Numerical Methods
for Some Fractional-Order
Partial Differential Equations

Bill McLean, UNSW

Kassem Mustapha, UNSW

Vidar Thomée, Chalmers

Last updated on January 5, 2006
Outline:

1. The continuous problem

2. A generalised Crank–Nicolson scheme

3. Convergence analysis

4. Simple numerical examples

5. Laplace transformation and quadrature
1. The continuous problem

Initial-value problem: for \(-1 < \alpha < 1\), find \(u = u(x,t)\) satisfying
\[
\frac{\partial u}{\partial t} + D_t^{-\alpha} Au = f(t) \quad \text{for } t > 0, \quad \text{with } u(0) = u_0.
\]
In the simplest case,
\[
Au = -\frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < 1,
\]
with \(u(0,t) = 0 = u(1,t)\) for \(t > 0\). Under Laplace transformation,
\[
\hat{u}(z) = \mathcal{L}\{u(t)\} := \int_0^\infty e^{-zt} u(t) \, dt,
\]
the fractional time derivative becomes
\[
\mathcal{L}\{D^{-\alpha} v(t)\} = z^{-\alpha} \hat{v}(z).
\]
Since
\[
\mathcal{L}\left\{ \frac{t^{\mu-1}}{\Gamma(\mu)} \right\} = z^{-\mu} \quad \text{for } \mu > 0,
\]
and
\[
\mathcal{L}\left\{ \frac{\partial u}{\partial t} \right\} = z\hat{u}(z) - u(0),
\]
we can express \( D^{-\alpha}v \) in terms of a Riemann–Liouville fractional integral:
\[
D^{-\alpha}v(t) = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} v(s) \, ds, \quad 0 < \alpha < 1,
\]
and
\[
D^{-\alpha}v(t) = D^1D^{-(1+\alpha)}v(t) = \frac{\partial}{\partial t} \int_0^t \frac{(t - s)^{\alpha}}{\Gamma(1 + \alpha)} v(s) \, ds, \quad -1 < \alpha < 0,
\]
Laplace transformation of
\[ \frac{\partial u}{\partial t} + D_t^{-\alpha} Au = f(t) \]
gives
\[ (z + z^{-\alpha} A) \hat{u}(z) = u_0 + \hat{f}(z) \]
so
\[ \hat{u}(z) = \hat{\mathcal{E}}(z)(u_0 + \hat{f}(z)) \quad \text{where} \quad \hat{\mathcal{E}}(z) = (z + z^{-\alpha} A)^{-1}. \]
Hence, the solution of the \textit{homogeneous problem} \((f \equiv 0)\) is
\[ u(t) = \mathcal{E}(t)u_0 := \mathcal{L}^{-1}\{\hat{\mathcal{E}}(z)u_0\}, \]
and in the general case
\[ u(t) = \mathcal{E}(t)u_0 + \int_0^t \mathcal{E}(t - s)f(s) \, ds. \]
Consider $Au = -u_{xx}$ for $0 < x < 1$ subject to homogeneous Dirichlet boundary conditions. We have orthonormal eigenfunctions and corresponding eigenvalues

$$\phi_m(x) = \sqrt{2} \sin(m\pi x) \quad \text{and} \quad \lambda_m = (m\pi)^2 \quad \text{for } m = 1, 2, 3, \ldots.$$ 

Taking the $L_2$-inner product of $\phi_m$ with the equation

$$\frac{\partial u}{\partial t} + D_t^{-\alpha} Au = f(t)$$

gives a sequence of scalar initial-value problems

$$\frac{du_m}{dt} + \lambda_m D^{-\alpha} u_m = f_m(t) \quad \text{for } t > 0, \quad \text{with } u_m(0) = u_{0m}.$$ 

Each of these problems has an explicit solution in terms of the Mittag-Leffler function

$$E_\mu(t) := \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(1 + \mu p)}, \quad \mu > 0.$$
In fact,
\[
\mathcal{L}^{-1}\left\{ \frac{1}{z + \lambda_m z^{-\alpha}} \right\} = \mathcal{L}^{-1} \sum_{p=0}^{\infty} (-\lambda_m)^p z^{-(1+\alpha)p-1}
\]
\[
= \sum_{p=0}^{\infty} \frac{(-\lambda_m t^{1+\alpha})^p}{\Gamma(1 + (1 + \alpha)p)} = E_{1+\alpha}(-\lambda_m t^{1+\alpha}),
\]
so
\[
\mathcal{E}(t)u_0 = \sum_{m=1}^{\infty} \langle u_0, \phi_m \rangle E_{1+\alpha}(-\lambda_m t^{1+\alpha})\phi_m.
\]
Can show \( |E_{1+\alpha}(-t)| \leq 1 \) for \( t \geq 0 \), so \( \|\mathcal{E}(t)v\| \leq \|v\| \) for every \( v \in L_2(0,1) \) and hence
\[
\|u(t)\| \leq \|u_0\| + \int_0^t \|f(s)\| \, ds.
\]
Thus, the initial-value problem is well-posed.
2. A generalised Crank–Nicolson scheme

Background:


Introduce time levels $0 = t_0 < t_1 < t_2 < \cdots$ and put

$$t_{n-1/2} := \frac{1}{2}(t_{n-1} + t_n) \quad \text{and} \quad k_n := t_n - t_{n-1} \quad \text{for } n \geq 1.$$  

Given a grid function $V^n$ write $V^{n-1/2} := \frac{1}{2}(V^{n-1} + V^n)$, define a piecewise-constant function

$$\overline{V}(t) := \begin{cases} V^1 & \text{for } t_0 < t < t_1, \\ V^{n-1/2} & \text{for } t_{n-1} < t < t_n \text{ and } n \geq 2. \end{cases}$$

and a discrete fractional derivative ($-1 < \alpha < 0$) or fractional integral ($0 < \alpha < 1$),

$$\left(D^{-\alpha}V\right)^{n-1/2} := \frac{1}{k_n} \int_{t_{n-1}}^{t_n} D^{-\alpha}\overline{V}(t) \, dt$$

$$= \omega_{n1}V^1 + \sum_{j=2}^{n} \omega_{nj}V^{j-1/2}.$$
When $\alpha = 0$ we have $D^0 V = \overline{V}$ so

$$(D^0 V)^{n-1/2} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \overline{V}(t) \, dt = \begin{cases} V^1 & \text{if } n = 1, \\ V^{n-1/2} & \text{if } n \geq 2. \end{cases}$$

If $-1 < \alpha < 0$ then we find that

$$\omega_{nn} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \frac{(t_n - s)^\alpha}{\Gamma(1 + \alpha)} \, ds = \frac{k_n^\alpha}{\Gamma(2 + \alpha)} > 0$$

and, for $1 \leq j \leq n - 1$,

$$\omega_{nj} = \frac{1}{k_n} \int_{t_{j-1}}^{t_j} \frac{(t_n - s)^\alpha - (t_{n-1} - s)^\alpha}{\Gamma(1 + \alpha)} \, ds < 0.$$

If $0 < \alpha < 1$ then

$$\omega_{nn} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds \, dt = \frac{k_n^\alpha}{\Gamma(2 + \alpha)} > 0$$

and, for $1 \leq j \leq n - 1$,

$$\omega_{nj} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_{j-1}}^{t_j} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds \, dt > 0.$$
Recall our continuous problem
\[
\frac{\partial u}{\partial t} + D_t^{-\alpha} Au = f(t) \quad \text{for } t > 0, \quad \text{with } u(0) = u_0.
\]
Starting from \( U^0 \approx u_0 \) we generate a discrete-time solution \( U^n \approx u(t_n) \) using
\[
\frac{U^n - U^{n-1}}{k_n} + (D^{-\alpha} AU)^{n-1/2} = f^{n-1/2} \quad \text{for } n \geq 1,
\]
for a suitable approximation
\[
f^{n-1/2} \approx \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(t) \, dt;
\]
e.g., \( f^{n-1/2} = f(t_{n-1}/2) \) or \( \frac{1}{2}(f(t_{n-1}) + f(t_n)) \).

At the first step, we have
\[
(I + \omega_1 k_1 A) U^1 = U^0 + f^{1/2} k_1.
\]
For $n \geq 2$,

$$(I + \frac{1}{2}\omega_{nn}k_n A)U^n = \left( I - \frac{1}{2}\omega_{nn}k_n A \right)U^{n-1} + f^{n-1/2}k_n$$

$$- \left( \omega_{n1}AU^1 + \sum_{j=2}^{n-1} \omega_{nj}AU^{j-1}/2 \right)k_n.$$

Hence, at each step we must solve an elliptic problem, so the scheme is *implicit*.

Formally, the scheme is second-order accurate. However, the $m$th Fourier mode of the $E(t)u_0$ involves a factor

$$E_{1+\alpha}(-\lambda mt^{1+\alpha}) = 1 - \frac{\lambda mt^{1+\alpha}}{\Gamma(2 + \alpha)} + O(t^{2(1+\alpha)}) \quad \text{as } t \to 0^+,$$

so the solution of the continuous problem is not smooth at $t = 0$. To compensate, we employ a *graded mesh*, e.g.,

$$t_n = (nk)^{\gamma}, \quad \text{with } \gamma \geq 1.$$
3. Convergence Analysis

Positivity of $D^{-\alpha}$: for real-valued $v$,

$$\int_0^\infty v(t)D^{-\alpha}v(t)\,dt = \frac{1}{\pi} \cos\left(\frac{\alpha\pi}{2}\right) \int_0^\infty \xi^{-\alpha}|\hat{v}(i\xi)|^2\,d\xi \geq 0.$$ 

Take the $L_2$-inner product of $u(t)$ with the FPDE,

$$\langle u(t), \frac{\partial u}{\partial t}\rangle + \langle u(t), D^{-\alpha}Au \rangle = \langle u(t), f(t) \rangle$$

and observe that

$$\langle u(t), \frac{\partial u}{\partial t}\rangle = \frac{\partial}{\partial t} \frac{1}{2} \|u(t)\|^2$$

and, with $u_m(t) := \langle \phi_m, u(t) \rangle$,

$$\langle u(t), D^{-\alpha}Au \rangle = \sum_{m=1}^\infty \lambda_m \langle u_m(t), D^{-\alpha}u_m \rangle.$$
Thus, integration from $t = 0$ to $t = T$ gives
\[
\frac{1}{2} \left( \|u(T)\|^2 - \|u(0)\|^2 \right) + \left[ \int_0^T \langle u(t), D^{-\alpha}Au \rangle dt \right]_{\geq 0} = \int_0^T \langle u(t), f(t) \rangle dt.
\]
and so
\[
\|u(T)\|^2 \leq \|u_0\|^2 + 2 \int_0^T \langle u(t), f(t) \rangle dt,
\]
from which
\[
\|u(T)\| \leq \|u_0\| + 2 \int_0^T \|f(t)\| dt.
\]
Can mimic this \textit{energy argument} for our discrete-time scheme once we know
\[
\sum_{n=1}^{N} \langle V^{n-1/2}, (D^{-\alpha}V)^{n-1/2} \rangle k_n \geq 0.
\]
In fact,

\[
\sum_{n=1}^{N} \langle V^{n-1/2}, (D^{-\alpha}V)^{n-1/2} \rangle k_n = \sum_{n=1}^{N} \left\langle V^{n-1/2}, \int_{t_{n-1}}^{t_n} D^{-\alpha}V(t) \, dt \right\rangle \\
= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \langle \nabla(t), D^{-\alpha}V(t) \rangle \, dt = \int_{0}^{t_N} \langle \nabla(t), D^{-\alpha}V(t) \rangle \, dt \geq 0,
\]

and \textit{stability} of the generalised Crank–Nicolson method follows:

\[
\|U^N\| \leq \|U^0\| + 2 \sum_{n=1}^{N} \|f^{n-1/2}\| k_n.
\]

Integrating the FPDE from \( t = t_{n-1} \) to \( t = t_n \) we have

\[
u(t_n) - u(t_{n-1}) + \int_{t_{n-1}}^{t_n} D^{-\alpha}Au(t) \, dt = \int_{t_{n-1}}^{t_n} f(t) \, dt,
\]

whereas

\[
U^n - U^{n-1} + \int_{t_{n-1}}^{t_n} D^{-\alpha}A\nabla(t) \, dt = f^{n-1/2} k_n.
\]
Therefore, the error $e^n = U^n - u(t_n)$ satisfies

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^{-\alpha} A\bar{e}(t) \, dt = \eta^{n-1/2} k_n$$

where $\eta^{n-1/2} = \eta_1^{n-1/2} + \eta_2^{n-1/2}$ with

$$\eta_1^{n-1/2} = f^{n-1/2} - \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(t) \, dt,$$

$$\eta_2^{n-1/2} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} D^{-\alpha} A(u - \bar{u}) \, dt.$$ 

Stability yields an *a priori* error bound

$$\|e^n\| \leq \|U^0 - u_0\| + 2 \sum_{j=1}^{n} \|\eta^{j-1/2}\| k_j.$$
3.1 Theorem (McLean and Mustapha, 2005). Let $0 < \alpha < 1$. If the exact solution and source term satisfy

$$t\|Au'(t)\| + t^2\|Au''(t)\| \leq Mt^{\sigma-1} \quad \text{and} \quad t\|f'(t)\| + t^2\|f''(t)\| \leq Mt^{\sigma-1},$$

for $t > 0$, with $\sigma > 0$, then for $0 \leq t \leq T$,

$$\|U^n - u(t_n)\| \leq \|U^0 - u_0\| + C_{\alpha,\gamma,\sigma,T} M \times \begin{cases} k^{\gamma\sigma}, & 1 \leq \gamma < 2/\sigma, \\ k^2 \log(t_n/t_1), & \gamma = 2/\sigma, \\ k^2, & \gamma > 2/\sigma. \end{cases}$$

If we discretise in space using continuous, piecewise-linear finite elements then the error bound has an additional term of order $h^2 \log(t_n/t_1)$, where $h$ is the maximum element size.
4. Simple Numerical Examples

Consider the scalar problem

\[ \frac{du}{dt} + D^{-\alpha}u = f(t) \quad \text{for } t > 0, \quad \text{with } u(0) = u_0. \]

We present results for two cases:

\[ u(t) = t, \quad f(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad u_0 = 0, \]

and

\[ u(t) = \frac{t^{1+\alpha}}{\Gamma(2 + \alpha)}, \quad f(t) = \frac{t^\alpha}{\Gamma(1 + \alpha)} + \frac{t^{1+2\alpha}}{\Gamma(2 + 2\alpha)}, \quad u_0 = 0. \]
In the first case, \( u(t) = t \) is smooth for \( t \geq 0 \) and the maximum errors (over \( [0, T] = [0, 1] \)) using a uniform mesh (\( \gamma = 1 \)) are as follows:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \alpha = -0.6 )</th>
<th>( \alpha = -0.2 )</th>
<th>( \alpha = +0.2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>3.07e-03</td>
<td>1.89e-03</td>
<td>1.02e-03</td>
</tr>
<tr>
<td>40</td>
<td>1.24e-03</td>
<td>5.66e-04</td>
<td>2.61e-04</td>
</tr>
<tr>
<td>80</td>
<td>4.95e-04</td>
<td>1.66e-04</td>
<td>6.59e-05</td>
</tr>
<tr>
<td>160</td>
<td>1.95e-04</td>
<td>4.84e-05</td>
<td>1.66e-05</td>
</tr>
<tr>
<td>320</td>
<td>7.63e-05</td>
<td>1.40e-05</td>
<td>4.19e-06</td>
</tr>
<tr>
<td>640</td>
<td>2.96e-05</td>
<td>4.04e-06</td>
<td>1.05e-06</td>
</tr>
<tr>
<td>1280</td>
<td>1.15e-05</td>
<td>1.16e-06</td>
<td>2.65e-07</td>
</tr>
</tbody>
</table>

We observe \( O(k^2) \) convergence for \( 0 \leq \alpha \leq 1 \), but this deteriorates to \( O(k^{2+\alpha}) \) when \( -1 \leq \alpha \leq 0 \).
In the second case, when $u(t) \propto t^{1+\alpha}$ is not smooth at $t = 0$, the convergence is much worse:

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\alpha = -0.6$</th>
<th>$\alpha = -0.2$</th>
<th>$\alpha = +0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.19e-01</td>
<td>1.03e-02</td>
<td>7.04e-04</td>
</tr>
<tr>
<td>40</td>
<td>9.25e-02 0.37</td>
<td>5.38e-03 0.94</td>
<td>4.17e-04 0.76</td>
</tr>
<tr>
<td>80</td>
<td>7.15e-02 0.37</td>
<td>2.89e-03 0.89</td>
<td>2.12e-04 0.98</td>
</tr>
<tr>
<td>160</td>
<td>5.50e-02 0.38</td>
<td>1.59e-03 0.86</td>
<td>9.99e-05 1.09</td>
</tr>
<tr>
<td>320</td>
<td>4.22e-02 0.38</td>
<td>9.09e-04 0.81</td>
<td>4.54e-05 1.14</td>
</tr>
<tr>
<td>640</td>
<td>3.23e-02 0.39</td>
<td>5.23e-04 0.80</td>
<td>2.03e-05 1.16</td>
</tr>
<tr>
<td>1280</td>
<td>2.47e-02 0.39</td>
<td>3.01e-04 0.80</td>
<td>8.95e-06 1.18</td>
</tr>
</tbody>
</table>
Using a graded mesh ($\gamma = 2$) we can restore $O(k^2)$ convergence for $0 \leq \alpha \leq 1$, but the improvement is weaker for $-1 < \alpha \leq 0$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\alpha = -0.6$</th>
<th>$\alpha = -0.2$</th>
<th>$\alpha = +0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>3.88e-02</td>
<td>1.08e-03</td>
<td>3.96e-04</td>
</tr>
<tr>
<td>40</td>
<td>2.32e-02 0.74</td>
<td>4.15e-04 1.38</td>
<td>1.05e-04 1.91</td>
</tr>
<tr>
<td>80</td>
<td>1.42e-02 0.71</td>
<td>1.54e-04 1.43</td>
<td>2.77e-05 1.93</td>
</tr>
<tr>
<td>160</td>
<td>8.50e-03 0.74</td>
<td>5.55e-05 1.47</td>
<td>7.24e-06 1.94</td>
</tr>
<tr>
<td>320</td>
<td>5.03e-03 0.76</td>
<td>1.96e-05 1.50</td>
<td>1.88e-06 1.95</td>
</tr>
<tr>
<td>640</td>
<td>2.96e-03 0.77</td>
<td>6.78e-06 1.53</td>
<td>4.85e-07 1.95</td>
</tr>
<tr>
<td>1280</td>
<td>1.72e-03 0.78</td>
<td>2.33e-06 1.54</td>
<td>1.25e-07 1.96</td>
</tr>
</tbody>
</table>
For $-1 < \alpha < 0$ we obtain better results with a different approximation to $D^{-\alpha}$. Using integration by parts we find

$$D^{-\alpha}v(t) = \frac{t^\alpha}{\Gamma(1 + \alpha)} v(0) + \int_0^t \frac{(t-s)^\alpha}{\Gamma(1 + \alpha)} v'(s) \, ds,$$

motivating the definition

$$(\tilde{D}^{-\alpha}V)^{n-1/2} := \frac{t^\alpha}{\Gamma(1 + \alpha)} V^0 + \int_0^t \frac{(t-s)^\alpha}{\Gamma(1 + \alpha)} \tilde{V}'(s) \, ds$$

$$= \tilde{\omega}_{n0} V^0 + \sum_{j=1}^n \tilde{\omega}_{nj} (V^j - V^{j-1}),$$

with all positive weights

$$\tilde{\omega}_{n0} = \frac{t_1^{1+\alpha} - t_1^{1+\alpha}}{k_n \Gamma(2 + \alpha)},$$

$$\tilde{\omega}_{nj} = \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \int_{t_{j-1}}^{t_j} \frac{(t-s)^\alpha}{\Gamma(1 + \alpha)} \, ds \, dt, \quad 1 \leq j \leq n - 1,$$

$$\tilde{\omega}_{nn} = \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^t \frac{(t-s)^\alpha}{\Gamma(1 + \alpha)} \, ds \, dt.$$
The results below were obtained using $\tilde{D}^{-\alpha}$, $\gamma = 3$ and

$$f^{n-1/2} := \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(t) \, dt.$$ 

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\alpha = -0.6$</th>
<th>$\alpha = -0.4$</th>
<th>$\alpha = -0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.96e-04</td>
<td>2.93e-04</td>
<td>1.81e-04</td>
</tr>
<tr>
<td>40</td>
<td>7.67e-05 1.95</td>
<td>7.42e-05 1.98</td>
<td>4.55e-05 1.99</td>
</tr>
<tr>
<td>80</td>
<td>1.97e-05 1.96</td>
<td>1.87e-05 1.99</td>
<td>1.14e-05 2.00</td>
</tr>
<tr>
<td>160</td>
<td>5.04e-06 1.97</td>
<td>4.71e-06 1.99</td>
<td>2.85e-06 2.00</td>
</tr>
<tr>
<td>320</td>
<td>1.28e-06 1.98</td>
<td>1.18e-06 1.99</td>
<td>7.14e-07 2.00</td>
</tr>
<tr>
<td>640</td>
<td>3.24e-07 1.98</td>
<td>2.96e-07 2.00</td>
<td>1.79e-07 2.00</td>
</tr>
<tr>
<td>1280</td>
<td>8.17e-08 1.99</td>
<td>7.42e-08 2.00</td>
<td>4.47e-08 2.00</td>
</tr>
</tbody>
</table>
5. Laplace transformation and quadrature

Background (sinc quadrature, heat equation):

- I. P. Gavrulyuk and V. L. Makarov, ????.
- Weideman, ????.

The **Laplace inversion formula**

\[
    u(t) = \mathcal{L}^{-1}\{\hat{u}(z)\} = \frac{1}{2\pi i} \int_{\Gamma} e^{zt} \hat{u}(z) \, dz, \quad t > 0,
\]

provides an integral representation of the solution of our FPDE in terms of its Laplace transform

\[
    \hat{u}(z) = \hat{E}(z)(u_0 + \hat{f}(z)) = (z + z^{-\alpha}A)^{-1}(u_0 + \hat{f}(z)).
\]

Here, the contour \( \Gamma \) must pass to the right of any singularities of \( \hat{u}(z) \).

We require the spectrum of \( A \) to lie in a sector \( |\arg z| < \chi \) with **spectral angle** \( \chi \) and to satisfy a **resolvent estimate**

\[
    \|(z - A)^{-1}\| \leq \frac{M}{1 + |z|} \quad \text{for } |\arg z| > \chi.
\]
We have
\[ \tilde{u}(z) = z^\alpha (z^{1+\alpha} + A)^{-1} (u_0 + \tilde{f}(z)) \]
and
\[ \|(z^{1+\alpha} + A)^{-1}\| \leq \frac{M}{1 + |z|^{1+\alpha}} \text{ for } |\arg z| < \frac{\pi}{2} + \delta_0, \]
where
\[ \delta_0 = \min \left( \frac{\pi}{2}, \frac{\pi - \chi}{1 + \alpha} - \frac{\pi}{2} \right). \]

To guarantee \( 0 < \delta_0 \leq \frac{1}{2}\pi \) we require \( 0 < \chi < \frac{1}{2}(1 - \alpha)\pi. \)

For \( \Gamma \) we may take the contour with parametric representation
\[ z(\xi) = a - b \sin(\delta - i\xi) \]
\[ = (a - b \sin \delta \cosh \xi) + ib \cos \delta \sinh \xi \]
for \( -\infty < \xi < \infty. \)
Putting $z = x + iy$ we find
\[
\left( \frac{x(\xi) - a}{b \sin \delta} \right)^2 - \left( \frac{y(\xi)}{b \cos \delta} \right)^2 = 1,
\]
implying that $\Gamma$ is the left branch of the hyperbola with asymptotes
\[y = \pm(x - a) \cot \delta.\]
Thus, $\Gamma$ lies in the sector $|\arg z| < \frac{1}{2}\pi + \delta_0$ provided
\[0 < \delta < \delta_0, \quad a > b \sin \delta, \quad b > 0.
\]

We want to compute
\[
u(t) = \int_{-\infty}^{\infty} v(\xi, t) \, d\xi \quad \text{for} \quad v(\xi, t) = \frac{1}{2\pi i} e^{z(\xi)t} \hat{u}(z) \frac{dz}{d\xi}
\]
and since
\[|e^{z(\xi)t}| = \exp \left( \text{Re} \, z(\xi)t \right) = \exp \left( (a - b \sin \delta \cosh \xi)t \right)
\]
the integrand $v(\xi, t)$ exhibits a *double exponential decay* as $|\xi| \to \infty$ for $t$ bounded away from 0.
Take a simple equal-weight quadrature rule of the form

\[ I(v) := \int_{-\infty}^{\infty} v(\xi) \, d\xi \approx Q_N(v) := k \sum_{j=-N}^{N} v(\xi_j), \quad \xi_j = jk. \]

5.1 Lemma (López-Fernández and Palencia, 2004). If \( v(\zeta) \) is analytic for \( |\text{Im} \, \zeta| < r \), continuous for \( |\text{Im} \, \zeta| \leq r \), and satisfies

\[ |v(\xi + i\eta)| \leq V e^{-\mu \cosh \xi} \quad \text{for} \ -\infty < \xi < \infty \text{ and } -r \leq \eta \leq r, \]

then

\[ |Q_N(v) - I(v)| \leq CL(\mu)V \left( \exp\left(-2\pi r/k\right) + \exp\left(-\mu \cosh(Nk)\right) \right), \]

where \( L(\mu) = 1 + \log_+(1/\mu). \)

Thus, if \( k = \log N/N \) then the quadrature error is of order \( e^{-cN/\log N} \).
Numerical method: solve (in parallel) the $2N + 1$ elliptic problems

$$(z_j + z_j^{-\alpha} A) \tilde{u}(z_j) = u_0 + \tilde{f}(z_j), \quad z_j = z(\xi_j) \in \Gamma, \quad -N \leq j \leq N,$$

and then put

$$U_N(t) = k \sum_{j=-N}^{N} v(\xi_j, t) = \sum_{j=-N}^{N} w_j e^{\xi_j t} \tilde{u}(z_j), \quad w_j = \frac{k}{2\pi i} \frac{dz}{d\xi} \bigg|_{\xi = \xi_j}.$$

The formula

$$z = T(\zeta) = a - b \sin(\delta - i\zeta)$$

defines a conformal mapping $T$ of the strip $|\text{Im } z| \leq r$ onto a region

$$S_r = \{ T(\zeta) : |\text{Im } z| \leq r \}$$

bounded by two hyperbolas. Note that $T : \mathbb{R} \to \Gamma$.  

31
If
\[ 0 < r < \delta, \quad \delta + r < \delta_0, \quad a > b \sin(\delta + r), \]
then \( S_r \) lies in the sector \( |\arg z| < \frac{1}{2}\pi + \delta_0 \) so \( v(\zeta, t) \) is analytic for \( |\text{Im} \zeta| < r \) and we can show that
\[
|U_N(t) - u(t)| \leq C e^{at} L(t) \left( \|u_0\| + \max_{\zeta \in S_r} \|\breve{f}(z)\| \right)
\times \left( \exp\left(-2\pi r/k\right) + \exp\left(-\frac{1}{2}at \sin(\delta - r) \cosh(Nk)\right) \right).
\]

Put
\[
k = \frac{\log(N/\tau)}{N} \quad \text{with} \quad 0 < \tau < N,
\]
then the error in \( U_N(t) \approx u(t) \) is of order
\[
\exp\left(-cN/\log(N/\tau)\right) + \exp\left(-c(t/\tau)N\right).
\]