$ such that $w = Qv.$
2. Line Integrals

Arc Length

Let \( C \) be an oriented curve (or path) with parametric representation

\[ x = x(u), \quad a \leq u \leq b. \]

We will always assume that \( x(u) \) is continuous and piecewise continuously differentiable with respect to the parameter \( u \).

Choose \( M + 1 \) points in the parameter interval \([a, b]\),

\[ a = u_0 < u_1 < u_2 < \cdots < u_M = b \]

and define corresponding points on \( C \),

\[ x_k = x(u_k), \quad 0 \leq k \leq M. \]

Also choose \( \bar{u}_k \in [u_k, u_{k+1}] \) and put

\[ \bar{x}_k = x(\bar{u}_k), \quad 0 \leq k \leq M - 1. \]
The length of the line segment from \( x_k \) to \( x_{k+1} \) is

\[
\Delta s_k = |\Delta x_k| \quad \text{where} \quad \Delta x_k = x_{k+1} - x_k.
\]

2.1 Motivation. The line integral of \( f : C \rightarrow \mathbb{R} \) with respect to arc length should be

\[
\int_C f \, ds = \lim_{M \to \infty} \sum_{k=0}^{M-1} f(\bar{x}_k) \Delta s_k
\]

taking the ‘limit’ as \( M \to \infty \) and

\[
\max_{0 \leq k \leq M-1} \Delta s_k \to 0.
\]

[One may think of \( C \) as a thin ‘wire’ of mass density \( f(x) \) (= mass per unit length). Then, total mass = \( \int_C f \, ds \).]
2.2 Definition. 

\[ \int_C f \, ds = \int_a^b f \frac{dx}{du} \, du = \int_a^b f(x(u)) |x'(u)| \, du. \]

Notice that by taking \( f \equiv 1 \) we have

\[ \text{length of } C = \lim_{M \to \infty} \sum_{k=0}^{M-1} \Delta s_k = \int_C 1 \, ds = \int_a^b |x'(u)| \, du. \]

2.3 Exercise. Find the length of the parabolic curve

\[ y = 1 - x^2, \quad -1 < x < 1. \]

2.4 Theorem. The value of the line integral

\[ \int_C f \, ds \]

does not depend on the choice of parametric representation for the oriented curve \( C \).
Integration of Vector Fields

2.5 Motivation. The line integral of the vector field

\[ \mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n \]

should be

\[ \int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \lim_{M \to \infty} \sum_{k=0}^{M-1} \mathbf{F}(\mathbf{x}_k) \cdot \Delta \mathbf{x}_k, \]

taking the ‘limit’ as \( M \to \infty \) and

\[ \max_{0 \leq k \leq M-1} |\Delta \mathbf{x}_k| \to 0. \]

[One may think of \( C \) as the path of a ‘particle’ which moves under the influence of a force \( \mathbf{F}(\mathbf{x}) \). Then, work done = \( \int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} \).]

2.6 Definition. \( \int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(u)) \cdot \frac{d\mathbf{x}}{du} \, du. \)
2.7 Exercise. For

\[ F = (\log x) \mathbf{i} - y \mathbf{k}, \]

find

\[ \int_C F(x) \cdot dx, \]

if \( C \) is described by

\[ x = (x, y, z) = (1 + u) \mathbf{i} - u^2 \mathbf{j} + 2u \mathbf{k}, \quad 0 \leq u \leq 2. \]

2.8 Theorem. The value of the line integral

\[ \int_C F(x) \cdot dx \]

does not depend on the choice of parametric representation for the oriented curve \( C \).
Alternative notation for a line integral:

\[ \int_C (P \, dx + Q \, dy + R \, dz) = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \]

where

\[ \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \]

and

\[ \mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j} + z(u) \mathbf{k} \]

The motive for this notation is that the dot product of \( \mathbf{F} \) with

\[ d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \]

gives the differential form \( P \, dx + Q \, dy + R \, dz \). Notice that, e.g.,

\[ \int_C P \, dx = \lim_{M \to \infty} \sum_{k=0}^{M-1} P(\bar{x}_k, \bar{y}_k, \bar{z}_k) \Delta x_k = \int_a^b P(x(u), y(u), z(u)) \frac{dx}{du} \, du \]

so

\[ \int_C (P \, dx + Q \, dy + R \, dz) = \int_a^b \left( P \frac{dx}{du} + Q \frac{dy}{du} + R \frac{dz}{du} \right) du. \]
Properties of Line Integrals

If $C_1$ ends at the point where $C_2$ begins, then we can combine the two curves into a single curve denoted by $C_1 + C_2$.

Let $-C$ denote the curve obtained by parameterizing $C$ in the opposite direction. Thus, if $C$ is given by $x = \phi(u)$ for $a \leq u \leq b$ then $-C$ is given by $x = \phi(a + b - u)$ for $a \leq u \leq b$.

2.9 Theorem.

1. $\int_{C_1+C_2} F(x) \cdot dx = \int_{C_1} F(x) \cdot dx + \int_{C_2} F(x) \cdot dx$.

2. $\int_{-C} F(x) \cdot dx = -\int_{C} F(x) \cdot dx$. 
Similarly,

\[ \int_{C_1+C_2} f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds \]

but

\[ \int_{-C} f \, ds = \int_{C} f \, ds. \]

2.10 Theorem. For any vector fields \( F, G \) and any scalar constants \( \lambda, \mu \), we have

\[ \int_{C} \left( \lambda F(x) + \mu G(x) \right) \cdot dx = \lambda \int_{C} F(x) \cdot dx + \mu \int_{C} G(x) \cdot dx. \]

Similarly,

\[ \int_{C} (\lambda f + \mu g) \, ds = \lambda \int_{C} f \, ds + \mu \int_{C} g \, ds. \]
For line integrals, the fundamental theorem of calculus takes the following form.

**2.11 Theorem.** If the curve $C$ starts at $x_a = x(a)$ and finishes at $x_b = x(b)$, then

$$\int_C \text{grad} \, f(x) \cdot dx = f(x_b) - f(x_a).$$

We say that $C$ is a **closed curve** if it finishes at the same point where it starts, i.e., if $x(b) = x(a)$. It is customary to put a circle on the integral sign to indicate that the curve is closed:

$$\oint_C F(x) \cdot dx.$$

The theorem above shows that

$$\oint_C \text{grad} \, f(x) \cdot dx = 0$$

for any scalar field $f$. 
Work and Energy

The work $W$ done by a constant force $F$ on an object that moves in a straight line from $r$ to $r + \Delta r$ is defined by

$$W = F \cdot \Delta r.$$ 

Thus, the work done by a variable force $F(r)$ on an object that moves along a curve $C$ is

$$W = \lim_{M \to \infty} \sum_{k=1}^{M} F(r_k) \cdot \Delta r_k = \int_C F \cdot dr.$$ 

If a force field $F$ may be derived from a scalar field $U$, that is

$$F = -\text{grad } U,$$

then we call $U$ a potential for $F$. Notice that we may add an arbitrary constant to $U$ and it remains a potential for $F$. 

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If $F$ is derived from a potential and if $C$ is any path from $r_a$ to $r_b$, then
\[ \int_C F \cdot dr = - \int_C \text{grad} \, U \cdot dr = -[U(r_b) - U(r_a)] \]

or in words:
\[
\text{work done by force} = -(\text{change in potential})
\]

so that, in particular, \( \int_C F(r) \cdot dr = 0 \).

Consider a particle of mass $m$ with

- position $r(t)$
- velocity $v = \frac{dr}{dt}$
- speed $v = |v|$.

The kinetic energy $T$ of $m$ is defined by
\[
T = T(v) = \frac{1}{2}mv^2.
\]
2.12 Theorem. If $F = F(r)$ is the net force on $m$, so that

$$m \frac{dv}{dt} = F,$$

and if $C$ is the trajectory of $m$ for $a \leq t \leq b$, then

$$\int_C F \cdot dr = T(v_b) - T(v_a).$$

In words,

work done by force = change in kinetic energy.

2.13 Corollary. In the case of a net force $F = -\nabla U$ there is a constant $E$, called the total energy, such that $T + U = E$ throughout the motion, i.e.,

$$T(v(t)) + U(r(t)) = E \quad \text{for } a \leq t \leq b.$$
2.14 Exercise. Consider the ‘central’ force field

\[ F = f(|r|)r. \]

Show that \( F \) may be derived from a potential by explicitly constructing an associated potential energy function.

Determine the total energy and establish how the speed \( v \) depends on the radial ‘distance’ \(|r|\).
3. Surface Integrals

Surface Area

Recall that the area of the parallelogram

\[ \{ u\mathbf{a} + v\mathbf{b} : 0 < u < 1, 0 < v < 1 \} \]

equals

\[ |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta \]
Consider a surface $S$ in $\mathbb{R}^3$ with parametric representation

$$S : \quad x = x(u, v), \quad (u, v) \in U,$$

for some closed, bounded region $U$ in $\mathbb{R}^2$. We will always assume that $x(u, v)$ is continuously differentiable on $U$.

It may be necessary (or convenient) to dissect the surface $S$ into several pieces, each with its own parametric representation.

Construct a rectangular mesh in $\mathbb{R}^2$ that covers $U$ and write

$$x_{ij} = x(u_i, v_j),$$

with

$$\Delta u_i = u_{i+1} - u_i, \quad \Delta u x_{ij} = x_{i+1,j} - x_{ij},$$

$$\Delta v_j = v_{j+1} - v_j, \quad \Delta v x_{ij} = x_{i,j+1} - x_{ij}.$$ 

We choose $\bar{u}_i \in [u_{i-1}, u_i]$ and $\bar{v}_j \in [v_{j-1}, v_j]$ and put

$$\bar{x}_{ij} = x(\bar{u}_i, \bar{v}_j).$$
The vector

\[ \Delta S_{ij} = (\Delta u \vec{x}_{ij}) \times (\Delta v \vec{x}_{ij}) \]

is approximately normal to \( S \) at \( \vec{x}_{ij} \) and its length \( |\Delta S_{ij}| \) is approximately the area of the piece of \( S \) corresponding to \( u_i < u < u_{i+1} \) and \( v_j < v < v_{j+1} \).

3.1 Motivation. The integral of \( f : S \to \mathbb{R} \) with respect to surface area should be

\[ \int_S f \, dS = \lim_{U \to \text{number of grid points}} \sum_{(\bar{u}_i, \bar{v}_j) \in U} f(\bar{x}_{ij}) |\Delta S_{ij}|. \]

taking the ‘limit’ as the number of grid points \( \to \infty \) and \( \max(\Delta u_i, \Delta v_j) \to 0 \).

[One may think of \( S \) as a thin material ‘sheet’ of mass density \( f(\vec{x}) \) (= mass per unit area). Then, total mass = \( \int_S f \, dS \).]
3.2 Definition.

\[ \int_S f \, dS = \iint_U f(x(u,v)) \left| \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right| \, du \, dv \]

3.3 Theorem. Any two parametric representations \( x(u,v) \) and \( x(u',v') \) of a surface \( S \) give rise to the same surface integral \( \int_S f \, dS \).

3.4 Theorem. If \( S \) is given by

\[ z = \varphi(x,y) \quad \text{for} \ (x,y) \in U, \]

for some function \( \varphi \), then

\[ \int_S f \, dS = \iint_U f(x,y,\varphi(x,y)) \sqrt{1 + |\text{grad} \varphi(x,y)|^2} \, dx \, dy. \]
Notice that taking $f \equiv 1$ gives

$$\text{area of } S = \lim_{(\tilde{u}_i, \tilde{v}_j) \in U} \sum |\Delta S_{ij}| = \int_S f \, dS = \iiint_U \sqrt{1 + |\text{grad } \varphi(x,y)|^2} \, dx \, dy.$$ 

3.5 Exercise. Find the surface integral

$$\iiint_S z \, dS,$$

where $S$ is the upper hemisphere defined by $x^2 + y^2 + z^2 = 1, \ z \geq 0.$
Orientation

A vector field $\mathbf{N}$ is said to be \textit{normal} to the surface $S$ if, for every point $x \in S$,

$$\mathbf{N}(x) \perp \text{tangent plane of } S \text{ at } x.$$ 

If $\mathbf{N}(x) \neq 0$ then we can define a \textit{unit} normal vector

$$\mathbf{n}(x) = \frac{1}{|\mathbf{N}(x)|} \mathbf{N}(x).$$

The partial derivatives $\partial x/\partial u$ and $\partial x/\partial v$ are vectors tangent to the surface $S$ given by $x(u, v)$. Hence, the cross product

$$\mathbf{N}(x) = \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}$$

is normal to $S$. 
3.6 Example. If $S$ is given by

$$z = \varphi(x, y) \quad \text{for } (x, y) \in U,$$

then

$$N = -\frac{\partial \varphi}{\partial x} \mathbf{i} - \frac{\partial \varphi}{\partial y} \mathbf{j} + \mathbf{k}$$

is a normal vector field for $S$.

A surface $S$ is orientable if it has a continuous unit normal vector field $n$, in which case we speak of the orientation defined by $n$.

An oriented surface is a surface $S$ equipped with a chosen orientation, i.e., a chosen unit normal $n$. In this case, $-S$ denotes the oriented surface with unit normal $-n$.

Later, we will generalise the above discussion to cover cases when $S$ has sharp corners or edges and so $n$ is discontinuous.
3.7 Example. Let $0 < b < a$. The Möbius band $S$ described by

$$x = (a + w \cos \frac{1}{2} \theta) \cos \theta,$$
$$y = (a + w \cos \frac{1}{2} \theta) \sin \theta,$$
$$z = w \sin \frac{1}{2} \theta,$$

for

$$-b \leq w \leq b, \quad -\pi \leq \theta \leq \pi,$$

is not orientable. Although the normal vector field

$$N = \frac{\partial r}{\partial w} \times \frac{\partial r}{\partial \theta}$$

is a continuous function of $(w, \theta)$ on $[-b, b] \times [-\pi, \pi]$, it is not a continuous function of $(x, y, z)$ on $S$ because

$$\lim_{\theta \to \pi} N = \begin{bmatrix} a \\ w/2 \\ 0 \end{bmatrix}$$
$$\text{but} \quad \lim_{\theta \to -\pi} N = \begin{bmatrix} -a \\ w/2 \\ 0 \end{bmatrix}.$$
Flux

3.8 Motivation. The flux (integral) of the vector field $\mathbf{F}$ through the oriented surface $S$ should be

$$
\int_S \mathbf{F} \cdot d\mathbf{S} = \lim_{\max(\Delta u_i, \Delta v_j) \to 0} \sum_{(\tilde{u}_i, \tilde{v}_j) \in U} \mathbf{F}(\tilde{x}_{ij}) \cdot \Delta S_{ij},
$$

taking the limit as $\max(\Delta u_i, \Delta v_j) \to 0$.

[One may think of $S$ as an imaginary ‘membrane’ in a fluid flow of velocity $\mathbf{F}(x)$. Then, flux through membrane $= \int_S \mathbf{F} \cdot d\mathbf{S}$.]

3.9 Definition.

$$
\int_S \mathbf{F} \cdot d\mathbf{S} = \iint_U \mathbf{F}(x(u,v)) \cdot \left( \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right) du \, dv
$$
3.10 Exercise. Calculate the flux of the vector field

\[ F(x, y, z) = (3x - y)i + (x^2 + z)j - 3xyk \]
through the oriented surface

\[ S : \quad x(u, v) = (u - v)i + (u + v)j - 2uk, \]
for \(-1 \leq u \leq 1\) and \(-2 \leq v \leq 2\).

3.11 Theorem. The flux of \( F \) through \( S \) equals the integral of \( F \cdot n \) with respect to surface area:

\[ \int_S F \cdot dS = \int_S F \cdot n \, dS. \]
Note that $\int_S f \, dS$ makes sense even when $S$ is not orientable.

Moreover, if $S$ is orientable then

$$\int_{-S} f \, dS = \int_S f \, dS \quad \text{but} \quad \int_{-S} F \cdot dS = -\int_S F \cdot dS.$$  

3.12 Exercise. Consider the vector field

$$F = f(|x|)x.$$  

Determine the flux through the sphere of radius $R$ centred at the origin. What does one conclude in the case $f \sim |x|^{-3}$?
Consider a fluid with velocity field \( u = u(x, t) \) passing through a fixed, oriented surface \( S \) given by \( x(\xi, \eta) \) for \( (\xi, \eta) \in U \). Assume for simplicity that \( u \cdot n > 0 \) everywhere on \( S \).

Let \( V(t) \) be the volume of fluid that passes through \( S \) between time 0 and time \( t \).

Let \( P_{ij} \) be a small parallelogram spanned by \( \Delta_\xi x_{ij} \) and \( \Delta_\eta x_{ij} \). At time \( t + \Delta t \), the fluid particles that passed through \( P_{ij} \) since time \( t \) make up a (slightly distorted) parallelepiped with sides \( \Delta_\xi x_{ij} \), \( \Delta_\eta x_{ij} \) and \( \Delta t u \). The volume of these particles is therefore given (to first order in \( \Delta t \), \( \Delta_\xi \) and \( \Delta_\eta \)) by the scalar triple product

\[
(\Delta t u) \cdot (\Delta_\xi x_{ij} \times \Delta_\eta x_{ij}) = \Delta t u \cdot \Delta S_{ij}.
\]
Summing the contributions from all parallelograms,

$$\Delta V = V(t + \Delta t) - V(t) = \Delta t \sum_{i,j} u \cdot \Delta S_{ij} \left( 1 + O(\Delta t + \Delta \xi + \Delta \eta) \right),$$

and in the limit as $\Delta \xi$ and $\Delta \eta$ tend to zero,

$$\Delta V = \Delta t \int_S u \cdot dS + O(\Delta t^2),$$

so

$$\frac{\Delta V}{\Delta t} = \int_S u \cdot dS + O(\Delta t).$$

Thus, in the limit $\Delta t \to 0$, we obtain

$$\int_S u \cdot dS = \frac{dV}{dt}.$$

Conclusion: the flux of the fluid velocity equals the instantaneous rate at which fluid volume passes through the surface.
Grad, curl and div

The vector differentiation operator $\nabla$, called *nabla* or *del*, is defined by

$$\nabla = e_i \frac{\partial}{\partial x_i} = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$  

Acting on a scalar function, $\nabla$ gives the gradient:

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix}.$$  

Abbreviations:

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \nabla = e_i \partial_i, \quad \nabla f = e_i \partial_i f = \partial_i f e_i.$$
4.1 Definition. The curl of the vector field

\[ F(r) = F_1 e_1 + F_2 e_2 + F_3 e_3 \]

is the vector field

\[ \text{curl } F = \nabla \times F = \epsilon_{ijk} \partial_i F_j e_k = e_i \times \partial_i F \]

\[ = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{bmatrix} = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{bmatrix}. \]

Written out explicitly,

\[ \text{curl } F = \left( \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \right) e_1 + \left( \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right) e_2 + \left( \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) e_3. \]
4.2 Exercise. Find curl $\mathbf{F}$ if

$$\mathbf{F} = e^{xy} \mathbf{i} + (x - z) \mathbf{j} + 2xy \mathbf{k}.$$ 

4.3 Definition. The divergence of the vector field

$$\mathbf{F}(r) = F_1 e_1 + F_2 e_2 + F_3 e_3$$

is the scalar field

$$\text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \partial_i F_i = \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$ 

4.4 Exercise. Find div $\mathbf{F}$ if

$$\mathbf{F} = (yz + \sin \pi x) \mathbf{i} + e^x \cos \pi z \mathbf{j} + (x + 2y - 3z) \mathbf{k}.$$
The following lemma is pivotal in potential theory.

4.5 Lemma.

\[ \text{curl}(\text{grad} \, f) = 0, \quad \text{div}(\text{curl} \, F) = 0. \]

If \( F \) is really a 2D vector field, say

\[ F(x, y, z) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}, \]

then

\[ \text{curl} \, F = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \]

and

\[ \text{div} \, F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}. \]
For any vector field $A$ we define the scalar differential operator $A \cdot \nabla$ in the obvious way. For a scalar function $f$,

$$(A \cdot \nabla)f = A_i \partial_i f = A_i \frac{\partial f}{\partial x_i} = A \cdot (\nabla f),$$

and for a vector field $F$,

$$(A \cdot \nabla)F = A_i \partial_i F = A_i \frac{\partial F}{\partial x_i} = A_i \frac{\partial F_j}{\partial x_i} e_j = (A \cdot \nabla F_j)e_j.$$

### 4.6 Exercise

Show that

$$(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C) = B \cdot [(A \cdot C)D] - A \cdot [(B \cdot C)D].$$

### 4.7 Exercise

Show that

$$(A \times B) \cdot \text{curl } F = B \cdot [(A \cdot \nabla)F] - A \cdot [(B \cdot \nabla)F].$$
Interpretation

If \( n \) denotes a unit vector then \( n \cdot \nabla f \) measures the change of \( f \) in the direction \( n \) at any point \( P \).

Let \( v \) denote the velocity field of a fluid. Then, the fluid tends to ‘accumulate’ in or ‘move away’ from a ‘small’ neighbourhood of a point \( P \) if \( \text{div} \, v < 0 \) or \( \text{div} \, v > 0 \) at that point respectively.

\( \text{curl} \, v \) measures the ‘rotational tendency of a fluid’, that is, if the fluid in a ‘small’ neighbourhood of a point \( P \) were suddenly solidified and the surrounding fluid simultaneously annihilated then this ‘solid’ neighbourhood would rotate with an angular velocity \( \frac{1}{2} |\text{curl} \, v| \) at \( P \).

These statements may be made precise by means of the integral theorems discussed in the following sections.
4.8 Exercise. Determine the divergence and the curl of the vector fields

\[ v = \alpha r, \quad v = -\omega y \mathbf{i} + \omega x \mathbf{j}. \]

Interpret the results.

4.9 Exercise. Show that

\[ \text{curl}(\text{curl} \, A) = -\nabla^2 A + \nabla (\text{div} \, A). \]
5. Integral Theorems

Boundary of a Set

Let \( U \subset \mathbb{R}^n \). We say that \( x \in \mathbb{R}^n \) is a boundary point of \( U \) if every neighbourhood of \( x \) intersects both \( U \) and its complement \( U^c = \mathbb{R}^n - U \). The set of all boundary points is called the topological boundary of \( U \), denoted \( \text{Bdry}(U) \).

It is easy to see that \( \mathbb{R}^n \) is the disjoint union of the interior of \( U \), the topological boundary of \( U \) and the interior of \( U^c \).

5.1 Example. In \( \mathbb{R}^1 \), if \( U = (a, b) \) or \([a, b) \) or \((a, b] \) or \([a, b] \) then \( \text{Bdry}(U) = \{a, b\} \).
5.2 Example. In $\mathbb{R}^2$, the boundary of a disc

$$U = \{(x, y) : x^2 + y^2 < a^2\}$$

is a circle

$$\text{Bdry}(U) = \{(x, y) : x^2 + y^2 = a^2\}.$$ 

5.3 Example. In $\mathbb{R}^3$, the boundary of a ball

$$U = \{(x, y) : x^2 + y^2 + z^2 < a^2\}$$

is a sphere

$$\text{Bdry}(U) = \{(x, y, z) : x^2 + y^2 + z^2 = a^2\}.$$
Let $1 \leq k \leq n$. We say that $S$ is a \textit{simple} $k$-dimensional surface in $\mathbb{R}^n$ if $S$ has a parametric representation

$$x = \psi(u), \quad u \in U \subseteq \mathbb{R}^k,$$

such that the mapping $\psi$ is one-to-one and continuously differentiable on the closed set $U$. In this case, we define the \textit{boundary} of $S$ to be the $k-1$-dimensional set

$$\partial S = \{ \psi(u) : u \in \text{Bdry}(U) \}.$$

Notice that

$$\partial S = \text{Bdry}(S) \quad \text{only if } k = n.$$

\textbf{5.4 Example.} If $C$ is a simple curve $x(u) \ (a \leq u \leq b)$ in $\mathbb{R}^n$ from $x_a = x(a)$ to $x_b = x(b)$, then $x_a \neq x_b$ and $\partial C = \{x_a, x_b\}$ but $\text{Bdry}(C) = C$ unless $n = 1$.

\textbf{5.5 Exercise.} Find $\partial S$ and $\text{Bdry}(S)$ if

$$S = \{ (x, y, z) : x^2 + y^2 \leq a^2, z = 0 \}.$$
**Green’s Formula**

We say that a function is $C^k$ if all its (partial) derivatives or order $\leq k$ exist and are continuous.

Let $U$ be a closed region in $\mathbb{R}^2$ whose topological boundary $\text{Bdry}(U)$ consists of finitely many continuous, piecewise $C^1$ curves.

The *standard orientation* for $\partial U = \text{Bdry}(U)$ is determined by the requirement that as you move along any piece of $\partial U$ the region $U$ is always on the *left*.

Thus, in the simplest case when $\partial U$ consists of a single, closed curve, the standard orientation goes *counterclockwise*.

The following result is a two-dimensional version of the fundamental theorem of calculus.
5.6 Theorem (Green’s Formula). If $P$ and $Q$ are $C^1$ on $U$, then
\[
\iint_U \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{\partial U} (P \, dx + Q \, dy).
\]

If $F$ is a 2D vector field, say
\[
F(x, y, z) = P(x, y) \, i + Q(x, y) \, j,
\]
then
\[
\text{curl } F = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) \times \left( P(x, y) \, i + Q(x, y) \, j \right) = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k
\]
and so we can write Green’s formula as
\[
\int_U \text{curl } F \cdot dS = \oint_{\partial U} F \cdot dr
\]
if we give $U$ the usual orientation for a region in $\mathbb{R}^2$, i.e., with unit normal $n = k$. 
5.7 Exercise. Verify Green’s formula for

\[ P = 1 + 10xy + y^2, \quad Q = 6xy + 5x^2 \]

with \( C \) being the square with vertices \((0, 0), (a, 0), (a, a), (0, a)\).
Stokes’s Formula

Let $S$ be a simple oriented surface (4) with unit normal $n$, and assume that the parameter set $U$ satisfies the assumptions for Green’s formula.

The standard orientation for its boundary curve $\partial S$ is defined as follows: if we imagine walking around $\partial S$ with the “up” direction given by $n$, then $S$ is always on the left.

5.8 Theorem (Stokes Formula). If $F$ is $C^1$ on a neighbourhood of $S$ in $\mathbb{R}^3$, then

$$\int_S \text{curl} \ F \cdot dS = \oint_{\partial S} F(x) \cdot dx.$$

5.9 Exercise. Verify Stokes’s theorem for

$$F = z^2 \mathbf{i} - 2x \mathbf{j} + y^3 \mathbf{k},$$

where $S$ is the hemisphere defined by $x^2 + y^2 + z^2 = 1, \ z \geq 0$. 

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Let us say that two oriented curves $C_1$ and $C_2$ cancel if $C_2 = -C_1$ and so, for any vector field $F$,

$$\int_{C_1} F \cdot dx + \int_{C_2} F \cdot dx = 0.$$

Consider a surface $S$ consisting of several pieces $S_1, \ldots, S_r$, where each piece is a simple, oriented surface. We say that $S$ is orientable if wherever a part of $\partial S_i$ coincides with a part of $\partial S_j$, for $i \neq j$, the two curves cancel. In this case, we define the boundary $\partial S$ to consists of all parts of $\partial S_1, \ldots, \partial S_r$ that do not cancel.

The above definition ensures that Stokes's theorem is valid for a non-simple, oriented surface.

We say that $S$ is a closed surface if $\partial S = \emptyset$. If $S$ is a closed surface then

$$\int_S \text{curl } F \cdot dS = 0.$$
We will use the next lemma to characterise curl $\mathbf{F}$ in a way that is clearly independent of the choice of Euclidean coordinates.

**5.10 Lemma.** Let $S_\epsilon \subseteq \mathbb{R}^2$ be the disc with centre 0 and radius $\epsilon$. If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$, then

$$\lim_{\epsilon \to 0^+} \frac{1}{\text{area}(S_\epsilon)} \int_{S_\epsilon} f \, dS = f(0).$$

In words: the average value of $f$ over $S_\epsilon$ converges to $f(0,0)$ as $\epsilon$ tends to zero.

**5.11 Theorem.** If $S_\epsilon \subseteq \mathbb{R}^3$ is the disc with centre $x_0$, radius $\epsilon$ and unit normal $n_0$, then

$$n_0 \cdot \text{curl} \, \mathbf{F}(x_0) = \lim_{\epsilon \to 0^+} \frac{1}{\text{area}(S_\epsilon)} \oint_{\partial S_\epsilon} \mathbf{F}(x) \cdot \mathbf{n} \, dx.$$
Circulation and Vorticity

Consider a fluid with velocity \( u = u(x,t) \).

The vorticity \( \omega \) is defined by

\[
\omega = \text{curl } u = \nabla \times u.
\]

For any closed, oriented curve \( C \) within the flow region, we call

\[
\oint_C u \cdot dx
\]

the circulation around \( C \).

Stokes’s formula gives

\[
\int_S \omega \cdot dS = \oint_{\partial S} u \cdot dx,
\]

or in words,

flux of vorticity through \( S \) = circulation around \( \partial S \).
If $S_\epsilon$ is the oriented disc with centre $x_0$, radius $\epsilon$ and unit normal $n_0$, then

$$n_0 \cdot \omega(x) = \lim_{\epsilon \to 0^+} \frac{1}{\text{area}(S_\epsilon)} \oint_{\partial S_\epsilon} u(x) \cdot dx = \lim_{\epsilon \to 0^+} \frac{\text{circulation around } \partial S_\epsilon}{\text{area}(S_\epsilon)}.$$  

5.12 Exercise. Consider the ‘velocity’ field

$$u = -\omega y \mathbf{i} + \omega x \mathbf{j}.$$  

Use the (vorticity flux) = (velocity circulation) identity to show that the area enclosed by a planar simple curve $C$ with position vector $r$ is given by

$$A = \frac{1}{2} \left| \oint_C r \times dr \right|.$$
**Divergence Theorem**

Let $U$ be a closed region in $\mathbb{R}^3$ such that its boundary $\partial U = \text{Bdry}(U)$ is an orientable, two-dimensional surface. The *standard orientation* of $\partial U$ is the one obtained by choosing $n$ to be the *outward* unit normal for $U$.

We will use the notation

$$\int_U f \, dV = \int_U f(x, y, z) \, dx \, dy \, dz$$

for the volume integral of a function.

**5.13 Divergence Theorem.** If $F$ is $C^1$ on $U$, then

$$\int_U \text{div} \, F \, dV = \int_{\partial U} F \cdot n \, dS = \int_{\partial U} F \cdot dS.$$
5.14 Exercise. Use the divergence theorem to find the total flux out of the region $U$ defined by $0 \leq x^2 + y^2 \leq 1$, $0 \leq z \leq 1$ for

$$F = x \mathbf{i} + 2y^2 \mathbf{j} + 3z^2 \mathbf{k}.$$ 

As with curl $F$, we can characterise $\text{div } F$ in a way that is clearly independent of the choice of Euclidean coordinates.

5.15 Lemma. Let $U_\varepsilon \subseteq \mathbb{R}^3$ be the ball with centre $0$ and radius $\varepsilon$. If $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous at $0$, then

$$\lim_{\varepsilon \to 0^+} \frac{1}{\text{volume}(U_\varepsilon)} \int_{U_\varepsilon} f \, dV = f(0).$$

5.16 Corollary. If $U_\varepsilon \subseteq \mathbb{R}^3$ is the ball with centre $x_0$ and radius $\varepsilon$ then

$$\text{div } F(x_0) = \lim_{\varepsilon \to 0^+} \frac{1}{\text{volume}(U_\varepsilon)} \oint_{\partial U_\varepsilon} F \cdot dS.$$
Consider a fluid with velocity field \( u = u(x, t) \) and density \( \rho = \rho(x, t) \).

For a fixed region \( U \), the fluid mass inside \( U \) at time \( t \) is

\[
M(t) = \int_U \rho \, dV = \int_U \rho(x, t) \, dV_x.
\]

On the one hand,

\[
\frac{dM}{dt} = \int_U \frac{\partial \rho}{\partial t} \, dV,
\]

and on the other hand, conservation of mass implies that

\[
\frac{dM}{dt} = -\text{rate at which fluid mass leaves } U \text{ through } \partial U
\]

\[
= -\int_{\partial U} \rho \mathbf{u} \cdot d\mathbf{S} = -\int_U \nabla \cdot (\rho \mathbf{u}) \, dV,
\]

so

\[
\int_U \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \, dV = 0.
\]
Taking $U = U_\epsilon$, dividing by volume($U_\epsilon$) and letting $\epsilon \to 0^+$, we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \quad (5)$$

5.17 Exercise. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$. Show that

$$\nabla \cdot (\phi F) = (\nabla \phi) \cdot F + \phi \nabla \cdot F.$$ 

A flow is said to be incompressible if the density is constant, i.e., independent of $x$ and $t$. Then, the continuity equation (5) reduces to

$$\nabla \cdot u = 0.$$
6. Curvilinear Coordinates

Examples

Cartesian coordinates \((x_1, x_2) = (x, y)\) for \(\mathbb{R}^2\).

Cartesian coordinates \((x_1, x_2, x_3) = (x, y, z)\) for \(\mathbb{R}^3\).

Polar coordinates \((r, \theta)\) for \(\mathbb{R}^2\): \((x_1, x_2) = (r \cos \theta, r \sin \theta)\)

Cylindrical coordinates \((r, \theta, z)\) for \(\mathbb{R}^3\):
\[
\begin{align*}
x_1 &= r \cos \theta, \\
x_2 &= r \sin \theta, \\
x_3 &= z.
\end{align*}
\]

Spherical coordinates \((\rho, \phi, \theta)\) for \(\mathbb{R}^3\):
\[
\begin{align*}
x_1 &= \rho \sin \phi \cos \theta, \\
x_2 &= \rho \sin \phi \sin \theta, \\
x_3 &= \rho \cos \phi.
\end{align*}
\]
Basis Vectors

Let

\[(x_1, x_2, x_3) = \Phi(\xi_1, \xi_2, \xi_3).\]

We say that \((\xi_1, \xi_2, \xi_3)\) are curvilinear coordinates for \(x = (x_1, x_2, x_3)\) if the transformation \(\Phi\) is such that

\[
\left| \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right| \neq 0. \tag{6}
\]

By the inverse function theorem, \(\Phi\) is locally invertible so there is a one-to-one correspondence between a point and its curvilinear coordinates, provided the \(\xi_i\) are suitably restricted.
Assumption (6) holds iff the vectors

\[ b_{\xi_i} = \frac{\partial x}{\partial \xi_i} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_i} \\ \frac{\partial x_2}{\partial \xi_i} \\ \frac{\partial x_3}{\partial \xi_i} \end{bmatrix}, \quad i = 1, 2, 3, \]

form a basis for \( \mathbb{R}^3 \). We say that the curvilinear coordinates are \textit{orthogonal} if

\[ b_{\xi_i} \cdot b_{\xi_j} = 0 \quad \text{whenever} \ i \neq j. \]

In this case, we define the \textit{scale factors}

\[ h_i = h_{\xi_i} = |b_{\xi_i}| = \left| \frac{\partial x}{\partial \xi_i} \right| > 0 \]

and the \textit{unit} basis vectors

\[ e_{\xi_i} = \frac{1}{h_i} b_{\xi_i} = \frac{1}{h_i} \frac{\partial x}{\partial \xi_i} \quad \text{(no sum over} \ i). \]
Thus, if \((\xi_1, \xi_2, \xi_3)\) are orthogonal (curvilinear) coordinates then \(e_{\xi_1}, e_{\xi_2}, e_{\xi_3}\) form an orthonormal basis for \(\mathbb{R}^3\):

\[
e_{\xi_i} \cdot e_{\xi_j} = \delta_{ij}.
\]

We will always order the coordinates \(\xi_1, \xi_2, \xi_3\) so that the orthonormal frame is right-handed; thus,

\[
e_{\xi_i} \times e_{\xi_j} = \epsilon_{ijk} e_{\xi_k}
\]

and

\[
\left| \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right| = |b_{\xi_1}, b_{\xi_2}, b_{\xi_3}| > 0.
\]

6.1 Lemma.

\[
\left| \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right| = h_1 h_2 h_3.
\]
6.2 Example. Cartesian coordinates

\[(x_1, x_2, x_3) = (x, y, z)\]

generate the standard basis

\[e_1 = e_x = i, \quad e_2 = e_y = j, \quad e_3 = e_z = k\]

for \(\mathbb{R}^3\). In this case the scale factors are just

\[h_1 = h_2 = h_3 = 1\]

or, with the alternative notation,

\[h_x = h_y = h_z = 1.\]
6.3 Example. For cylindrical coordinates \((r, \theta, z)\) we have the orthogonal basis vectors

\[
b_r = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}, \quad b_\theta = r \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix}, \quad b_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\]

the scale factors

\[
h_r = 1, \quad h_\theta = r, \quad h_z = 1,
\]

and the orthonormal basis vectors

\[
e_r = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}, \quad e_\theta = \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix}, \quad e_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]
6.4 Example. For spherical coordinates \((\rho, \phi, \theta)\) we have orthogonal basis vectors

\[
\begin{align*}
\mathbf{b}_\rho &= \begin{bmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{bmatrix}, & \mathbf{b}_\phi &= \rho \begin{bmatrix} \cos \phi \cos \theta \\ \cos \phi \sin \theta \\ -\sin \phi \end{bmatrix}, & \mathbf{b}_\theta &= \rho \sin \phi \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix},
\end{align*}
\]

the scale factors

\[
\begin{align*}
h_\rho &= 1, & h_\phi &= \rho, & h_\theta &= \rho \sin \phi,
\end{align*}
\]

and the orthonormal basis vectors

\[
\begin{align*}
\mathbf{e}_\rho &= \begin{bmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{bmatrix}, & \mathbf{e}_\phi &= \begin{bmatrix} \cos \phi \cos \theta \\ \cos \phi \sin \theta \\ -\sin \phi \end{bmatrix}, & \mathbf{e}_\theta &= \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix}.
\end{align*}
\]
Line Integrals

Consider a curve $C$ given in terms of orthogonal coordinates,

\[
\begin{align*}
\xi_1 &= \xi_1(u) \\
\xi_2 &= \xi_2(u) \\
\xi_3 &= \xi_3(u)
\end{align*}
\]

for $a \leq u \leq b$.

If $f$ is a scalar function of $\xi_1$, $\xi_2$, $\xi_3$, we can compute

\[
\int_C f \, ds = \int_a^b f \left| \frac{dx}{du} \right| \, du
\]

by observing that

\[
\frac{dx}{du} = \frac{\partial x}{\partial \xi_i} \frac{d\xi_i}{du} = \sum_{i=1}^{3} \frac{d\xi_i}{du} h_i e_{\xi_i}
\]

so that

\[
\left| \frac{dx}{du} \right| = \sqrt{\sum_{i=1}^{3} \left( h_i \frac{d\xi_i}{du} \right)^2}.
\]

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The (purely formal) expression
\[ ds^2 = \sum_{i=1}^{3} h_i^2 \, d\xi_i^2 \]
is called the *metric* for the curvilinear coordinate system, and provides a convenient way to list the scale factors.

**Cartesian coordinates:**
\[ ds^2 = dx^2 + dy^2 + dz^2. \]

**Cylindrical coordinates:**
\[ ds^2 = dr^2 + r^2 \, d\theta^2 + dz^2. \]

**Spherical coordinates:**
\[ ds^2 = d\rho^2 + \rho^2 \, d\phi^2 + \rho^2 \sin^2 \phi \, d\theta^2. \]
6.5 Exercise. Find the length of the helix given by

\[ r = 1, \quad z = \theta, \quad 0 \leq \theta \leq 4\pi. \]

Since \( e_{\xi_1}, e_{\xi_2}, e_{\xi_3} \) are orthonormal, if we define

\[ F_{\xi_i}(\xi) = F \cdot e_{\xi_i} \]

then

\[ F = F_{\xi_i} e_{\xi_i} = F_{\xi_1} e_{\xi_1} + F_{\xi_2} e_{\xi_2} + F_{\xi_3} e_{\xi_3}. \]

To compute

\[ \int_C F \cdot dr = \int_C F \cdot dx = \int_a^b F \cdot \frac{dx}{du} du \]

use

\[ F \cdot \frac{dx}{du} = \sum_{i=1}^3 F_{\xi_i} \frac{d\xi_i}{du} h_i. \]
Alternatively, write

\[ \int_C F \cdot dx = \int_C \sum_{i=1}^{3} F_{\xi_i} h_i\, d\xi_i. \]

6.6 Exercise. Find \( \int_C F \cdot dx \) if \( C \) is the curve

\[ \rho = 1, \quad \theta = \phi, \quad 0 \leq \phi \leq \frac{\pi}{2} \]

and if \( F = \rho e_\rho - 2\rho \cos \theta e_\phi + \rho \sin \phi e_\theta \).
Surface Integrals

Consider a surface $S$ given in terms of orthogonal coordinates,

$$
\begin{align*}
\xi_1 &= \xi_1(u,v) \\
\xi_2 &= u_2(u,v) \\
\xi_3 &= \xi_3(u,v)
\end{align*}
$$

for $(u,v) \in U$.

We want to compute

$$
\int_S f \, dS = \int_U f \left| \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right| \, du \, dv
$$

or

$$
\int_S \mathbf{F} \cdot dS = \int_U \mathbf{F} \cdot \left( \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \right) \, du \, dv.
$$
Since

\[
\frac{\partial x}{\partial u} = \frac{\partial x}{\partial \xi_i} \frac{\partial \xi_i}{\partial u} = \sum_{i=1}^{3} \frac{\partial \xi_i}{\partial u} h_i e_{\xi_i},
\]

\[
\frac{\partial x}{\partial v} = \frac{\partial x}{\partial \xi_i} \frac{\partial \xi_i}{\partial v} = \sum_{i=1}^{3} \frac{\partial \xi_i}{\partial v} h_i e_{\xi_i},
\]

it follows that

\[
\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} = \det \begin{bmatrix}
    e_{\xi_i} & e_{\xi_2} & e_{\xi_3} \\
    h_1 \frac{\partial \xi_1}{\partial u} & h_2 \frac{\partial \xi_2}{\partial u} & h_3 \frac{\partial \xi_3}{\partial u} \\
    h_1 \frac{\partial \xi_1}{\partial v} & h_2 \frac{\partial \xi_2}{\partial v} & h_3 \frac{\partial \xi_3}{\partial v}
\end{bmatrix}.
\]
In particular, if the surface has the form

$$\xi_3 = g(\xi_1, \xi_2), \quad (\xi_1, \xi_2) \in U,$$

then

$$\frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2} = \det \begin{bmatrix}
e_{\xi_1} & e_{\xi_2} & e_{\xi_3} \\
h_1 & 0 & h_3 \frac{\partial g}{\partial \xi_1} \\
0 & h_2 & h_3 \frac{\partial g}{\partial \xi_2}
\end{bmatrix}
= -h_2 h_3 \frac{\partial g}{\partial \xi_1} e_{\xi_1} - h_1 h_3 \frac{\partial g}{\partial \xi_2} e_{\xi_2} + h_1 h_2 e_{\xi_3}.$$ 

6.7 Exercise. Find the mass of a hemispherical dome with radius $a$ and surface density $\mu(1 + \sin \phi)$ per unit area.
Volume Integrals

If \( U = \Phi(R) = \{ x : x = \Phi(\xi), \xi \in R \} \) then

\[
\int_U f \, dV = \iiint_U f \, dx_1 \, dx_2 \, dx_3 = \iiint_R f \left| \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right| \, d\xi_1 \, d\xi_2 \, d\xi_3
\]

Thus, for orthogonal coordinates,

\[
\int_U f \, dV = \iiint_R f h_1 h_2 h_3 \, d\xi_1 \, d\xi_2 \, d\xi_3.
\]

6.8 Exercise. Let \( 0 < \alpha < \pi \). Find the volume of the region \( T \) given by

\[
0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \alpha, \quad 0 \leq \theta \leq 2\pi.
\]
Grad, curl and div

6.9 Theorem.

\[
\text{grad } f = \frac{1}{h_1} \frac{\partial f}{\partial \xi_1} e_{\xi_1} + \frac{1}{h_2} \frac{\partial f}{\partial \xi_2} e_{\xi_2} + \frac{1}{h_3} \frac{\partial f}{\partial \xi_3} e_{\xi_3},
\]

\[
\text{curl } F = \frac{1}{h_1 h_2 h_3} \, \text{det} \begin{vmatrix}
  h_1 e_{\xi_1} & h_2 e_{\xi_2} & h_3 e_{\xi_3} \\
  \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\
  h_1 F_{\xi_1} & h_2 F_{\xi_2} & h_3 F_{\xi_3}
\end{vmatrix},
\]

\[
\text{div } F = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial \xi_1} (h_2 h_3 F_{\xi_1}) + \frac{\partial}{\partial \xi_2} (h_3 h_1 F_{\xi_2}) + \frac{\partial}{\partial \xi_3} (h_1 h_2 F_{\xi_3}) \right).
\]

6.10 Example. In spherical coordinates,

\[
\text{grad } f = \frac{\partial f}{\partial \rho} e_{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} e_{\phi} + \frac{1}{\rho \sin \phi} \frac{\partial f}{\partial \theta} e_{\theta}.
\]
6.11 Exercise. Verify in cylindrical coordinates that
\[ \text{div } F = 0, \quad \text{curl } F = 2\omega e_z \]
for the vector field \( F = -\omega y \mathbf{i} + \omega x \mathbf{j} \).
7. Potential Theory

The Laplace Operator

The *Laplace operator* (or *Laplacian*) $\nabla^2$ is defined by

$$\nabla^2 f = \nabla \cdot (\nabla f) = \text{div}(\text{grad } f).$$

In Cartesian coordinates,

$$\nabla f = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3$$

so

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

or

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$
7.1 Exercise. Show that if \((\xi_1, \xi_2, \xi_3)\) are orthogonal curvilinear coordinates with metric

\[
ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2
\]

then

\[
\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial \xi_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial \xi_3} \right) \right].
\]

7.2 Exercise. Verify that for spherical coordinates \((\rho, \phi, \theta)\) we have

\[
\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial f}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2}
\]
Green’s Identities

Let $U$ be a region in $\mathbb{R}^3$ satisfying our assumptions for the Divergence Theorem. (Note: $U$ is a closed set so $\partial U \subseteq U$.)

We say that a function $\varphi$ is $C^k$ on $U$ if all partial derivatives of $\varphi$ of order $\leq k$ exist and are continuous on a neighbourhood of $U$. Thus, for $\varphi$ to be $C^2$ we require continuity of

$$
\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}, \quad \frac{\partial^2 \varphi}{\partial y \partial z} = \frac{\partial^2 \varphi}{\partial z \partial y}, \quad \frac{\partial^2 \varphi}{\partial z \partial x} = \frac{\partial^2 \varphi}{\partial x \partial z}.
$$

If $\varphi$ is $C^1$ on $U$ then we define its normal derivative by

$$
\frac{\partial \varphi}{\partial n} = n(x) \cdot \nabla \varphi(x) \quad \text{for } x \in \partial U,
$$

where, as usual, $n$ is the outward unit normal for $U$. 

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The following identities are consequences of the Divergence Theorem.

7.3 Theorem (1st Green identity). If $\varphi$ is $C^2$ on $U$ and if $\psi$ is $C^1$ on $U$ then

$$\int_U \psi \nabla^2 \varphi \, dV = \int_{\partial U} \psi \frac{\partial \varphi}{\partial n} \, dS - \int_U \nabla \psi \cdot \nabla \varphi \, dV.$$ 

7.4 Theorem (2nd Green identity). If $\varphi$ and $\psi$ are $C^2$ on $U$, then

$$\int_U \left( \psi \nabla^2 \varphi - \varphi \nabla^2 \psi \right) \, dV = \int_{\partial U} \left( \psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) \, dS.$$
Whether the Divergence Theorem may be applied to ‘singular’ functions depends crucially on the nature of the singularities.

7.5 Theorem. Let \( \varphi \) be a continuous function and \( B(x_0, \epsilon) \) be the ball of radius \( \epsilon \) about a point \( x_0 \). Then

\[
\lim_{\epsilon \to 0^+} \int_{\partial B} \varphi(x) \nabla \left( \frac{1}{|x - x_0|} \right) \cdot dS = -4\pi \varphi(x_0).
\]

Note that if the Divergence Theorem were applicable then the corresponding volume integral would be zero! This is a consequence of the following observation:

7.6 Exercise. Show that

\[
\nabla^2 \left( \frac{1}{|x|} \right) = 0, \quad x \neq 0.
\]
7.7 Corollary (3rd Green identity). If $\varphi$ is $C^2$ on $U$, then

$$\int_{\partial U} \left( \frac{1}{|x-y|} \frac{\partial \varphi}{\partial n}(y) - \varphi(y) \frac{\partial}{\partial n y} \left| \frac{1}{x-y} \right| \right) dS_y - \int_U \frac{1}{|x-y|} \nabla^2 \varphi(y) dV_y$$

$$= \begin{cases} 
4\pi \varphi(x), & \text{if } x \in \text{interior of } U, \\
0, & \text{if } x \notin U.
\end{cases}$$

In particular, if $\varphi$ and its derivatives decay sufficiently fast at $\infty$ then

$$-\int_{\mathbb{R}^3} \frac{1}{|x-y|} \nabla^2 \varphi(y) dV_y = 4\pi \varphi(x),$$
Application: The Poisson Equation

The *Dirichlet problem* is to find a function $\varphi$ satisfying the *Poisson equation*

$$-\nabla^2 \varphi = f \quad \text{in } U, \quad (7)$$

subject to the *boundary condition*

$$\varphi = g \quad \text{on } \partial U,$$

for given functions $f$ and $g$.

The first Green identity immediately gives rise to the following uniqueness result.

**7.8 Theorem.** The Dirichlet problem has at most one $C^2$ solution.
7.9 Exercise. In terms of spherical coordinates, the $C^2$ solution of the boundary value problem

$$\nabla^2 \varphi = 1 \quad \text{in } U,$$

$$\varphi = 1 \quad \text{on } \partial U,$$

where $U$ is the closed unit ball, is given by

$$\varphi = \frac{5 + \rho^2}{6}.$$
The function
\[ G(x) = \frac{1}{4\pi|x|}, \quad x \neq 0, \]
is called the fundamental solution of the Laplace equation
\[ \nabla^2 G(x) = 0, \quad x \neq 0. \]

If \( \varphi \) is a solution of Poisson’s equation (7) which, along with its derivatives, decays sufficiently fast at \( \infty \) then the 3rd Green identity implies that
\[ \varphi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dV_y. \quad (8) \]

Conversely, the following key theorem holds:

**7.10 Theorem.** If \( f: \mathbb{R}^3 \to \mathbb{R} \) is \( C^2 \) with sufficiently rapid decay at \( \infty \), then a solution of the Poisson equation
\[ -\nabla^2 \varphi = f \quad \text{on } \mathbb{R}^3 \quad (9) \]
is given by the \( C^2 \) function (8).

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Scalar and Vector Potentials

7.11 Theorem. In a convex region, the following are equivalent:

1. \( F = \nabla \varphi \) for some scalar potential \( \varphi \);

2. \( F \) is irrotational or curl-free, i.e., \( \nabla \times F = 0 \);

3. \( F \) is conservative, i.e., \( \int_{C} F \cdot dr = 0 \) for any closed curve \( C \);

4. \( \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr \) whenever the curves \( C_1 \) and \( C_2 \) share the same starting point and the same finishing point.
7.12 Exercise. Verify that
\[ \nabla \times (\phi F') = (\nabla \phi) \times F + \phi \nabla \times F. \]

7.13 Exercise. Show that the central force \( F = f(|x|)x \) is conservative in a convex region.
7.14 Corollary. If \( \nabla \cdot v = 0 \) and \( \nabla \times v = 0 \), then \( v = \nabla \varphi \) for some scalar field \( \varphi \) satisfying Laplace’s equation:
\[
\nabla^2 \varphi = 0.
\]

7.15 Theorem. In a convex region, the following are equivalent:

1. \( F = \nabla \times A \) for some vector potential \( A \);

2. \( F \) is solenoidal or divergence-free, i.e., \( \nabla \cdot F = 0 \).

7.16 Exercise. Show that any two vector potentials \( A_1 \) and \( A_2 \) for the same solenoidal vector field \( F \) are necessarily related by
\[
A_1 - A_2 = \nabla \varphi,
\]
where \( \varphi \) is a scalar potential. Demonstrate that the frequently adopted gauge \( \text{div } A = 0 \) is admissible.
The following integral representations of irrotational or solenoidal vector fields are of importance, for instance, in electromagnetic theory.

7.17 Theorem (Helmholtz; particular case). Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field which is $C^2$ and, together with its derivatives, decays ‘sufficiently rapidly’ at $\infty$.

1. If $\nabla \times F = 0$ then

$$F = \nabla \varphi, \quad \varphi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla \cdot F(y)}{|x - y|} \, dV_y.$$ 

2. If $\nabla \cdot F = 0$ then

$$F = \nabla \times A, \quad A(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla \times F(y)}{|x - y|} \, dV_y$$

and $\text{div} \, A = 0$. 
8. Electromagnetic Theory

Static Electric Fields

In a vacuum, a stationary particle with electric charge \( q_1 \) located at \( x_1 \) creates an electric field

\[
E(x) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|x - x_1|^3} (x - x_1), \quad x \neq x_1,
\]

where \( \epsilon_0 \) is a constant called the permittivity of free space. A second stationary particle with charge \( q \) located at \( x \) experiences an electrostatic force

\[
qE(x).
\]

This force law was discovered experimentally by Coulomb around 1775.
The electric field due to \( n \) charges \( q_1, q_2, \ldots, q_n \) located at \( x_1, x_2, \ldots, x_n \), respectively, is a simple vector sum of the fields due to the individual charges:

\[
E(x) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{|x - x_i|^3} (x - x_i).
\]

Since

\[
\frac{x - y}{|x - y|^3} = -\nabla x \left( \frac{1}{|x - y|} \right)
\]

we have

\[
E = -\text{grad } \Phi
\]

where \( \Phi \), the \textit{electrostatic (scalar) potential} generated by the \( n \) charges, is given by

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{|x - x_i|}.
\]

In particular, the electrostatic field \( E \) is conservative.
Recall that
\[
\text{curl}(\text{grad } \Phi) = \nabla \times (\nabla \Phi) = 0
\]
for any scalar function \( \Phi \), so the electrostatic field is curl-free:
\[
\text{curl } E = 0.
\]

### 8.1 Theorem (Gauss's Law for discrete charges).

For any region \( U \) satisfying the assumptions of the divergence theorem,
\[
\varepsilon_0 \oint_{\partial U} E \cdot dS = \sum_{x_i \in U} q_i,
\]
provided no charges lie on \( \partial U \).

In words,
\[
\varepsilon_0 \times (\text{flux of electric field through } \partial U) = \text{total charge enclosed by } U.
\]
Now consider a continuous charge distribution with volume density \( \rho \). The electric field due to the charge in a small region of volume \( \Delta V_y \) around a point \( y \) is approximately

\[
\frac{1}{4\pi\varepsilon_0} \frac{\rho(y) \Delta V_y}{|x - y|^3} (x - y),
\]

so the electric field due to all of the electric charge is

\[
E(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(y)}{|x - y|^3} (x - y) dV_y
\]

and once again \( E = -\text{grad} \Phi \), but with the electrostatic potential given by

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(y)}{|x - y|} dV_y
\]

[since \( \nabla_x \int = \int \nabla_x \) is admissible].
8.2 Lemma. Throughout $\mathbb{R}^3$,\[ \epsilon_0 \text{div} \, \mathbf{E} = \varrho. \]

8.3 Theorem (Gauss's Law). For any region $U$ satisfying the assumptions of the divergence theorem,\[ \epsilon_0 \oint_{\partial U} \mathbf{E} \cdot d\mathbf{S} = \int_U \varrho \, dV. \]

8.4 Exercise. Show that if the charge distribution is radially symmetric, i.e., if \( \varrho = \varrho(\|x\|) \), and \( \varrho = 0 \) for \( \|x\| > R \) then the electrostatic potential $\Phi$ is given by\[ \Phi = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\|x\|} \quad \text{for} \quad \|x\| > R, \]where $Q$ is the total charge.
Electric Currents

Suppose there are electric charges with density \( \rho = \rho(x, t) \) moving with velocity \( v = v(x, t) \). The current density of these charges is

\[
J = \rho v.
\]

In time \( \Delta t \) the charge flowing through a small oriented parallelogram with area vector \( \Delta S = n \Delta S \) is

\[
\rho(v \Delta t) \cdot n \Delta S = (J \cdot \Delta S) \Delta t.
\]

Thus, the flux integral

\[
\int_S J \cdot dS
\]

gives the rate at which charge flows through the oriented surface \( S \), i.e., the current flowing through \( S \).

8.5 Theorem (Continuity Equation). Conservation of electric charge implies that

\[
\frac{\partial \rho}{\partial t} + \text{div } J = 0.
\]
Static Magnetic Fields

If a current flows along a wire with a (very small) cross-section $\Delta S$ then we call $J \cdot \Delta S$ simply the current flowing in the wire.

Consider a steady electric current $I_1$ flowing in a stationary wire loop in the shape of a closed curve $C_1$ and surrounded by a vacuum. Orient $C_1$ in the direction of the current. The Biot–Savart Law (1820) states that the current creates a magnetic flux density

$$B(x) = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 \, dy \times (x - y)}{|x - y|^3},$$

where $\mu_0$ is a constant called the permeability of free space, and that a second steady current $I_2$ flowing in a second stationary wire loop $C_2$ experiences a force

$$F = \oint_{C_2} I_2 \, dx \times B.$$
If the current is not confined to a wire then we replace $I_1 \, dy$ by $J \, dV_y$ to obtain the magnetic flux density

$$B(x) = \frac{\mu_0}{4\pi} \int \frac{J(y) \times (x - y)}{|x - y|^3} \, dV_y.$$  

Since

$$\nabla \times \frac{J(y)}{|x - y|} = \frac{J(y) \times (x - y)}{|x - y|^3}$$

we have

$$B(x) = \frac{\mu_0}{4\pi} \int \nabla \times \frac{J(y)}{|x - y|} \, dV_y = \nabla \times \frac{\mu_0}{4\pi} \int \frac{J(y)}{|x - y|} \, dV_y.$$
Thus,

\[ B = \text{curl} \ A \]

where the \textit{magnetic vector potential} is

\[ A(x) = \frac{\mu_0}{4\pi} \int \frac{J(y)}{|x - y|} \, dV_y. \]

Recalling that

\[ \text{div}(\text{curl} \ A) = \nabla \cdot (\nabla \times A) = 0 \]

for \textit{any} vector field \( A \), we see that the magnetostatic flux density is divergence-free:

\[ \text{div} \ B = 0. \]

\textbf{8.6 Theorem.} The magnetostatic flux density \( B \) due to the steady motion of charges with current density \( J \) satisfies

\[ \text{curl} \ B = \mu_0 J. \]
8.7 Corollary (Ampère’s Law). For any oriented surface $S$, 
\[ \oint_{\partial S} B \cdot dx = \mu_0 \int_S J \cdot dS, \]

If the current is confined to a wire linking $\partial S$, then 
\[ \oint_{\partial S} B \cdot dx = \mu_0 I. \]

8.8 Example. An axially-symmetric current density 
\[ J = f(r)e_z \]
generates a magnetic flux density 
\[ B = \frac{\mu_0}{2\pi} \frac{I(r)}{r} e_\theta, \]
where 
\[ I(r) = \int_{\tilde{r}}^{r} J \cdot dS = 2\pi \int_0^r \tilde{r} f(\tilde{r}) d\tilde{r}. \]
Electromagnetic Induction

Around 1830, Faraday discovered that a time-varying (i.e., non-steady) magnetic flux density $B$ induces an electric field $E$ satisfying

$$\oint_{\partial S} E \cdot d\mathbf{x} = -\frac{d}{dt} \int_S B \cdot dS$$

for any (stationary) oriented surface $S$. Since

$$\oint_{\partial S} E \cdot d\mathbf{x} = \int_S \text{curl } E \cdot dS \quad \text{and} \quad \frac{d}{dt} \int_S B \cdot dS = \int_S \frac{\partial B}{\partial t} \cdot dS$$

we see that

$$\int_S \left( \frac{\partial B}{\partial t} + \text{curl } E \right) \cdot dS = 0.$$ 

The surface $S$ is arbitrary and hence

$$\frac{\partial B}{\partial t} + \text{curl } E = 0.$$ 

In particular, a non-steady electric field is generally not curl-free.
Maxwell’s Equations

In a vacuum, the *electric displacement* $D$ and the *magnetic field* $H$ are given by

$$D = \varepsilon_0 E \quad \text{and} \quad H = \frac{1}{\mu_0} B. \quad (10)$$

For *static* fields, we have seen that

$$\text{div } D = \varrho \quad \text{(Gauss’s law)}, \quad (11)$$
$$\text{curl } E = 0 \quad \text{(Existence of electrostatic potential)}, \quad (12)$$
$$\text{div } B = 0 \quad \text{(Existence of magnetic vector potential)}, \quad (13)$$
$$\text{curl } H = J \quad \text{(Ampère’s law)}. \quad (14)$$

Thus, if $E = -\nabla \phi$ and $B = \nabla \times A$ with the gauge $\text{div } A = 0$ then Gauss’s and Ampère’s laws reduce to

$$\nabla^2 \phi = -\varrho, \quad \nabla^2 A = -J,$$

where $\varepsilon_0 = \mu_0 = 1$ without loss of generality.
For time-varying fields, (12) is no longer true because Faraday’s law of induction implies that

\[ \frac{\partial B}{\partial t} + \text{curl} \ E = 0. \]  
\[ (15) \]

Also, (14) cannot hold in general because it implies that \( \text{div} \ J = 0 \) which violates the continuity equation

\[ \frac{\partial \varrho}{\partial t} + \text{div} \ J = 0. \]  
\[ (16) \]

In 1865, Maxwell realised that if (11) were valid for time-varying fields then

\[ \frac{\partial \varrho}{\partial t} + \text{div} \ J = \text{div} \left( \frac{\partial D}{\partial t} + J \right) \]

must vanish by (16). Thus, we can make (14) consistent with (16) by adding a term, \( \frac{\partial D}{\partial t} \), called the \emph{displacement current}, to the right hand side, so that

\[ \text{curl} \ H = J + \frac{\partial D}{\partial t}. \]
In this way, we arrive at **Maxwell’s equations**:

\[
\begin{align*}
\text{div } D &= \varrho, \\
\text{curl } E + \frac{\partial B}{\partial t} &= 0, \\
\text{div } B &= 0, \\
\text{curl } H - \frac{\partial D}{\partial t} &= J.
\end{align*}
\]

**8.9 Theorem.** Maxwell’s equations may be brought into the compact form

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi &= \varrho, \\
\frac{\partial^2 A}{\partial t^2} - \nabla^2 A &= J
\end{align*}
\]  

subject to the **Lorentz condition**

\[
\frac{\partial \phi}{\partial t} + \text{div } A = 0,
\]

where \( E = -\nabla \phi - \frac{\partial A}{\partial t} \) and \( B = \nabla \times A \) with \( \epsilon_0 = \mu_0 = 1 \).
8.10 Exercise. Show that the ‘wave equation’

$$\frac{\partial^2 \varphi}{\partial t^2} = \nabla^2 \varphi$$

is invariant under the Lorentz transformation

$$x \rightarrow (\cosh \lambda) x + (\sinh \lambda) t$$
$$y \rightarrow y$$
$$z \rightarrow z$$
$$t \rightarrow (\cosh \lambda) t + (\sinh \lambda) x,$$

where $\lambda$ is an arbitrary constant.

The above exercise implies that Maxwell’s equations are compatible with Einstein’s *Theory of Special Relativity* (1905)!