Efficient Spherical Designs with Good Geometric Properties

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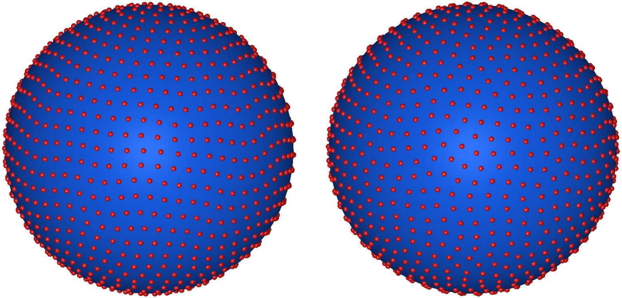


45-design with $N = 1059$ and symmetrtic 45-deisgn with $N = 1038$

Outline

Unit sphere

Unit sphere

$$\mathbb{S}^d = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : |\mathbf{x}| = 1 \right\}$$

- Sets of points $\mathcal{X}_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^d$
  - $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{d+1} x_i y_i, \quad |\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$
- Distance
  - Euclidean distance: $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d, \quad |\mathbf{x} - \mathbf{y}|^2 = 2(1 - \mathbf{x} \cdot \mathbf{y})$
  - Geodesic distance: $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d, \quad \mathrm{dist}\,(\mathbf{x}, \mathbf{y}) = \arccos(\mathbf{x} \cdot \mathbf{y})$
- Can choose points or given points (scattered data)
- Want sequences of point sets $\mathcal{X}_N$, often as part of integration/approximation problem

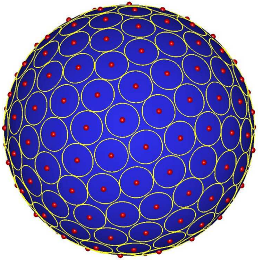
Geometric quality of point set

Spherical cap centre $\mathbf{z} \in \mathbb{S}^d$, radius α

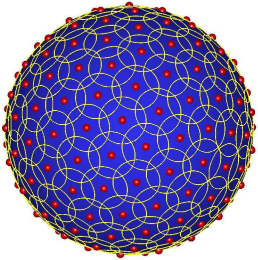
$$\mathcal{C}(\mathbf{z}; \alpha) = \left\{ \mathbf{x} \in \mathbb{S}^d : \mathrm{dist}(\mathbf{x}, \mathbf{z}) \leq \alpha \right\}$$

- Separation (twice packing radius): $\delta_{\mathcal{X}_N} = \min_{i \neq j} \mathrm{dist}(\mathbf{x}_i, \mathbf{x}_j)$
- Mesh norm (covering radius): $h_{\mathcal{X}_N} = \max_{\mathbf{x} \in \mathbb{S}^d} \min_{j=1,\ldots,N} \mathrm{dist}(\mathbf{x}, \mathbf{x}_j)$
- Mesh ratio: $\rho_{\mathcal{X}_N} = \frac{2h_{\mathcal{X}_N}}{\delta_{\mathcal{X}_N}} \geq 1$ Desire: $\rho_{\mathcal{X}_N} \leq c$

Packing: N = 169, δ = 0.2711



Covering: N = 169, h = 0.1905



Aims

- Numerical integration (cubature)

$$Q_N(f) := \sum_{j=1}^{N} w_j f(\mathbf{x}_j) \approx I(f) := \int_{\mathbb{S}^d} f(\mathbf{x}) d\omega(\mathbf{x})$$

  - Equal weights $w_j = |\mathbb{S}^d|/N, j = 1, \ldots, N$ (Quasi Monte-Carlo rules)
  - Degree of precision t if exact for all polynomials of degree $\leq t$

- Spherical t-design is a set $\mathcal{X}_N$ of N points such that

$$\frac{1}{N} \sum_{j=1}^{N} p(\mathbf{x}_j) = \frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} p(\mathbf{x}) d\omega(\mathbf{x}) \qquad \forall p \in \mathbb{P}_t(\mathbb{S}^d),$$

  - N point, equal weight $w_j = \frac{|\mathbb{S}^d|}{N}$ cubature rule, degree of precision t

- Efficient: Low number of points N
- Good geometric properties: Quasi-uniform: Mesh ratio $\rho_{\mathcal{X}} \leq c$

Spherical Polynomials

- Space $\mathbb{P}_t \equiv \mathbb{P}_t\left(\mathbb{S}^d\right)$ of spherical polynomials of degree at most t
- Dimension of space of homogeneous harmonic polynomials of degree ℓ

$$Z(d,0) = 1; \qquad Z(d,\ell) = \frac{(2\ell+d-1)\Gamma(\ell+d-1)}{\Gamma(d)\Gamma(\ell+1)},$$

- Orthonormal basis $Y_{\ell,k}, \quad \ell = 0,1,2,\ldots, \quad k = 1,\ldots,Z(d,\ell)$
- Dimension $\mathbb{P}_t\left(\mathbb{S}^d\right)$ is $D(d,t) = Z(d+1,t) \asymp t^d$
- Addition Theorem

$$\sum_{k=1}^{Z(d,\ell)} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}) = \frac{Z(d,\ell)}{|\mathbb{S}^d|} P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}),$$

  - Normalized Gegenbauer polynomial $P_\ell^{(d+1)}(z) = \frac{P_\ell^{\left(\frac{d-2}{2},\frac{d-2}{2}\right)}(z)}{P_\ell^{\left(\frac{d-2}{2},\frac{d-2}{2}\right)}(1)}$
  - Jacobi polynomial $P_\ell^{(\alpha,\beta)}(z)$ for $z \in [-1,1]$

Spherical t-designs – Number of points N

- Delsarte, Goethals and Seidel (1977)N point t-design on $\mathbb{S}^d$

$$N \geq N^*(d,t) := \begin{cases} 2\binom{d+m}{d} & \text{if } t = 2m+1, \\ \binom{d+m}{d} + \binom{d+m-1}{d} & \text{if } t = 2m. \end{cases}$$

  - Positive weight cubature, degree of precision $t \Longrightarrow N \geq \dim \mathbb{P}_{\lfloor t/2 \rfloor}(\mathbb{S}^d)$
  - On $\mathbb{S}^2$: $N^*(2,t) = (t+1)(t+3)/4$ for t odd; $(t+2)^2/4$ for t even
  - Improved by Yudin (1997) by exponential factor $(e/4)^{d+1}$ as $t \to \infty$.
- Bannai and Damerell (1979, 1980)
  - Tight spherical t-designs if achieve lower bounds
  - Cannot exist on $\mathbb{S}^2$ except for $t = 1, 2, 3, 5$
- Seymour and Zaslavsky (1984) t-designs exist for N sufficiently large
- Bondarenko, Radchenko and Viazovska (2011, 2013, 2015) On $\mathbb{S}^d$
  - spherical t-designs exist for $N \geq c_d\, t^d$
  - well-separated spherical t-designs exist for $N \geq c'_d\, t^d$

Existence Results for $\mathbb{S}^2$

- Bajnok (1991)construction with $N = O(t^3)$
  - n points $z_1, \ldots, z_n$, t-design on $[-1, 1]$
  - Regular m-gon at latitudes z_j
  - $N = mn$ point t-design if $m \geq t + 1$
- Korevaar and Meyers (1993)
  - $N = O(t^3)$
- Both depend on t-designs for interval $[-1, 1]$
  - Set of n points $z_j \in [-1, 1]$:

$$\frac{2}{n} \sum_{j=1}^{n} p(z_j) = \int_{-1}^{1} p(z) dz \quad \forall p \in \mathbb{P}_t([-1, 1])$$

  - Equal weights $\Longrightarrow n = O(t^2)$ points
  - Survey Gautschi (2004)
- Tensor product constructions based on 1-D existence result

Evidence for $\mathbb{S}^2$

- Hardin and Sloane (1996)
  - Summary of known results for $\mathbb{S}^2$
  - Conjecture

$$N = \frac{t^2}{2}\,(1 + o(1))$$

- $N = (t+1)^2 = \dim\left(\mathbb{P}_t(\mathbb{S}^2)\right)$
  - Start from extremal (maximum determinant) points
    Sloan, W. (2004)
  - **Under-determined** system of equations
  - Use interval methods to verify a nearby solution
    - Chen and W. (2006)
    - Chen, Frommer, Lang (2009)
    - An, Chen, Sloan, W. (2010)

Number of points, dimension of space

- $D(d,t) =$ dimension of space of polynomials of degree $\leq t$ on $\mathbb{S}^d$
- DGS lower bound $N^*(d,t)$
- Ratio of leading terms of $D(d,t)/N^*(d,t) = 2^d$
- Efficient if $N < D(d,t)$

d	$N^*(d,t)$	N	$D(d,t)$
2	$\frac{t^2}{4} + t + O(1)$		$(t+1)^2$
3	$\frac{t^3}{24} + \frac{3t^2}{8} + O(t)$		$\frac{t^3}{3} + O(t^2)$
4	$\frac{t^4}{192} + \frac{t^3}{12} + O(t^2)$		$\frac{t^4}{12} + O(t^3)$
5	$\frac{t^5}{1920} + \frac{5t^4}{384} + O(t^3)$		$\frac{t^5}{60} + O(t^4)$

Spherical Harmonic Basis matrix

- Spherical Harmonic Basis matrix

$$\mathbf{Y} = \begin{bmatrix} Y_{0,1}\mathbf{e}^T \\ \widehat{\mathbf{Y}} \end{bmatrix} \in \mathbb{R}^{D(d,t)\times N}$$

  - Rows = basis functions, Columns = points
  - $\mathbf{e} = (1, 1, \ldots, 1)^T \in \mathbb{R}^N$
- Consider case $N \leq D(d,t)$
- Gram matrix

$$\mathbf{G} = \mathbf{Y}^T\mathbf{Y} = Y_{0,1}^2\mathbf{e}\mathbf{e}^T + \widehat{\mathbf{Y}}^T\widehat{\mathbf{Y}} \in \mathbb{R}^{N\times N}$$

- Addition Theorem implies

$$G_{ii} = \sum_{\ell=0}^{t} \frac{Z(d,\ell)}{|\mathbb{S}^d|} P_\ell^{(d+1)}(\mathbf{x}_i \cdot \mathbf{x}_i) = \frac{D(d,t)}{|\mathbb{S}^d|}$$

  - Fixed diagonal elements so trace$(G) = \frac{N D(d,t)}{|\mathbb{S}^d|}$ constant

Spherical designs – nonlinear equations

- Delsarte, Goethals and Seidel (1977) $\mathcal{X}_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^d$ is a spherical t-design if and only if

$$r_{\ell,k}(\mathcal{X}_N) := \sum_{j=1}^{N} Y_{\ell,k}(\mathbf{x}_j) = 0$$

for $k = 1, \ldots, Z(d, \ell), \quad \ell = 1, \ldots, t.$

  - Constant ($\ell = 0$) polynomial $Y_{0,1} = 1/\sqrt{|\mathbb{S}^d|}$ not included in (12)
  - Integral of all spherical harmonics of degree $\ell \geq 1$ is zero
  - Weyl sums: In matrix form

$$\mathbf{r}(\mathcal{X}_N) := \widehat{\mathbf{Y}}\mathbf{e} = \mathbf{0}$$

    - $\mathbf{e} = (1, \ldots, 1)^T \in \mathbb{R}^N$
    - $\widehat{\mathbf{Y}} \in \mathbb{R}^{D(d,t)-1 \times N}$, Spherical harmonic basis matrix excluding first row

Polynomials with positive Legendre coefficients

- Polynomial $\psi_t \in \mathbb{P}_t[-1,1]$ with positive coefficients

$$\psi_t(z) := \sum_{\ell=1}^{t} a_{t,\ell} P_\ell^{(\tau,\tau)}(z),$$
$$a_{t,\ell} > 0 \quad \text{for} \quad \ell = 1, \dots, t.$$

  - $P_\ell^{(\tau,\tau)}(z)$ for $z \in [-1,1]$ Jacobi polynomial, parameter $\tau = \frac{d-2}{2}$
  - $\int_{-1}^{1} \psi_t(z) dz = 0$

- Variational form

$$A_{t,N,\psi}(\mathcal{X}_N) := \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_t(\mathbf{x}_i \cdot \mathbf{x}_j)$$

Spherical designs – variational characterizations

$t \geq 1$, $\mathcal{X}_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^d$, Then

$$0 \leq A_{t,N,\psi}(\mathcal{X}_N) \leq \sum_{\ell=1}^{t} a_{t,\ell} = \psi_t(1)$$

$$\overline{A}_{t,N,\psi} := \frac{1}{(|\mathbb{S}^d|)^N} \int_{\mathbb{S}^d} \cdots \int_{\mathbb{S}^d} A_{t,N,\psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N) d\omega(\mathbf{x}_1) \cdots d\omega(\mathbf{x}_N) = \frac{\psi_t(1)}{N}$$

$\mathcal{X}_N$ is a spherical design if and only if

$$A_{t,N,\psi}(\mathcal{X}_N) = 0.$$

- Weighted sum of squares, strictly positive coefficients

$$A_{t,N,\psi}(\mathcal{X}_N) = \frac{|\mathbb{S}^d|}{N^2} \sum_{\ell=1}^{t} \frac{a_{t,\ell}}{Z(d,\ell)} \sum_{k=1}^{Z(d,\ell)} \left(r_{\ell,k}(\mathcal{X}_N)\right)^2$$

- $A_{t,N,\psi}(\mathcal{X}_N) = 0 \iff \mathcal{X}_N$ spherical t-design
- Global min $A_{t,N,\psi}(\mathcal{X}_N) > 0 \implies$ no spherical t-design with N points

Examples

- Grabner and Tichy (1993)

$$\psi_t(z) = z^t + z^{t-1} - a_{t,0}$$

$$a_{t,0} = \begin{cases} \frac{1}{t} & t \text{ odd}, \\ \frac{1}{t+1} & t \text{ even}. \end{cases}$$

- Cohn and Kumar (2007)

$$\psi_t(z) = (1+z)^t - \frac{2^t}{t+1}.$$

- Sloan and W. (2009)

$$\psi_t(z) = \frac{1}{4\pi} P_t^{(1,0)}(z) - 1 = \sum_{\ell=1}^{t} Z(d,\ell) P_\ell(z)$$

  - $P_t^{(1,0)}$ Jacobi polynomial

Evaluating $A_{t,N,\psi}(\mathcal{X}_N)$

- Matrix $\mathbf{\Psi}$: $\mathbf{\Psi}_{ij} = \psi_t(\mathbf{x}_i \cdot \mathbf{x}_j), \qquad i,j = 1,\ldots,N$
- Spherical t-design $\iff$ $D(d,t) - 1$ equations

$$\mathbf{r} := \widehat{\mathbf{Y}}\mathbf{e} = \mathbf{0},$$

- Diagonal matrix $\mathbf{D}$ of weights

$$\mathbf{\Psi} = |\mathbb{S}^d|\widehat{\mathbf{Y}}^T\mathbf{D}\widehat{\mathbf{Y}}$$

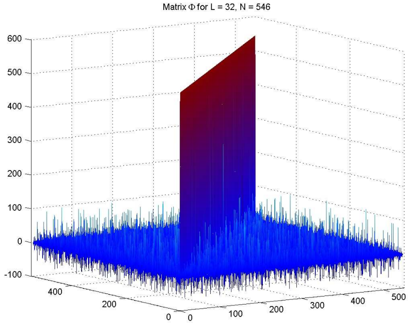
$$\mathbf{D} = \text{diag}\left(\frac{a_{t,\ell}}{Z(d,\ell)}, k = 1,\ldots,Z(d,\ell), \ell = 1,\ldots,t\right)$$

- Any symmetric positive definite $\mathbf{D}$ possible
- Minimize

$$A_{t,N,\psi}(\mathcal{X}_N) = \frac{1}{N^2}\mathbf{e}^T\mathbf{\Psi}\mathbf{e} = \frac{|\mathbb{S}^d|}{N^2}\mathbf{e}^T\widehat{\mathbf{Y}}^T\mathbf{D}\widehat{\mathbf{Y}}\mathbf{e} = \frac{|\mathbb{S}^d|}{N^2}\mathbf{r}^T\mathbf{D}\mathbf{r}$$

Evaluating $A_{t,N,\psi}(\mathcal{X}_N)$ using $\mathbf{\Psi}$

- N by N matrix $\mathbf{\Psi}_{ij} = \psi_t(\mathbf{x}_i \cdot \mathbf{x}_j)$
- Constant diagonal elements $\psi_t(1) = \sum_{\ell=1}^{t} a_{t,\ell}$
- Matrix $\mathbf{\Psi}$ for $a_{t,\ell} = Z(d,\ell) \iff \mathbf{D} = \mathbf{I}$



- Advantages: simple, (trivially) parallel
- Issue: cancelation errors in summing off diagonal elements

Degrees of freedom for $\mathbb{S}^2$

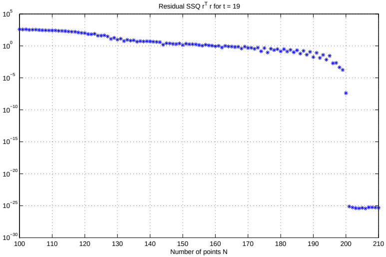
- Spherical parametrization, normalization $\Longrightarrow n = 2N - 3$ variables
- $m = \dim(\mathbb{P}_t) - 1 = (t+1)^2 - 1$ equations
- Threshold $n \geq m \Longrightarrow$

$$N \geq \widehat{N}(2,t) := \lceil (t+1)^2)/2 \rceil + 1$$

- $\widehat{N}$ less than twice the DGS lower bound N^*

$$2N^*(2,t) - \widehat{N}(2,t) = t,$$

- Sum of squares for $t = 19$, varying N, $\widehat{N}(2,19) = 201$



Symmetric designs

- Exploit symmetry to reduce conditions: Sobolev (1962)
- Symmetric: N even, $\mathbf{x} \in \mathcal{X}_N \iff -\mathbf{x} \in \mathcal{X}_N$
- Equal weights, ℓ odd $\Longrightarrow Y_{\ell,k}$ integrated exactly
- Constraints from even degrees $\leq t$, t odd

$$m = \sum_{k=1}^{(t-1)/2} 2(2k) + 1 = \frac{(t-1)(t+2)}{2}$$

- $N = 2K$ points $\Longrightarrow 2K - 3 = N - 3$ degrees of freedom
- Degrees of freedom $\geq$ number of equations $\Longrightarrow$

$$N \geq \overline{N}(2,t) := 2\left\lceil \frac{t^2+t+4}{4} \right\rceil$$

- Slightly less than $\widehat{N}(2,t)$

$$2N^*(2,t) - \overline{N}(2,t) = \tfrac{3}{2}t - \begin{cases} \frac{3}{2} & \text{if} \quad \mod(t,4) = 1, \\ \frac{1}{2} & \text{if} \quad \mod(t,4) = 3. \end{cases}$$

Degrees of freedom $\mathbb{S}^d$

- $\mathbb{S}^d \subset \mathbb{R}^{d+1}$: Spherical parametrization $\Longrightarrow$ d variables
- N points $\mathbf{x}_j, j = 1, \ldots, N \Longrightarrow Nd$ variables
- Orthogonal invariance $\Longrightarrow Q\mathbf{x}_j$ so $d(d+1)/2$ zero elements
- Number of variables $n = Nd - d(d+1)/2$
- Number of equations for t-design

$$m = \sum_{\ell=1}^{t} Z(d, \ell) = Z(d+1, t) - 1$$

- Number of variables $\geq$ number of conditions $\Longrightarrow \widehat{N}(d,t)$
- Symmetric point set (both $\mathbf{x}_j, -\mathbf{x}_j$ in set) automatically integrates odd degree polynomial $\Longrightarrow \overline{N}(d,t)$

Number of points, dimension of space

- $D(d,t)$ = dimension of space of polynomials of degree $\le t$ on $\mathbb{S}^d$
- DGS lower bound $N^*(d,t)$
- $\widehat{N}(d,t)$ ensures $n \ge m$
- $\overline{N}(d,t)$ ensures symmetric point set has $n \ge m$
- Ratio of leading terms of $D(d,t)/\widehat{N}(d,t) = d$

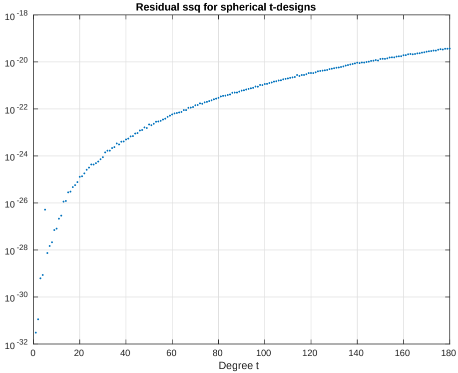
d	$N^*(d,t)$	$\overline{N}(d,t)$	$\widehat{N}(d,t)$	$D(d,t)$
2	$\frac{t^2}{4} + t + O(1)$	$\frac{t^2}{2} + \frac{t}{2} + O(1)$	$\frac{t^2}{2} + t + O(1)$	$(t+1)^2$
3	$\frac{t^3}{24} + \frac{3t^2}{8} + O(t)$	$\frac{t^3}{9} + \frac{t^2}{3} + O(t)$	$\frac{t^3}{9} + \frac{t^2}{2} + O(t)$	$\frac{t^3}{3} + O(t^2)$
4	$\frac{t^4}{192} + \frac{t^3}{12} + O(t^2)$	$\frac{t^4}{48} + \frac{t^3}{8} + O(t^2)$	$\frac{t^4}{48} + \frac{t^3}{6} + O(t^2)$	$\frac{t^4}{12} + O(t^3)$
5	$\frac{t^5}{1920} + \frac{5t^4}{384} + O(t^3)$	$\frac{t^5}{300} + \frac{t^4}{30} +)(t^3)$	$\frac{t^5}{300} + \frac{t^4}{24} +)(t^3)$	$\frac{t^5}{60} + O(t^4)$

Least squares

- Spherical parametrization of variables $x \in \mathbb{R}^n$
- Residuals $\mathbf{r} : \mathbb{R}^n \to \mathbb{R}^m$, $m = \sum_{\ell=1}^{t} Z(d,\ell)$
- Number of points chosen so $n = m$ or $n = m + 1$.
- Sum of squares objective $f(x) = \mathbf{r}(x)^T \mathbf{r}(x) = \sum_{i=1}^{m} [r_i(x)]^2$
- Jacobian $A(x) \in \mathbb{R}^{m \times n}$
- Gradient $\nabla f(x) = 2A(x)r(x)$
- Hessian $\nabla^2 f(x) = 2A(x)^T A(x) + 2\sum_{i=1}^{m} r_i(x) \nabla^2 r_i(x)$
- $\mathbf{r}(x^*) = \mathbf{0}$ and $A(x^*)$ rank $n \implies$ strict (isloated) global minimum
- $n > m$ have some freedom
- Algorithm: Levenberg-Marquardt $(A^T A + \lambda I)d = -A^T r$
- Issues
  - Singular Jacobians $A(x)$
  - Stuck with $\nabla f(x) = \mathbf{0}$ but $f(x) > 0$, perhaps small
  - Different solutions: depends on starting point, algorithm parameters, ...

Spherical designs - numerical results

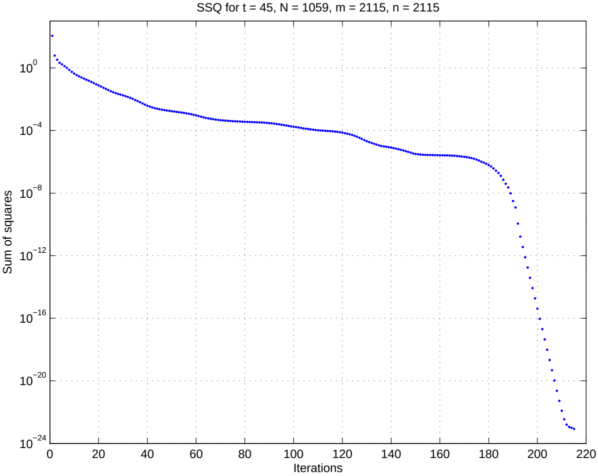
- Use $N = \widehat{N}(2,t)$, $\Longrightarrow$ t odd, $n = m$, t even, $n = m + 1$
- Rounding error limits achievable accuracy in $A_{t,N}$
- Both $A_{t,N,\psi}(\mathcal{X}_N)$, $\mathbf{r}^T\mathbf{r}$ order of rounding error $\Longrightarrow$ what confidence?
- $t = 180 \Longrightarrow \widehat{N}(2,t) = 16382, \quad m = 32760, \quad n = 32761$



http://web.maths.unsw.edu.au/~rsw/Sphere/EffSphDes/

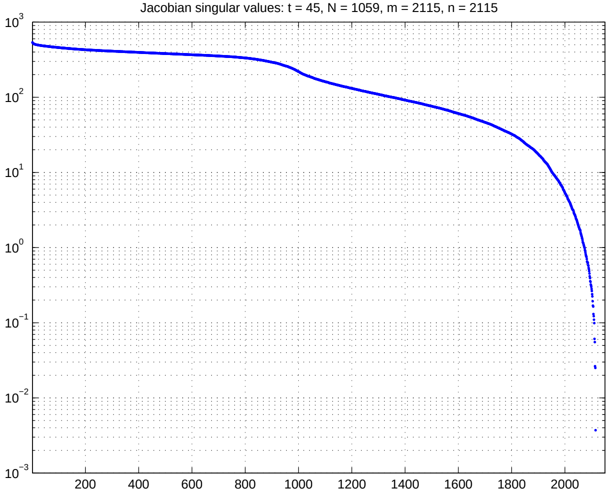
Spherical designs - rate of convergence

- Rate of convergence: $t = 45$, $N = 1059$, $m = 2115$, $n = 2115$



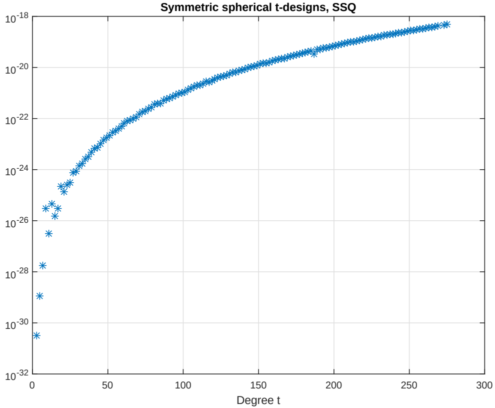
Spherical designs - Jacobian singular values

- Jacobian singular values: $t = 45$, $N = 1059$, $m = 2115$, $n = 2115$



Symmetric spherical designs - numerical results

- For t odd, use $N = \overline{N}(2, t)$
  - $\mod (t, 4) = 3 \Longrightarrow n = m, \qquad \mod (t, 4) = 1 \Longrightarrow n = m + 1$
- $t = 277 \Longrightarrow \overline{N}(2, t) = 38506, \quad m = 38502, \quad n = 38503$



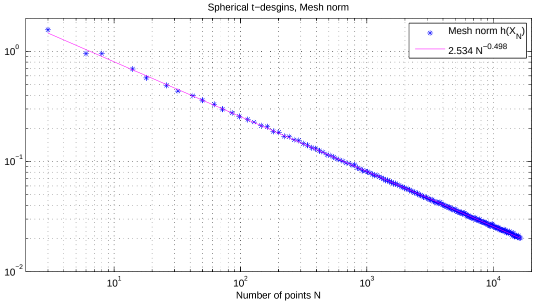
http://web.maths.unsw.edu.au/~rsw/Sphere/EffSphDes/

Mesh norm

- Mesh norm (covering radius)

$$h_{\mathcal{X}_N} = \max_{\mathbf{x}\in\mathbb{S}^2} \min_{j=1,\ldots,N} \text{dist}\left(\mathbf{x}, \mathbf{x}_j\right) \geq \frac{c_{\text{cov}}}{\sqrt{N}}$$

  - Yudin (1995) Mesh norm h given by largest zero $z_t = \cos(h)$ of $P^{(1,0)}(z)$
  - Reimer (2003) extended to any positive weight cubature rule with degree of precision t

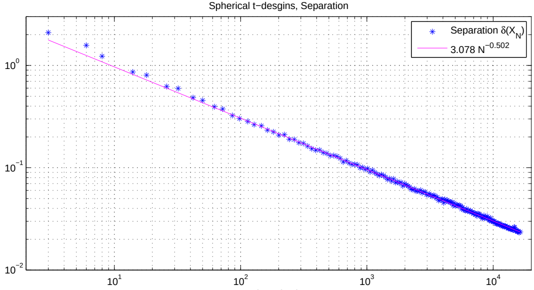


Separation

- Separation (twice packing radius)

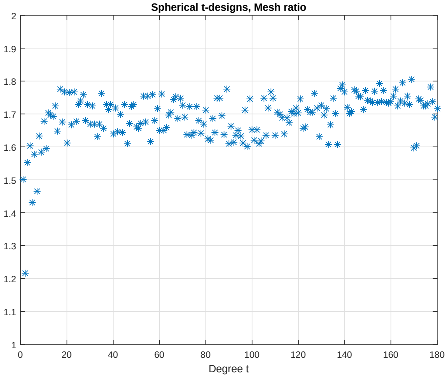
$$\delta_{\mathcal{X}_N} = \min_{i \neq j} \text{dist}\left(\mathbf{x}_i, \mathbf{x}_j\right) \leq \frac{c_{\text{pack}}}{\sqrt{N}}$$

  - Union of two spherical t-designs is a spherical t-design
  - $\mathcal{X}_N \cup Q\mathcal{X}_N$ is $2N$ point spherical t-design with arbitrary separation
  - $\overline{N} < 2N^*$, $\widehat{N} < 2N^*$ so cannot occur
- If N sufficiently small get separation ?



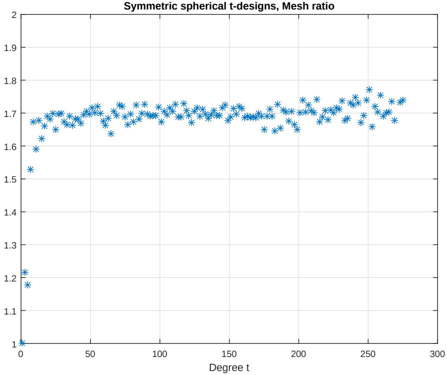
Mesh ratio

- Mesh ratio
$$\rho_{\mathcal{X}_N} = \frac{2h_{\mathcal{X}_N}}{\delta_{\mathcal{X}_N}} = \frac{\text{Covering radius}}{\text{Packing radius}} \geq 1$$
- $\rho_{\mathcal{X}_N}$ bounded $\Longrightarrow$ $\mathcal{X}_N$ quasi-uniform
- Spherical t-designs with $\overline{N}$ points



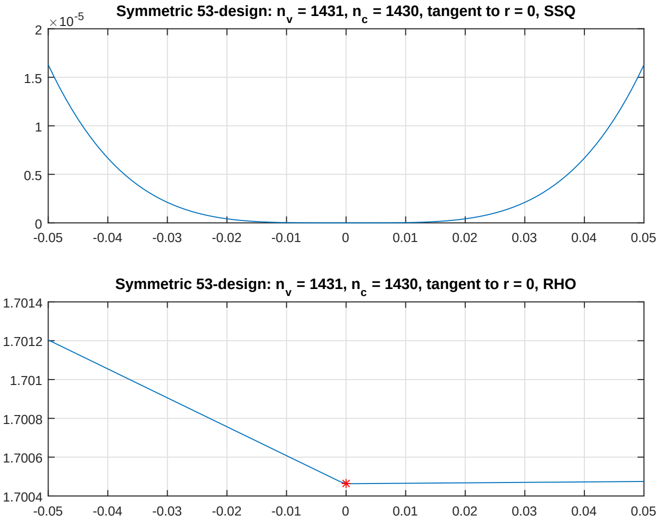
Mesh ratio - Symmetric t-designs

- Mesh ratio for symmetric t-designs with $\overline{N}$ points

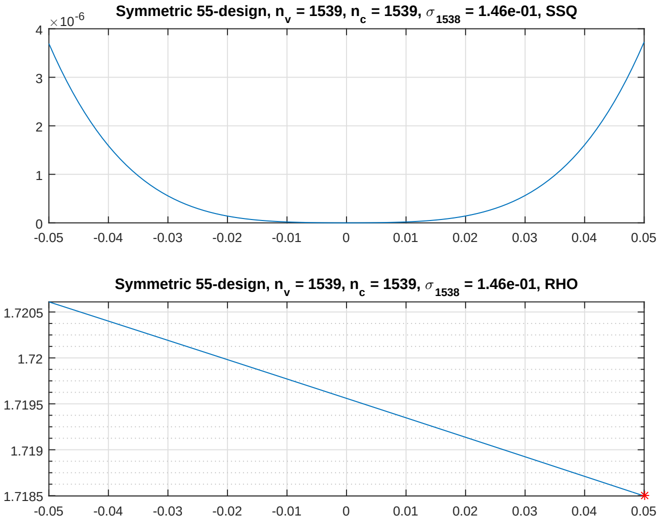


- Converge to different spherical designs from different starting points
- If $n = m$, $\mathbf{r}^* = \mathbf{0}$ and $\sigma_n(A^*) > 0 \Longrightarrow$ strict global minimizer
- If $n = m + 1$ use freedom to reduce mesh ratio

Reducing mesh ratio - one degree of freedom

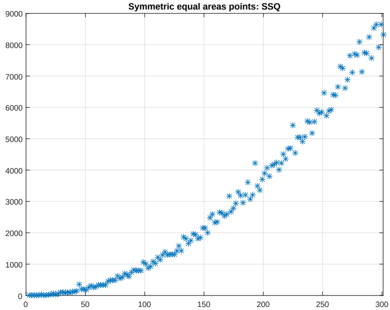


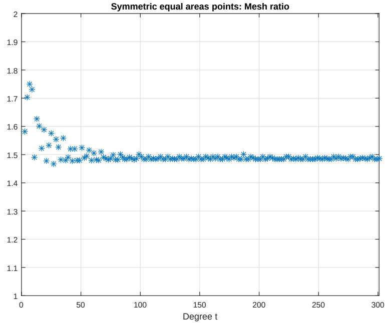
Reducing mesh ratio - isolated zero



Starting points

- Close to spherical design (low sums of squares)
- Good mesh ratio
- Generalized spiral points
- Equal area points





Conclusions

- Efficient sets of (numerical) t-designs for $\mathbb{S}^2$
  - http://web.maths.unsw.edu.au/~rsw/Sphere/EffSphDes/
  - Equal weight cubature rule, degree of precision t with $\widehat{N} = (t^2 + 2t)/2 + O(1)$ points for $t = 1, \ldots, 180$
  - Symmetric equal weight cubature rule, degree of precision t with $\overline{N} = (t^2 + t)/2 + O(1)$ points for $t = 1, \ldots, 277$
  - Good geometric properties: mesh norm, separation, mesh ratio < 1.8
  - Larger N: Use extra degrees of freedom to satisfy other criteria
- Issues
  - Rounding errors in evaluating criteria, speed of extended precision
  - Convergence difficulties with close to singular Jacobians
  - No proof of nearby exact spherical designs when $N < (t+1)^2$
  - No proof of existence for all t
  - There exist t-designs with $N < \overline{N}(d, t)$; special symmetries
  - Calculation by optimization for each t, N
  - Finding points sets with better mesh ratio ad-hoc
  - Point sets $\mathcal{X}_N$ not nested
  - Higher dimensions d