Meshless boundary element methods for exterior problems on spheroids

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Abstract. In geophysical applications one is interested in the Neumann problem exterior to a spheroid where the orbits of satellites are located. The satellites create data which amount to boundary conditions in scattered points. As a model problem, we consider the exterior Neumann problem of the Laplacian with boundary condition on an oblate or prolate spheroid. We propose to use spherical radial basis functions in the solution of the boundary integral equation arising from the Dirichlet-to-Neumann map. Our meshless approach with radial basis functions is particularly suitable for handling scattered satellite data. We also propose a preconditioning technique based on an overlapping domain decomposition method to deal with ill-conditioned matrices arising from the approximation problem.

Here we report from [14, 11] on a meshless method with radial basis functions for the Neumann problem for the Laplacian exterior to an oblate or a prolate spheroid. In geophysical applications [1, 2] one is interested in such exterior Neumann problems where the orbits of satellites are located on spheroids. A key tool of our approach is the use of the Dirichlet-to-Neumann map which directly converts the boundary value problem into a pseudodifferential equation on the spheroid. This integral equation is then handled with Fourier techniques by expansion into appropriate spherical harmonics. Huang and Yu [6] solved this pseudodifferential equation numerically with standard boundary elements on a regular grid on the angular domain of the spherical coordinates. Our approach uses spherical radial basis functions (SRBF’s) instead, allowing for better handling of scattered data. Originally we introduced the meshless boundary element method for integral equations on the sphere [4] and we used it to solve the linearized Molodensky problem [3]. The error analysis for the meshless method with radial basis functions on the sphere as done in [4] has been extended to spheroids in [14, 11]. Again the smooth solution of the pseudodifferential equation can be approximated with high convergence rates by the Galerkin solution consisting of radial basis functions. The numerical solution is obtained by an appropriate implementation [9] of the prolate and oblate spheroidal harmonics and by truncating the resulting infinite dimensional discrete Galerkin system.

We present numerical results of our meshless boundary element method for equidistributed points, so-called Saff points, and for scattered data points from satellite observations. Furthermore we present iteration numbers for the conjugate gradient method applied to solve the discrete systems in case of scattered data. We list the iteration numbers for the preconditioned conjugate gradient method when an overlapping additive Schwarz method is used as preconditioner. For both oblate and prolate spheroids we obtain only mildly growing iteration numbers for the overlapping additive Schwarz method. This technique was firstly analysed for pseudodifferential equations on the sphere in [5].

Let \( \Gamma_0 = \{(x_1, x_2, x_3): \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1, a > b > 0\} \) be an oblate spheroid and \( \Omega^c \) be the unbounded domain outside the boundary \( \Gamma_0 \).

We consider the exterior Neumann problem: Given \( g \in L_2(\Gamma_0) \), find \( U \in \Omega^c \) satisfying

\[
\begin{align*}
\Delta U &= 0 & \text{in } & \Omega^c \\
\partial_{\nu} U &= g & \text{on } & \Gamma_0 \\
U(x) &= O(|x|^{-1}) & \text{as } & |x| \to \infty
\end{align*}
\]
where \(||\mathbf{x}|||\) denotes the Euclidean norm of \(\mathbf{x}\) and \(\nu\) denotes the unit outward normal vector on \(\Gamma_0\).

The solution is given by the series
\[
U(\mu, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{T_n^m(\sinh\mu)}{T_n^m(\sinh\mu_0)} \hat{u}_{nm}(\theta, \varphi), \quad \mu \geq \mu_0 > 0,
\]
where \((\mu, \theta, \varphi)\) denote the oblate spheroidal coordinates [6]. Here
\[
T_n^m = i^{m+n} \exp \left( \frac{im}{2} \right) Q_n^m(i\chi), \quad i^2 = -1,
\]
and \(Q_n^m\) are the associated Legendre functions of second kind and \(Y_{nm}\) the spherical harmonics of degree \(n\) and \(\hat{u}_{nm}\) the expansion coefficients.

In [6] it is shown that eq (1) is equivalent to
\[
\mathcal{K}u = g \quad \text{on} \quad \Gamma_0
\]
with the Dirichlet-to-Neumann map (Steklov-Poincaré operator) \(\mathcal{K}\). Its weak formulation is: Find \(u \in H^{1/2}(\Gamma_0)\) satisfying
\[
D(u, v) = \int_{\Gamma_0} (\mathcal{K}u)v \, ds = \int_{\Gamma_0} gv \, ds \quad \forall v \in H^{1/2}(\Gamma_0)
\]
with
\[
D(u, v) = f_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} G_n^m(\sinh\mu_0) \hat{u}_{nm} \hat{v}_{nm}^*,
\]
where
\[
G_n^m(x) = -\frac{(1+x^2)^{1/2}T_n^m(x)}{T_n^m(x)}.
\]

**Proposition:** There exists a unique solution for the variational problem (3).

This is due to the Lax-Milgram theorem since \(D(\cdot, \cdot)\) is continuous and coercive on the Sobolev space \(H^{1/2}(\Gamma_0)\) and the right hand side in eq (3) defines a continuous linear functional on \(H^{1/2}(\Gamma_0)\). Here \(u \in H^s(\Gamma_0), s \in \mathbb{R}\) if and only if
\[
\| u \|^2_{H^s(\Gamma_0)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (1 + n^2)^s |\hat{u}_{nm}|^2 < \infty
\]
The approximate solution to eq (3) is sought in a finite dimensional subspace of \(H^{1/2}(\Gamma_0)\). In order to use SRBFs we take the bijection \(\omega: \Gamma_0 \rightarrow \mathbb{S}^2\)
\[
\omega(x) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]
and we define a reproducing kernel on \(\Gamma_0\) as
\[
\Psi(x, x') = \Phi(\omega(x), \omega(x')) , \quad x, x' \in \Gamma_0
\]
where \(\Phi\) is defined via a univariate function \(\Phi: [-1,1] \rightarrow \mathbb{R}\) by (see [7, 8])
\[
\Phi(y, z) = \phi(y \cdot z) \quad \forall y, z \in \mathbb{S}^2.
\]
Here \(\phi\) has a series expansion in terms of Legendre polynomials \(P_n\) of degree \(n\), as
\[
\phi(t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (2n+1) \hat{\phi}(n) P_n(t)
\]
with
\[
\hat{\phi}(n) = 2\pi \int_{-1}^{+1} \phi(t) P_n(t) \, dt.
\]
The kernel \(\Psi\) can be expanded into a series of spherical harmonics as
\[
\Psi(x, x') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \hat{\phi}(n) Y_{nm}(\omega(x)) Y_{nm}^*(\omega(x'))
\]
where we choose
\[ \hat{\phi}(n) \equiv (1 + n^2)^{-\tau}. \] (12)

Given a set of scattered data points \( X = \{x_1, \ldots, x_M\} \subset \Gamma_0 \), we take

\[ V^\tau := \text{span} \{ \Psi_j : = \Psi(x, \cdot) : x \in X \}. \]

Now, the solution of eq (3) is approximated by \( u_X \in V^\tau \) satisfying the Galerkin method

\[ D(u_X, v) = \int_{\Gamma_0} g v \, ds \quad \forall v \in V^\tau. \] (13)

To this end we have to solve a linear system

\[ Ac = g \] (14)

where \( A \) is a matrix with entries \( A_{ij} = D(\Psi_i, \Psi_j) \), \( i, j = 1, \ldots, M \) and \( g \) is a vector with entries

\[ g_j = \int_{\Gamma_0} g \Psi_j \, ds, \quad j = 1, \ldots, M. \]

Using eq (4) and eq (11) we can write

\[ A_{ij} = \int_0^\infty \sum_{m=0}^{\infty} \sum_{n=-n}^{n} [\hat{\phi}(n)]^2 G_n^m(\sinh \mu_0) Y_{nm}(\omega(x_i)) Y_{nm}^*(\omega(x_j)). \] (15)

Further let \( N \) denote the number of the series terms of the truncated matrix element \( A_{ij}^N \). Then we compute the actual Galerkin approximation \( u_X \) by solving eq (14) with the Galerkin matrix \( A^N \) obtained via the truncated entries \( A_{ij}^N \).

Let \( Y = y_1, \ldots, y_M \) be the image of \( X \) under the map \( \omega \), i.e. \( y_j = \omega(x_j) \) for \( j = 1, \ldots, M \). As \( Y \) is a set of scattered points on \( \mathbb{S}^2 \), we define the mesh norm \( h_Y \) of \( Y \) as usual \( h_Y^\tau = \sup_{y \in \mathbb{S}^2} \min_{y_j \in Y} \cos^{-1}(y \cdot y_j) \).

**Proposition [11]:** For an oblate spheroid and truncation number \( N \) chosen sufficiently large there holds the quasi-optimal error estimate for the difference between the exact solution \( u \in H^s(\Gamma_0) \) of eq (3) and the Galerkin solution \( u_X \in V^\tau, 2\tau \geq s \)

\[ \| u - u_X \|_{H^{1/2}(\Gamma_0)} \leq C h_Y^{-1/2}, \]

where the constant \( C \) is independent from \( N \) and the set \( X \) used to define \( \Psi_j \).

(The corresponding result for the prolate spheroid see [14].)

Our numerical experiments plotted in Fig. 2 for the equidistant mesh of Saff points (see Fig. 1) show these predicted convergence rates \( \alpha \), namely \( \alpha = 2.5 \) for \( 0 m = 0 (\tau = 1.5) \), \( \alpha = 4.5 \) for \( 0 m = 1 (\tau = 2.5) \), \( \alpha = 6.5 \) for \( 0 m = 2 (\tau = 3.5) \), cf. Table 1.

To compute the entries \( A_{ij} \) of the stiffness matrix given in eq (15), we need to compute the spherical harmonics \( Y_{nm} \) and the functions \( G_n^m \). The functions \( T_{nm} \) are calculated using the algorithm for oblate spheroidal harmonics presented in (see [9]) and we use \( G_n^m(x) = G_n^{m*}(x) \)

\[ G_n^m(x) = (n + 1)x + (n - m + 1) \frac{T_{n+1}^m(x)}{T_n^m(x)}, \quad m = 0, 1, \ldots, n; \quad n = 0, 1, 2, \ldots \]

The right hand side terms \( g_j \) are computed by using the Fourier coefficients of \( g \) and \( \Psi_j \) and Parseval’s identity.

In our numerical experiments, we use \( \Psi(x, x') = \Phi(\omega(x), \omega(x')) \) for arbitrary \( y, z \in \mathbb{S}^2 \) and \( \Phi(y, z) = \rho_m(\sqrt{2 - 2y \cdot z}) \) with \( \rho_m \) being locally supported radial basis functions defined by Wendland [10]; see Table 1. In this case eq (12) holds for \( \tau = m + 3/2 \) see[12].
Table 1: Wendland’s RBFs

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\rho_m(r)$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(1 - r)^2$</td>
<td>1.5</td>
</tr>
<tr>
<td>1</td>
<td>$(1 - r)^2(4r + 1)$</td>
<td>2.5</td>
</tr>
<tr>
<td>2</td>
<td>$(1 - r)^2(35r^2 + 18r + 3)$</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Example 1: Problem (1) with oblate spheroid $\Gamma_0$ ($f_0 = 4$, $\mu_0 = 1$), Neumann condition and exact solution

$$g = -\frac{\sinh \mu \sin \theta \cos \varphi (2\cosh^2 \mu + \cos^2 \theta)}{f_0^2(\cosh^2 \mu - \cos^2 \theta)^2}$$

$$U = \frac{\cosh \mu \sin \theta \cos \varphi}{f_0^2(\cosh^2 \mu - \cos^2 \theta)^2}$$

(16)

Let $e := u - u_X$ where $u(\theta, \varphi) = U(\mu_0, \theta, \varphi)$, solves eq (3) and $u_X$ solves eq (13) with truncation number $N_{trunc} = 100$. We compute $\| e \|_{L^2(\Gamma_0)}$ and $\| e \|_{H^{1/2}(\Gamma_0)}$ approximately by $N_{max} = 120$

$$\| e \|_{L^2(\Gamma_0)} \approx \left( \sum_{n=0}^{N_{max}} \sum_{m=-n}^{n} |\tilde{u}_{X_{nm}} - \tilde{u}_{nm}|^2 \right)^{1/2}$$

and

$$\| e \|_{H^{1/2}(\Gamma_0)} \approx \left( \sum_{n=0}^{N_{max}} (1 + n^2)^{1/2} \sum_{m=-n}^{n} |\tilde{u}_{X_{nm}} - \tilde{u}_{nm}|^2 \right)^{1/2}$$

with $\tilde{u}_{X_{nm}} = \sum_{i=1}^{M} \hat{\varphi}(n) c_i Y_{nm}(\omega(x_i))$

and $\tilde{u}_{nm}$ is computed by an appropriate quadrature formula [13].

Figure 1: Image of Saff points on oblate spheroid
Figure 2: Log-log plot for $H^{1/2}(\Gamma_0)$ errors using Wendland RBFs $\rho_m(r)$ (Saff points) for the oblate (Om) and prolate spheroid (Pm).

Table 2 gives the errors in the $L^2(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$ norms for scattered points. The matrix is ill conditioned and a preconditioner is required. Table 3 shows the corresponding numbers of iteration of the preconditioned conjugate gradient method (PCG) with an overlapping additive Schwarz preconditioner [mref]; stopping criteria in both cases is relative tolerance $\leq 10^{-10}$. Errors of the same order as in the non-preconditioned case are obtained but PCG needs much smaller iteration numbers. (cpu: computational times in seconds, iter: numbers of iterations)

<table>
<thead>
<tr>
<th>$M$</th>
<th>$q_Y$</th>
<th>$| e |_{L^2(\Gamma_0)}$</th>
<th>$| e |_{H^{1/2}(\Gamma_0)}$</th>
<th>ITER</th>
<th>CPU</th>
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<td>$\pi/140$</td>
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<td>5.01114E-005</td>
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<td>2.13257E-005</td>
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<td>10443</td>
<td>$\pi/240$</td>
<td>1.87142E-006</td>
<td>1.62031E-005</td>
<td>30931</td>
<td>17361.9</td>
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</tbody>
</table>

Table 2: Errors with scattered points from MAGSAT, using CG, $\rho_0(r)$, Ex.1

<table>
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<tr>
<th>$M$</th>
<th>$\cos\alpha$</th>
<th>$\cos\beta$</th>
<th>$| e |_{L^2(\Gamma_0)}$</th>
<th>$| e |_{H^{1/2}(\Gamma_0)}$</th>
<th>ITER</th>
<th>CPU</th>
</tr>
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Table 3: Errors with scattered points from MAGSAT, for PCG, $\rho_0(r)$, Ex.1
In [14] we have analysed the foregoing meshless boundary element method for a prolate spheroid. We obtain similar results both with respect to convergence of the meshless Galerkin solution as well as the behaviour of the iteration number of PCG. For brevity we present only some numerical results.

**Example 2**: Problem (1) with **prolate spheroid** $\Gamma_0$ ([14]), Neumann condition and exact solution

$$g(\mu, \theta, \varphi) = -\frac{\sqrt{2}\sin 2\theta \cos \varphi (7 - 3 \cosh 4\mu + 4 \cos 2\varphi)}{4f_0^4 \sqrt{\cosh \mu - \cos \theta (\cosh 2\mu + \cos 2\varphi)^7/2}}$$

$$U(\mu, \theta, \varphi) = \frac{\sqrt{2}\sin \mu \cosh \mu \cos \varphi}{2f_0^3 (\cosh 2\mu + \cos 2\varphi)^{5/2}}. \tag{17}$$

stopping criteria in both cases is relative tolerance $\leq 10^{-10}$.

Tables 4 and 5 give the errors in the $L^2(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$ norms and the iteration numbers for CG and PCG for scattered points for Example 2 (stopping criteria in both cases is relative tolerance $\leq 10^{-8}$). We observe for the prolate spheroid that the errors of the meshless boundary element method and the performance of the PCG with overlapping Schwarz preconditioner behave as in the case of the oblate spheroid.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(M)</th>
<th>(q_f)</th>
<th>$|e|_{L^2(\Gamma_0)}$</th>
<th>$|e|_{H^{1/2}(\Gamma_0)}$</th>
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Table 4: Errors with scattered points from MAGSAT, using CG, $\rho_m(r)$, Ex.2

<table>
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<th>(M)</th>
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<th>(\cos\beta)</th>
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Table 5: Errors with scattered points from MAGSAT, for PCG, $\rho_m(r)$, Ex.2
References


