Solving the Dirichlet-to-Neumann map on an oblate spheroid by a mesh-free method

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Abstract

In this paper, we study a mesh-free method using the Galerkin method with radial basis functions (RBFs) for the exterior Neumann problem of the Laplacian with boundary condition on an oblate spheroid. This problem is reformulated as a pseudo-differential equation on the spheroid by using the Dirichlet-to-Neumann map. We show convergence of the Galerkin scheme. Our approach is particularly suitable for handling scattered data. We also propose a fast solution technique based on a domain decomposition method (obtained by the additive Schwarz operator) to precondition the ill-conditioned matrices arising from the Galerkin scheme. We estimate the condition number of the preconditioned system. Numerical results supporting the theoretical results are presented.

Keywords: mesh free method, Dirichlet-to-Neumann map, oblate spheroid

1. Introduction

In this paper, we construct numerical solutions to the Neumann problem for the Laplacian exterior to an oblate spheroid. Applications of such

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problems can be found in geophysics [3, 5], when the earth is modelled as an oblate spheroid.

There are two key ingredients in our approach: the first one is the use of the Dirichlet-to-Neumann map which directly converts the boundary value problem into a pseudodifferential equation on the spheroid. The second ingredient is the use of spherical radial basis functions (RBFs) in the Galerkin method to solve approximately this pseudo-differential equation. The advantage of using RBFs is that they enable us to handle scattered data, e.g. satellite data. As a result, we prove that if the solution is smooth then a high rate of convergence of the approximate solution can be achieved by choosing appropriate RBFs.

Since the linear systems arising from our discretization are ill-conditioned, we also discuss a preconditioning strategy based on the additive Schwarz method which gives us a fast solution procedure.

The paper is organised as follows. In Section 2, we derive the pseudo-differential equation on the oblate spheroid which represents the Dirichlet-to-Neumann map. We show how to use the solution of this pseudo-differential equation to obtain the solution for the exterior Neumann problem. In Section 3, the symbol of this pseudo-differential equation is used explicitly to compute the entries of the stiffness matrix in the Galerkin scheme where we use radial basis functions. For smooth data, we give an optimal a priori error estimate for the Galerkin approximation to the exact solution. In Section 4 we introduce the additive Schwarz preconditioner and give an estimate for the condition number of the preconditioned stiffness matrix. In Section 5 we comment on the implementation with locally supported RBFs. The last section reports our numerical experiments which underly the theory.

2. The Dirichlet-to-Neumann map on the oblate spheroid

Let the oblate spheroid be given by

$$
\Gamma_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} = 1, \ a > b > 0\}
$$

and $\Omega^c$. A point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ can be represented in oblate spheroid coordinates as

$$
\begin{align*}
    x_1 &= f_0 \cosh \mu_0 \sin \theta \cos \varphi \\
    x_2 &= f_0 \cosh \mu_0 \sin \theta \sin \varphi \\
    x_3 &= f_0 \sinh \mu_0 \cos \theta
\end{align*}
$$

(2.1)
where \( \mu_0 > 0, f_0 = \sqrt{a^2 - b^2}, a = f_0 \cosh \mu_0, b = f_0 \sinh \mu_0, \theta \in [0, \pi] \) and \( \varphi \in [0, 2\pi) \). In oblate spherical coordinates, the Laplace operator is written as

\[
\Delta = \frac{1}{f_0^2 (\cosh^2 \mu - \sin^2 \theta)} \left\{ \frac{1}{\cosh \mu} \frac{\partial}{\partial \mu} \left( \cosh \mu \frac{\partial}{\partial \mu} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \left( \frac{1}{\sin^2 \theta} - \frac{1}{\cosh^2 \mu} \right) \frac{\partial^2}{\partial \varphi^2} \right\}.
\]

Let \( \Psi(\mu, \theta, \varphi) = F(\mu)G(\theta)H(\varphi) \) be such that \( \Delta \Psi = 0 \). By using the technique of separation of variables we obtain

\[
\frac{d^2}{d\varphi^2} H(\varphi) + m^2 H(\varphi) = 0
\]

\[
\frac{1}{\cos \theta} \frac{d}{d\theta} \left( \cos \theta \frac{dG(\theta)}{d\theta} \right) - \frac{m^2 G(\theta)}{\cos^2 \theta} + n(n + 1)G(\theta) = 0,
\]

\[
\frac{1}{\cosh \mu} \frac{d}{d\mu} \left( \cosh \mu \frac{dF(\mu)}{d\mu} \right) + \frac{m^2 F(\mu)}{\cosh^2 \mu} - n(n + 1)F(\mu) = 0,
\]

where \( m, n \) are integers. A solution of the above system is

\[
\Psi_{nm}^{\mu}(\mu, \theta, \varphi) = T_{nm}^{\mu}(\sinh \mu)Y_{nm}(\theta, \varphi), \quad m = -n, ..., n, \quad n = 0, 1, 2, ...
\]

with \( T_{nm}^{\mu} \) given by

\[
T_{nm}^{\mu} = i \exp\left(\frac{i\pi n}{2}\right)Q_{n}^{m}(ix), \quad i^2 = -1,
\]

with the associated Legendre functions of the second kind \( Q_{n}^{m}(x) \) (see [1], Chapter 8) and spherical harmonics \( Y_{nm}(\theta, \varphi) \) of degree \( n \) (see [9]) which form an orthonormal basis for \( L^2(S^2) \), where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \).

Exterior to the oblate spheroid \( \Gamma_0 \), we consider the Neumann problem:

\[
\begin{cases}
\Delta U = 0 & \text{in } \Omega^c \\
\partial_{\nu} U = g & \text{on } \Gamma_0 \\
U(x) = O(||x||^{-1}) \quad \text{as } ||x|| \to \infty
\end{cases} \tag{2.2}
\]

where \( ||x|| \) denotes the Euclidean norm of \( x \) and \( \nu \) denotes the unit outward normal vector on \( \Gamma_0 \). Suppose \( u(\theta, \varphi) := U(\mu_0, \theta, \varphi) \) is expanded into an
absolutely convergent series

\[ u(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \hat{u}_{nm} Y_{nm}(\theta, \varphi) \]  

(2.3)

where

\[ \hat{u}_{nm} = \int_{0}^{\pi} \int_{0}^{2\pi} u(\theta, \varphi) Y_{nm}^*(\theta, \varphi) \sin \theta \, d\varphi \, d\theta \]  

(2.4)

with \( Y_{nm}^* \) being the complex conjugate of \( Y_{nm} \). The solution of the Laplace equation in the unbounded domain \( \Omega^c \) outside the oblate spheroid \( \Gamma_0 \) is given by

\[ U(\mu, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} T_{nm}(\sinh \mu) \hat{u}_{nm} Y_{nm}(\theta, \varphi), \quad \mu \geq \mu_0 > 0. \]  

(2.5)

Since

\[ \frac{\partial}{\partial \mu} (\mu, \theta, \varphi) = f_0 \sqrt{\cosh^2 \mu - \sin^2 \theta}, \]

the outward normal derivative \( \partial_{\nu} U \) on \( \Gamma_0 \) can be computed as

\[ \partial_{\nu} U(\theta, \varphi) = -\frac{1}{f_0 \sqrt{\cosh^2 \mu_0 - \sin^2 \theta}} \partial_{\mu} (\mu_0, \theta, \varphi) \]

Therefore the normal derivative of the solution on \( \Gamma_0 \) is

\[ \partial_{\nu} U(\theta, \varphi) = -\frac{1}{f_0 \sqrt{\cosh^2 \mu_0 - \sin^2 \theta}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{dT_{nm}(\sinh \mu_0)}{dT_{nm}(\sinh \mu_0)} \hat{u}_{nm} Y_{nm}(\theta, \varphi). \]

We denote by \( K \) the Dirichlet-to-Neumann map (Steklov-Poincaré operator) defined for any \( v \in H^{1/2}(\mathbb{S}^2) \) by

\[ (Kv)(\theta, \varphi) := -\frac{1}{f_0 \sqrt{\cosh^2 \mu_0 - \sin^2 \theta}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{dT_{nm}(\sinh \mu_0)}{dT_{nm}(\sinh \mu_0)} \hat{v}_{nm} Y_{nm}(\theta, \varphi). \]  

(2.6)

It is known that (see e.g. [11]) (2.2) is equivalent to

\[ K u = g \quad \text{on} \quad \Gamma_0. \]  

(2.7)

In subsequent sections, we will develop a Galerkin approximation method to \( u \) by first writing \( u \) as a solution of a pseudo-differential operator equation, defined by the Dirichlet-to-Neumann map (2.6). Using this approximation, together with (2.4) and (2.5), we will obtain an approximation to the solution \( U \) of the Neumann problem.
3. Weak formulation and Galerkin approximation

For a given real number \( s \), the Sobolev space \( \mathcal{H}^s(\Gamma_0) \) is defined by

\[
\mathcal{H}^s(\Gamma_0) = \{ f \in D'(\Gamma_0) : \| f \|_{\mathcal{H}^s(\Gamma_0)}^2 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (1 + n^2)^s |\hat{f}_{nm}|^2 < \infty \},
\]

where \( D'(\Gamma_0) \) is the set of all distributions defined on \( \Gamma_0 \).

Let us denote the usual \( L^2(\Gamma_0) \) inner product by \( \langle \cdot, \cdot \rangle_{\Gamma_0} \). For two arbitrary functions \( u, v \in \mathcal{H}^{1/2}(\Gamma_0) \), let us define the bilinear form

\[
D(u, v) := \langle Ku, v \rangle_{\Gamma_0}.
\]

Since the surface measure of \( \Gamma_0 \) is \( ds = f_0^2 \cosh \mu_0 \sqrt{\cosh^2 \mu_0 - \sin^2 \theta} \sin \theta \, d\theta \, d\varphi \), from the definition of \( D(u, v) \) we have

\[
D(u, v) = f_0^2 \cosh \mu_0 \int_{0}^{2\pi} \int_{0}^{\pi} (Ku)v \sqrt{\cosh^2 \mu_0 - \sin^2 \theta} \sin \theta \, d\theta \, d\varphi
= f_0^2 \cosh \mu_0 \int_{S^2} (Ku)v \sqrt{\cosh^2 \mu_0 - \sin^2 \theta} \, d\sigma,
\]

where \( d\sigma = \sin \theta \, d\theta \, d\varphi \) is the surface measure on the unit sphere \( S^2 \). Using (2.6) and the Plancherel theorem, we have

\[
D(u, v) = -f_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{dT^m_n(\sinh \mu_0)/d\mu}{T^m_n(\sinh \mu_0)} \cosh(\mu_0) \hat{u}_{nm} \hat{v}^*_{nm}.
\]

Defining

\[
G^m_n(x) := -\frac{(1 + x^2) \frac{d}{dx} T^m_n(x)}{T^m_n(x)} \quad (3.1)
\]

so that

\[
G^m_n(\sinh \mu) = -\frac{d}{d\mu} \frac{T^m_n(\sinh \mu)}{T^m_n(\sinh \mu)} \cosh \mu,
\]

we can rewrite \( D(u, v) \) as

\[
D(u, v) = f_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} G^m_n(\sinh \mu_0) \hat{u}_{nm} \hat{v}^*_{nm}. \quad (3.2)
\]
By integrating (2.7) against test functions in $\mathcal{H}^{1/2}(\Gamma_0)$, we obtain the following weak formulation of the equation (2.7):

Find $u \in \mathcal{H}^{1/2}(\Gamma_0)$ satisfying $D(u, v) = \langle g, v \rangle_{\Gamma_0} \quad \forall v \in \mathcal{H}^{1/2}(\Gamma_0)$. \hfill (3.3)

The following result is proved in [6]:

**Proposition 3.1.** The bilinear forms $D(\cdot, \cdot)$ is continuous and coercive on $\mathcal{H}^{1/2}(\Gamma_0)$, i.e., there exists constants $C_1$ and $C_2$ such that

$$|D(u, v)| \leq C_1 \|u\|_{\mathcal{H}^{1/2}(\Gamma_0)} \|v\|_{\mathcal{H}^{1/2}(\Gamma_0)} \quad \forall u, v \in \mathcal{H}^{1/2}(\Gamma_0)$$

and

$$C_2 \|v\|_{\mathcal{H}^{1/2}(\Gamma_0)}^2 \leq |D(v, v)| \quad \forall v \in \mathcal{H}^{1/2}(\Gamma_0)$$

So by using Lax–Milgram theorem, there exists a unique solution for the variational problem (3.3).

In the following we consider solving the Dirichlet-to-Neumann map approximately by the Galerkin method with radial basis functions as in [8] where we have considered the prolate spheroid. We obtain the Galerkin solution by mapping functions defined on the spheroid to functions defined on the sphere. Therefore, let us first introduce the concept of positive definite kernel and radial basis functions on the sphere.

A continuous function $\Phi : S^2 \times S^2 \to \mathbb{R}$ is called a (strictly) positive definite kernel on $S^2$ if it satisfies: (i) $\Phi(y, z) = \Phi(z, y) \quad \forall y, z \in S^2$; (ii) for any set of distinct scattered points $\{y_1, ..., y_K\} \subset S^2$, the matrix $[\Phi(y_i, y_j)]$ is positive (definite)/semi-definite; see [13, 17].

We define the kernel $\Phi$ from a univariate function $\phi : [-1, 1] \to \mathbb{R}$ by

$$\Phi(y, z) = \phi(y \cdot z) \quad \forall y, z \in S^2,$$

where $\phi$ has a series expansion in terms of Legendre polynomials $P_n$ of degree $n$ as

$$\phi(t) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n + 1) \hat{\phi}(n) P_n(t), \quad \hat{\phi}(n) = 2\pi \int_{-1}^{1} \phi(t) P_n(t) dt. \hfill (3.5)$$

Since the approximate solution to (3.3) is sought in a finite dimensional subspace of $\mathcal{H}^{1/2}(\Gamma_0)$ on the oblate spheroid, we introduce the following bijection $\omega : \Gamma_0 \to S^2$ to make use of the RBFs on the sphere,

$$\omega(x) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \hfill (3.6)$$
where \( x \) is an arbitrary point on \( \Gamma_0 \) with oblate spheroidal coordinates
\[
  x(\theta, \varphi) = (f_0 \cosh \mu_0 \sin \theta \cos \varphi, f_0 \cosh \mu_0 \sin \theta \sin \varphi, f_0 \sinh \mu_0 \cos \theta).
\] (3.7)

Using this map, we define a kernel on \( \Gamma_0 \) as
\[
  \Psi(x, x') = \Phi(\omega(x), \omega(x')), \quad x, x' \in \Gamma_0
\] (3.8)
where \( \Phi \) is the kernel defined on \( S^2 \); see (3.4). The kernel \( \Psi \) can be expanded into a series of spherical harmonics as
\[
  \Psi(x, x') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \hat{\phi}(n) Y_{nm}(\omega(x)) Y_{nm}^*(\omega(x')).
\] (3.9)

For a set of scattered data points \( X = \{x_1, \ldots, x_M\} \subset \Gamma_0 \), we define \( V^\tau \) by
\[
  V^\tau := \text{span}\{\Psi_1, \ldots, \Psi_M\}
\] (3.10)
where \( \Psi_j := \Psi(x_j, \cdot), j = 1, \ldots, M \). The solution of (3.3) is approximated by \( u_X \in V^\tau \) satisfying
\[
  D(u_X, v) = \langle g, v \rangle_{\Gamma_0} \quad \forall v \in V^\tau.
\] (3.11)

To this end we have to solve a linear system
\[
  Ac = g
\] (3.12)
where \( A \) is a matrix with entries \( A_{ij} = D(\Psi_i, \Psi_j), \quad i, j = 1, \ldots, M \), and \( g \) is a vector with entries \( g_j = \langle g, \Psi_j \rangle_{\Gamma_0} \), \( j = 1, \ldots, M \). Using (3.2) and (3.9) we can write
\[
  A_{ij} = f_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \hat{\phi}(n)^2 G_n^m(\sinh \mu_0) Y_{nm}(\omega(x_i)) Y_{nm}^*(\omega(x_j)).
\] (3.13)

The positive definiteness of the matrix \( A \) is a direct consequence of coercivity of the bilinear form \( D(\cdot, \cdot) \) established in Theorem 3.1. In the calculation we use a truncated version of \( D(\cdot, \cdot) \) defined by
\[
  D_N(u, v) = f_0 \sum_{n=0}^{N} \sum_{m=-n}^{n} G_n^m(\sinh \mu_0) \hat{u}_{nm} \hat{v}_{nm}^*.
\] (3.14)
The matrix $A$ is approximated by $A^{(N)}$ with entries

$$A_{i,j}^{(N)} = f_0 \sum_{n=0}^{N} \sum_{m=-n}^{n} [\hat{\phi}(n)]^2 C_n^m (\sinh \mu_0) Y_{nm}(\omega(x_i)) Y_{nm}^*(\omega(x_j)).$$

We have to choose a sufficient large $N$ ($N = N_{\text{truncate}} = 100$ or $120$ are used in our numerical experiments) to guarantee the positive definiteness of the matrix $A^{(N)}$. This is termed a “variational” crime by Strang and Fix [14] and will be discussed later in the error analysis. The integral

$$\langle g, \Psi_j \rangle_{\Gamma_0} = f_0^2 \cosh \mu_0 \int_0^\pi \int_0^{2\pi} g(\theta, \varphi) \sqrt{\cosh^2 \mu_0 - \sin^2 \theta} \Psi_j(\theta, \varphi) \sin \theta \, d\varphi \, d\theta$$

can be evaluated by an appropriate cubature on the sphere $S^2$ (e.g. [2]), or by using the Fourier expansion of $g$ and $\Psi_j$.

Let $Y = \{y_1, \ldots, y_M\}$ be the image of $X$ under the map $\omega$, i.e. $y_j = \omega(x_j)$ for $j = 1, \ldots, M$. As $Y$ is a set of scattered points on $S^2$, we define the mesh norm $h_Y$ of $Y$ as usual,

$$h_Y = \sup_{y \in S^2} \min_{y_j \in Y} \cos^{-1}(y \cdot y_j).$$

We also define the separation radius of the set $Y$ by

$$q_Y = 0.5 \min_{y_j \neq y_i \in Y} \cos^{-1}(y \cdot y_j).$$

If $h_Y/q_Y \leq C$ for some universal constant $C$ we called $Y$ is a quasi-uniform set of points.

Similarly to [8, Theorem 3.2], we have the following approximation property for trial functions on the oblate spheroids. The proof carries over verbatim from Theorem 3.2 in [8], where one only has to substitute the map from the prolate to the sphere by the corresponding one of the oblate spheroid.

**Proposition 3.2.** Assume that the Fourier-Legendre coefficients of the kernel $\Phi$ satisfy

$$\hat{\psi}(n) \simeq (1 + n^2)^{-\tau}$$

holds for some $\tau > 1$. Let $V^\tau$ be defined as in (3.10). If $f \in H^s(\Gamma_0)$, then for $t \leq \tau$, $t \leq s \leq 2\tau$ there exists $\eta \in V^\tau$ so that

$$\|f - \eta\|_{H^t(\Gamma_0)} \leq ch_Y^{s-t} \|f\|_{H^s(\Gamma_0)},$$

where $c$ is a positive constant independent of $Y$. 

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In (3.15) and henceforth, \( a \simeq b \) means that there are two positive constants \( c_1 \) and \( c_2 \) so that \( c_1 a \leq b \leq c_2 a \). Using Proposition 3.2 we will derive an error estimate for the approximation of the solution \( u \) of (3.3) by the solution \( u_X \) of (3.11).

**Theorem 3.3.** Let \( V^* \) be defined by (3.10) with \( \Psi \) satisfying (3.15) where \( \tau > 1 \). Assume that the solution \( u \) to (3.3) belongs to \( H^s(\Gamma_0) \) for some \( s \) satisfying \( 1/2 < s < 2\tau - 1 \). Then there exists a positive constant \( C \) independent of the set \( X \) and an \( N_0 > 0 \) such that for all \( N \geq N_0 \) there holds

\[
\| u - u_X \|_{H^{1/2}(\Gamma_0)} \leq C(N^{s-1/2} + N^{-s+1/2})\| u \|_{H^s(\Gamma_0)}.
\]

**Proof.** By using Strang Lemma [14], we have the following estimate

\[
\| u - u_X \|_{H^{1/2}(\Gamma_0)} \leq \inf_{v \in V^*} \| u - v \|_{H^{1/2}(\Gamma_0)} + \max_{v \in V^*} \left| \frac{D_N(u, v) - D(u, v)}{D_N(v, v)} \right|^{1/2}. \tag{3.16}
\]

The first term on the right-hand side can be estimated using Theorem 3.1

\[
\inf_{v \in V^*} \| u - v \|_{H^{1/2}(\Gamma_0)} \leq ch^{s-1/2}\| u \|_{H^s(\Gamma_0)}. \tag{3.17}
\]

To estimate the second term on the right-hand side, firstly we notice that it is shown in [6, Lemma 3.1] that

\[
c(\mu_0)(n^2 + 1)^{1/2} < G_n^m(\sinh \mu_0) < C(\mu_0)(n^2 + 1)^{1/2}, \tag{3.18}
\]

and that \( G_n^m(\sinh \mu_0) = G_n^m(\sinh \mu_0) \) for \( 0 \leq m \leq n \) and \( n = 0, 1, 2, \ldots \).

By using the Cauchy-Schwarz inequality and (3.18) we have

\[
\left| D_N(u, v) - D(u, v) \right| = \left| f_0 \sum_{n=N+1}^{\infty} \sum_{m=-n}^{n} G_n^m(\sinh \mu_0) \hat{u}_{nm} \hat{v}_{nm}^* \right|
\leq C \left( \sum_{n=N+1}^{\infty} (n^2 + 1)^{1/2} |\hat{u}_{nm}|^2 \right)^{1/2} \left( \sum_{n=N+1}^{\infty} (n^2 + 1)^{1/2} |\hat{v}_{nm}|^2 \right)^{1/2}.
\]

It follows from

\[
\sum_{n=N+1}^{\infty} (n^2 + 1)^{1/2} |\hat{v}_{nm}|^2 \leq (1 + N^2)^{-s+1/2}\| v \|_{H^s(\Gamma_0)}^2 \quad \forall v \in H^s(\Gamma_0), \ s > 1/2,
\]
that
\[
|D_N(u, v) - D(u, v)| \leq C (1 + N^2)^{-s+1/2}\|u\|_{\mathcal{H}^s(\Gamma_0)}\|v\|_{\mathcal{H}^s(\Gamma_0)}. \tag{3.19}
\]

Using (3.18) again we have
\[
D_N(v, v) = f_0 \sum_{n=0}^{N} \sum_{m=-n}^{n} G_n^m (\sinh \mu_0) |\hat{v}_{nm}|^2 > c(\mu_0) \sum_{n=0}^{N} \sum_{m=-n}^{n} (n^2 + 1)^{1/2} |\hat{v}_{nm}|^2.
\]

Hence
\[
D_N(v, v) > c(\mu_0) \sum_{n=0}^{N} \sum_{m=-n}^{n} (n^2 + 1)^s(n^2 + 1)^{1/2-s} |\hat{v}_{nm}|^2
\]
\[
\geq c(N^2 + 1)^{1/2-s} \sum_{n=0}^{N} \sum_{|m| \leq n} (n^2 + 1)^s |\hat{v}_{nm}|^2 \quad \text{for } s > 1/2. \tag{3.20}
\]

Let
\[
R_N := \sum_{n=0}^{N} \sum_{|m| \leq n} (n^2 + 1)^s |\hat{v}_{nm}|^2.
\]

Since \( v \in V^\tau \) we write \( v \) as \( v = \sum_{i=1}^{M} \beta_i \Psi_i \). Hence \( \hat{v}_{nm} = \sum_{i=1}^{M} \beta_i (\hat{\Psi}_i)_{nm} \).

Note that
\[
(\hat{\Psi}_i)_{n,m} = |\hat{\phi}(n)| Y_{nm}(x_i). \tag{3.21}
\]

Therefore
\[
R_N = \sum_{n=0}^{N} (n^2 + 1)^s \sum_{i,j=1}^{M} \beta_i \beta_j |\hat{\phi}(n)|^2 \sum_{m=-n}^{n} Y_{nm}(x_i)Y_{nm}^*(x_j).
\]

By using the addition formula [9]
\[
\sum_{m=-n}^{n} Y_{nm}(y)Y_{nm}^*(z) = \frac{2n + 1}{4\pi} P_n(y \cdot z) \tag{3.22}
\]
and (3.15) we deduce
\[
R_N \simeq \sum_{n=0}^{N} (2n + 1)(n^2 + 1)^{s-2\tau} \sum_{i,j=1}^{M} \beta_i \beta_j P_n(x_i \cdot x_j). \tag{3.23}
\]
We need the following conjecture, which is strongly supported by our numerical experiments: for all $\epsilon > 1$, there exist $c > 0$ and $N_0 > 1$ such that for any set of quasi-uniform points $\{x_1, \ldots, x_M\}$, and any $\beta = (\beta_1, \cdots, \beta_M) \in \mathbb{R}^M$, if $N \geq N_0$ there holds

$$\sum_{n \geq N+1} n^{-\epsilon} \sum_{i,j=1}^M \beta_i \beta_j P_n(x_i \cdot x_j) \leq c \sum_{n=0}^N n^{-\epsilon} \sum_{i,j=1}^M \beta_i \beta_j P_n(x_i \cdot x_j). \quad (3.24)$$

The constant $c$ is independent of the set $X$ but $N_0$ may depend on the set $X$. The equivalent statement for (3.24) is that the matrix $cQ - R$ is positive-semidefinite, \( (3.25) \)

where $Q$ and $R$ are two $M \times M$ matrices with entries

$$Q_{i,j} = \sum_{n=0}^N n^{-\epsilon} P_n(x_i \cdot x_j), \quad R_{i,j} = \sum_{n \geq N+1} n^{-\epsilon} P_n(x_i \cdot x_j), \quad i, j = 1, \ldots, M.$$ 

We have carried out extensive numerical experiments to verify the conjecture. In particular, we computed the minimum eigenvalues of the matrix $100Q - R$ with $\epsilon = 1, 2, 3$ with up to 500 quasi-uniform points generated by Saff’s algorithm [12]. The results are summarized in Figures 1–3. As can be seen from the figures, the minimum eigenvalues of the matrix $100Q - R$ with $\epsilon = 1, 2, 3$ are always positive.

By using (3.24) with $\epsilon = -1 - 2(s - 2\tau) > 1$ we deduce that for $N \geq N_0$

$$R_N \geq c \sum_{n=N+1}^{\infty} (2n+1)(n^2+1)^{s-2\tau} \sum_{i,j=1}^M \beta_i \beta_j P_n(x_i \cdot x_j). \quad (3.26)$$

It is then clear that

$$R_N \geq c\|v\|^2_{H^s(\Gamma_0)}. \quad (3.27)$$

This together with (3.20) gives

$$D_N(v, v) = c(N^2 + 1)^{1/2-s}\|v\|^2_{H^s(\Gamma_0)}. \quad (3.28)$$

Combining all the estimates in (3.19) and (3.28), we obtain

$$\max_{u \in V^*} \frac{|D_N(u, v) - D(u, v)|}{[D_N(v, v)]^{1/2}} \leq cN^{-s+1/2}\|u\|_{H^s(\Gamma_0)}.$$ 

This together with (3.17) proves the assertion. \qed
4. Additive Schwarz preconditioners

The linear system (3.12) arising from the Galerkin approximation problem is usually ill-conditioned, especially when the separation radius of the set of scattered points is very small. In this situation, we propose a preconditioning algorithm based on the additive Schwarz operator, using a subspace decomposition of $V^\tau$ as

$$V^\tau = V_0 + \ldots + V_J$$

for some positive integer $J$. In the following, we will describe how to construct the subspaces $V_j$, for $j = 0, \ldots, J$.

Given a finite set of points $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$ on the oblate spheroid $\Gamma_0$, we define the set $Y$ on the sphere $S^2$ to be the image of $X$ via the map $\omega$ (cf. (3.6)), that is $Y = \{y_1 = \omega(x_1), \ldots, y_M = \omega(x_M)\}$.

We then decompose the set $Y$ into $J$ overlapping subsets $Y_j$ for $j = 0, \ldots, J$ by the following simple algorithm:

Select $\alpha \in (0, \pi/3)$, $\beta \in (0, \pi]$;
Let $p_1 = y_1 \in Y$;
$Y_0 := \{p_1\}$; 
$Y_1 := \{y \in Y : \cos^{-1}(y \cdot p_1) \leq \alpha\}$;
$J = 1, k = 1$;

while $Y_1 \cup \ldots \cup Y_k \neq Y$ do
\begin{align*}
k &= k + 1; \\
p_k &\text{ is chosen from } Y \setminus Y_0 \text{ such that } \cos^{-1}(p_{k-1} \cdot p_k) \geq \beta; \\
Y_0 &:= Y_0 \cup \{p_k\}; \\
Y_k &:= \{y \in Y : \cos^{-1}(y \cdot p_k) \leq \alpha\};
\end{align*}
end while

$J = k$

Roughly speaking, the algorithm selects from $Y$ a coarse set of points $Y_0 = \{p_1, \ldots, p_J\}$ across the whole sphere so that the geodesic distance between any two successive points is not less than $\beta$. Then each set $Y_k$ (for $k = 1, \ldots, J$) is a collection of points inside a spherical cap of radius $\alpha$ centered at $p_k$. We note that the subsets $Y_j$ may overlap. The subspaces $V_j$ can
now be defined by

\[ V_j = \text{span}\{\Psi_k = \Psi(y_k, \cdot) : y_k \in Y_j\}, \quad j = 0, \ldots, J. \]

Assume that functions in \( V_j \) have supports in \( \Omega_j \). We assume further that:

**Assumption 4.1.** We can partition the index set \( \{1, \ldots, J\} \) into \( \gamma \) (with \( 1 \leq \gamma \leq J \)) sets \( J_m \), for \( 1 \leq m \leq \gamma \), such that of \( i, j \in J_m \) and \( i \neq j \) then \( \Omega_i \cap \Omega_j = \emptyset \).

Let \( P_j : V^\tau \rightarrow V_j \), \( j = 0, \ldots, J \), be projections defined by

\[
D(P_j v, w) = D(v, w) \quad \forall v \in V^\tau, \quad \forall w \in V_j.
\]  

(4.1)

Then the additive Schwarz operator is defined by

\[
P := P_0 + \cdots + P_J.
\]  

(4.2)

The additive Schwarz method for equation (3.11) consists of solving, by an iterative method, the equation

\[
P \tilde{u} = g,
\]  

(4.3)
where the right-hand side is given by $g = \sum_{j=0}^{J} g_j$, with $g_j \in V_j$ being solutions of
\[ D(g_j, w) = \int_{\Gamma_0} f w ds, \quad \text{for any } w \in V_j. \]
Solving (4.3) is equivalent to solving (3.11), and in turn equivalent to solving
\[ P_j \tilde{u} = g_j, \quad j = 0, \ldots, J. \]

The operator $P$ can be considered as the preconditioned version of $K$, i.e. $P = BK$ for some preconditioner $B$. The preconditioning technique is in practice performed by computing the action of $B^{-1}$ on a residual $r \in V^\tau$. This consists of the solution of independent problems on each of the subspaces involved in the decomposition.

1. Correction on the global coarse set $Y_0$:
   
   Find $u_0 \in V_0$ satisfying $D(u_0, v) = \int_{\Gamma_0} rv ds \quad \forall v \in V_0$.

2. Corrections on the local sets $Y_j$, $j = 1, \ldots, J$:
   
   Find $u_j \in V_j$ satisfying $D(u_j, v) = \int_{\Gamma_0} rv ds \quad \forall v \in V_j$.

3. The residual $r$ in the conjugate gradient is preconditioned by:
\[ B^{-1} r := \sum_{j=0}^{J} u_j. \]

For the implementation details, see the pseudocode in [7, page 17].

By noting (3.1), the symbol of the operator $K$ defined in (2.6) behaves like $(n^2 + 1)^{1/2}$, see (3.18), i.e. $K$ is a pseudodifferential operator of order 1. This allows us to extend our analysis [15] for the overlapping Schwarz preconditioner on the sphere to the prolate spheroid.

**Lemma 4.2.** 1. Under Assumption 4.1, there exists a positive constant $c$ independent of the set $X$ such that for every $v \in V^\tau$ satisfying $v = \sum_{j=0}^{J} v_j$ with $v_j \in V_j$ for $j = 0, \ldots, J$ there holds
\[ D(v, v) \leq c \sum_{j=0}^{J} D(v_j, v_j). \]
2. For any \( u \in V^r \) there exist \( u_j \in V_j, \ j = 0, \ldots, J, \) satisfying \( u = \sum_{j=0}^J u_j \) and

\[
\sum_{j=0}^J D(u_j, u_j) \leq \left( 1 + \frac{J}{(1 - \|\tilde{Q}\|_D)^2} \right) D(u, u),
\]

where \( \tilde{Q} = Q_J \cdots Q_1. \) Here \( Q_i \) is the orthogonal projection from \( V^r \) onto \( V_i^\perp, \) the orthogonal complement of \( V_i \) with respect to the inner product induced by the bilinear form \( D(\cdot, \cdot). \)

**Proof.** By using the map \( \omega \) (cf. (3.6)) from the oblate spheroid to the sphere, for any function \( f \in H^{1/2}(\Gamma_0), \) if \( F = f(\omega^{-1}y) \) then \( F \) is a function in \( H^{1/2}(S^2). \) Using this, we can carry over the proof of [15, Lemma 4.4 and Lemma 4.8] for the functions defined on the sphere here. \( \Box \)

As a consequence of the previous lemma, we obtain the following bound for the condition number of the Schwarz operator.

**Theorem 4.3.** The condition number of the additive Schwarz operator \( P \) is bounded by

\[
\kappa(P) \leq c \gamma \left( 1 + \frac{J}{(1 - \|\tilde{Q}\|_D)^2} \right),
\]

where \( c \) is a constant independent of \( \gamma \) and the set \( X. \)

5. Implementation of the Galerkin method

In order to compute the entries \( A_{ij} \) of the stiffness matrix given in (3.12), we need to compute the spherical harmonics \( Y_{nm} \) and the functions \( G^m_n; \) see (3.13).

The spherical harmonics are computed by using the formula [1]

\[
Y_{nm}(\theta, \varphi) := \sqrt{\frac{2n + 1}{4\pi}} P_n^m(\cos \theta) e^{im\varphi}, \quad (5.1)
\]

where \( P_n^m(x) \) are associated Legendre functions of the first kind which can be computed using the following recurrence relations, for \( m = 0, 1, 2, \ldots, n \)
and $n = 0, 1, 2, \ldots$

\[ P_n^m(x) = \frac{\sqrt{(2n)!}}{2^n n!} (1 - x^2)^{n/2}, \]
\[ P_{n-1}^m(x) = 0, \]
\[ P_{n+1}^m(x) = v_n^m x P_n^m(x) + w_n^m P_{n-1}^m(x), \]

where

\[ v_n^m = \frac{(2n + 1)}{\sqrt{(n - m + 1)(n + m + 1)}} \quad \text{and} \quad w_n^m = \frac{\sqrt{(n - m)(n + m)}}{\sqrt{(n - m + 1)(n + m + 1)}}. \]

The functions $T_n^m$ given by

\[ T_n^m(x) = i \exp \left( \frac{i\pi n}{2} \right) Q_n^m(ix) \quad (5.2) \]

are calculated using the algorithm for oblate spheroidal harmonics presented in (see [4]).

In order to calculate $\frac{d}{dx} T_n^m$ we use (5.2). We have

\[ Q_{n+1}^m(ix) = -i \exp \left( -\frac{i\pi(n + 1)}{2} \right) T_{n+1}^m(x) \quad \text{and} \quad Q_n^m(ix) = -i \exp \left( -\frac{i\pi n}{2} \right) T_n^m(x). \]

Thus

\[ \frac{Q_{n+1}^m(ix)}{Q_n^m(ix)} = -i \frac{T_{n+1}^m(x)}{T_n^m(x)} \quad (5.3) \]

and with

\[ \frac{d}{dx} T_n^m(x) = - \exp \left( \frac{i\pi n}{2} \right) \frac{d}{dz} Q_n^m(ix) \quad (z = ix) \]

we finally obtain

\[ \frac{d}{dx} \frac{T_n^m(x)}{T_n^m(x)} = - \exp \left( \frac{i\pi n}{2} \right) \frac{d}{dz} \frac{Q_n^m(ix)}{Q_n^m(ix)} = \frac{d}{dz} \frac{Q_n^m(ix)}{Q_n^m(ix)}. \quad (5.4) \]

Furthermore, it is known that (see [6])

\[ -i \frac{d}{dz} \frac{Q_n^m(z)}{Q_n^m(z)} (1 - z^2) = (n + 1) (-iz) + (n - m + 1) i \frac{Q_n^m(z)}{Q_n^m(z)}. \quad (5.5) \]
and comparing (5.4) and (5.5) (with $z = ix \implies (1 - z^2) = (1 + x^2)$) we have

$$-(1 + x^2) \frac{d}{dx} T_n^m(x) \begin{array}{ll} = & -i \frac{d}{dz} Q_n^m(z) (1 - z^2) \\ = & (n + 1)(-iz) + (n - m + 1) i Q_{n+1}^m(z) \end{array}.$$  

Therefore the recurrence relation (with $iz = x$ and (5.3)) for $\frac{d}{dx} T_n^m$ can be expressed as

$$-(1 + x^2) \frac{d}{dx} T_n^m(x) = (n + 1)x + (n - m + 1) \frac{T_{n+1}^m(x)}{T_n^m(x)}.$$

With this relation the term $G_n^m(\sinh \mu_0)$ in the entry $A_{ij}$ of the stiffness matrix (see 3.13) and (3.1) can be computed by using the relation

$$G_n^m(x) = (n + 1)x + (n - m + 1) \frac{T_{n+1}^m(x)}{T_n^m(x)}, \quad m = 0, 1, \ldots, n; n = 0, 1, 2, \ldots$$

For negative values of $m$ we use the relation $G_n^m(x) = G_n^{-m}(x)$.

The right hand side terms $g_j$ are computed by using the Fourier coefficients of $g$ and $\Phi_j$ and Parseval’s identity.

In our experiments, we use the kernel $\Psi(x, x') = \Phi(\omega(x), \omega(x'))$ where for arbitrary $y, z \in S^2$,

$$\Phi(y, z) = \rho_m(\sqrt{2 - 2y \cdot z})$$

with $\rho_m$ being locally supported radial basis functions defined by Wendland [16]; see Table 1. It is shown by Narcowich and Ward [10] that in this case (3.15) holds for $\tau = m + 3/2$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\rho_m(r)$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(1 - r)^2$</td>
<td>1.5</td>
</tr>
<tr>
<td>1</td>
<td>$(1 - r)^{4/3}(4r + 1)$</td>
<td>2.5</td>
</tr>
<tr>
<td>2</td>
<td>$(1 - r)^6(35r^2 + 18r + 3)$</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Table 1: Wendland’s RBFs
6. Numerical experiments

In our experiments we choose the oblate spheroid $\Gamma_0$ such that $f_0 = 4$ and $\mu_0 = 1$. Let the Neumann condition be

$$g(\mu, \theta, \varphi) = -\frac{\sinh \mu \sin \theta \cos \varphi (2 \cosh^2 \mu + \cos^2 \theta)}{f_0^2 (\cosh^2 \mu - \cos^2 \theta)^{\frac{3}{2}}}$$

so that the exact solution to (2.2) is

$$U(\mu, \theta, \varphi) = \frac{\cosh \mu \sin \theta \cos \varphi}{f_0^2 (\cosh^2 \mu - \cos^2 \theta)^{\frac{3}{2}}}.$$  \hspace{1cm} (6.1)

This example is taken from [6] where the authors solve (3.3) using piecewise bilinear functions on grids of $\Lambda = \{(\theta, \varphi) : 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$.

We solved (3.11) in the space $V^\tau$ defined in (3.10) with three different types of sets of points $X$.

**Points of type 1.** As in [6], we divide the intervals $[0, \pi]$ and $[0, 2\pi]$ into $N_1$ and $N_2$ subintervals, respectively, by

$$\theta_s = s\pi/N_1, \quad s = 0, 1, 2, ..., N_1$$

and

$$\varphi_t = 2t\pi/N_2, \quad t = 0, 1, 2, ..., N_2.$$ 

Then we use $M := N_2(N_1 - 1)$ points on $\Gamma_0$,

$$x_{(N_2-1)s+t} = (f_0 \cosh \mu_0 \sin \theta_s \cos \varphi_t, f_0 \cosh \mu_0 \sin \theta_s \sin \varphi_t, f_0 \sinh \mu_0 \cos \theta_s),$$

where $1 \leq s \leq N_1 - 1$ and $1 \leq t \leq N_2$ to construct the basis functions.

**Points of type 2.** Next, we generate sets of points $X$ on $\Gamma_0$ as images under the mapping $\omega$, see (3.6), of sets of points $Y = \{y_1, ..., y_M\}$ on $S^2$ which are defined by using Saff’s algorithm [12]. This algorithm partitions $S^2$ into $M$ equal-area regions whose centre are $y_1, ..., y_M$; see Figure 4.

**Points of type 3.** Finally, we use sets of scattered points on the oblate spheroid which are obtained by mapping to the oblate spheroid the geocentric coordinates of data points taken from MAGSAT satellite data; see Figure 5. These sets are extracted from a full data set of about 26 million points in such a way that the separation radius of each set $q_Y = \frac{1}{2} \min_{y \neq y'} (y \cdot y')$ is not too small; see Table 6.
We solved the matrix equation (3.12) by the conjugate gradient method with relative tolerance $10^{-10}$, i.e. the stopping criteria is

$$\frac{\|Ae^{(m)} - g\|_2}{\|g\|_2} \leq 10^{-10}. \quad (6.2)$$

Here $e^{(m)}$ is the $m$th iterate.

Let $e := u - u_X$ where $u(\theta, \varphi) = U(\mu_0, \theta, \varphi)$, see (6.1), is the solution to (3.3) and $u_X$ is the solution to (3.11). We compute $\|e\|_{L^2(\Gamma_0)}$ and $\|e\|_{H^{1/2}(\Gamma_0)}$ approximately by

$$\|e\|_{L^2(\Gamma_0)} \approx \left( \sum_{n=0}^{120} \sum_{m=-n}^{n} |(u_X)_{nm} - \hat{u}_{nm}|^2 \right)^{1/2}$$

and

$$\|e\|_{H^{1/2}(\Gamma_0)} \approx \left( \sum_{n=0}^{120} \sum_{m=-n}^{n} \left(1 + n^2\right)^{1/2} |(u_X)_{nm} - \hat{u}_{nm}|^2 \right)^{1/2}$$

in which

$$\hat{u}_{nm} = \sum_{i=1}^{M} \hat{\phi}(n)c_i Y_{nm}(\omega(x_i))$$

and $\hat{u}_{nm}$ is computed by using a quadrature [2] for formula (2.4).

We also compute $l_2$ and $l_\infty$ errors for point sets of type 1. Let $\mathcal{G}$ be points of the grid of size $(N_1, N_2) = (160, 320)$, then

$$\|e\|_{l_\infty(\Gamma_0)} = \max_{y \in \mathcal{G}} |u_X(y) - u(y)|$$
Figure 5: Image of satellite points on the oblate spheroid

and

$$\|e\|_{L^2(\Gamma_0)} = \left( \frac{1}{|\mathcal{G}|} \sum_{y \in \mathcal{G}} |u_X(y) - u(y)|^2 \right)^{1/2}$$

where $|\mathcal{G}| = 50880$ is cardinality of $\mathcal{G}$ and $|\mathcal{G}| := N_2(N_1 - 1)$.

Table 2 shows the experimental orders of convergence (EOC) for the errors in the $H^{1/2}(\Gamma_0)$-norm (energy norm) when Saff points (points of type 2) are used.

The numbers in the Tables 3, 4 and 5 show fast convergence of our RBF Galerkin method applied to the Dirichlet-to-Neumann map with different values of $N$ on different grids of size $(N_1, N_2)$.

Table 6 gives the errors in the $L^2(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$ norms for scattered points (points of type 3). In this case, the matrix is ill conditioned and hence a preconditioner is required. Table 7 shows the corresponding numbers of iteration of the preconditioned conjugate gradient using the same stopping criteria as before, i.e. with the relative tolerance $\leq 10^{-10}$. Errors of the same order as in the non-preconditioned case are obtained. The advantage of the preconditioner can be observed.
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\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
$m$ & $M$ & $h_X$ & $\mathcal{H}^{1/2}\text{err}$ & EOC \\
\hline
0 & 100 & 0.2672 & 3.1112E-03 & 2.70 \\
 & 200 & 0.1942 & 1.3158E-03 & 2.38 \\
 & 500 & 0.1237 & 3.9392E-04 & 2.60 \\
 & 1000 & 0.0849 & 1.6113E-04 & 2.67 \\
 & 2000 & 0.0609 & 6.7877E-05 & 2.70 \\
 & 4000 & 0.0426 & 2.4877E-05 & 2.81 \\
\hline
1 & 100 & 0.2672 & 1.0988E-03 & \\
 & 200 & 0.1942 & 1.9053E-04 & 5.49 \\
 & 500 & 0.1237 & 2.4204E-05 & 4.57 \\
 & 1000 & 0.0849 & 4.7746E-06 & 4.31 \\
 & 2000 & 0.0609 & 9.9207E-07 & 4.73 \\
 & 4000 & 0.0426 & 1.9228E-07 & 4.59 \\
\hline
2 & 100 & 0.2672 & 1.1860E-03 & \\
 & 200 & 0.1942 & 8.0785E-05 & 8.42 \\
 & 500 & 0.1237 & 3.6622E-06 & 6.86 \\
 & 1000 & 0.0849 & 3.3840E-07 & 6.33 \\
 & 2000 & 0.0609 & 3.6577E-08 & 6.70 \\
 & 4000 & 0.0426 & 3.5598E-09 & 6.52 \\
\hline
\end{tabular}
\caption{Errors with Saff points and $\rho_m(r)$}
\end{table}
<table>
<thead>
<tr>
<th>$m$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$|e|_{L^2(\Gamma_0)}$</th>
<th>$|e|_{H^{1/2}(\Gamma_0)}$</th>
<th>$|e|_{H^1(\Gamma_0)}$</th>
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<td>0.1028E-03</td>
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<td>0.4083E-08</td>
</tr>
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Table 3: Errors on grid points, using conjugate gradient method for $N_{\text{truncate}} = 80$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$|e|_{L^2(\Gamma_0)}$</th>
<th>$|e|_{H^{1/2}(\Gamma_0)}$</th>
<th>$|e|_{H^1(\Gamma_0)}$</th>
<th>$|e|_{L^\infty(\Gamma_0)}$</th>
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<td>0.3248E-03</td>
<td>0.1343E-04</td>
<td>0.9091E-04</td>
</tr>
<tr>
<td>0</td>
<td>40</td>
<td>80</td>
<td>0.5814E-05</td>
<td>0.5303E-04</td>
<td>0.1888E-05</td>
<td>0.1116E-04</td>
</tr>
<tr>
<td>0</td>
<td>80</td>
<td>160</td>
<td>0.2402E-08</td>
<td>0.2470E-07</td>
<td>0.4065E-06</td>
<td>0.1425E-05</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>20</td>
<td>0.1033E-03</td>
<td>0.4509E-03</td>
<td>0.2828E-04</td>
<td>0.1099E-03</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>40</td>
<td>0.2922E-05</td>
<td>0.1848E-04</td>
<td>0.7933E-06</td>
<td>0.3357E-05</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
<td>80</td>
<td>0.9079E-07</td>
<td>0.8163E-06</td>
<td>0.2402E-07</td>
<td>0.1068E-06</td>
</tr>
<tr>
<td>1</td>
<td>80</td>
<td>160</td>
<td>0.8770E-09</td>
<td>0.5410E-08</td>
<td>0.1235E-08</td>
<td>0.4930E-08</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>20</td>
<td>0.6982E-04</td>
<td>0.2818E-03</td>
<td>0.2356E-04</td>
<td>0.6829E-04</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>40</td>
<td>0.4338E-06</td>
<td>0.2695E-05</td>
<td>0.1310E-06</td>
<td>0.4398E-06</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>80</td>
<td>0.3392E-08</td>
<td>0.2924E-07</td>
<td>0.1089E-08</td>
<td>0.4647E-08</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>160</td>
<td>0.1155E-08</td>
<td>0.6304E-08</td>
<td>0.6819E-09</td>
<td>0.4083E-08</td>
</tr>
</tbody>
</table>

Table 4: Errors on grid points, using conjugate gradient method for $N_{\text{truncate}} = 100$
\[ \parallel e \parallel_{L^2(\Gamma_0)} \parallel e \parallel_{H^{1/2}(\Gamma_0)} \parallel e \parallel_{L^2(\Gamma_0)} \parallel e \parallel_{H^{1/2}(\Gamma_0)} \]

## Table 5: Errors on grid points, using conjugate gradient method for \( N_{\text{truncate}} = 120 \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( N_1 )</th>
<th>( N_2 )</th>
<th>( \parallel e \parallel_{L^2(\Gamma_0)} )</th>
<th>( \parallel e \parallel_{H^{1/2}(\Gamma_0)} )</th>
<th>( \parallel e \parallel_{L^2(\Gamma_0)} )</th>
<th>( \parallel e \parallel_{H^{1/2}(\Gamma_0)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>20</td>
<td>0.3854E-03</td>
<td>0.1779E-02</td>
<td>0.1028E-03</td>
<td>0.6957E-03</td>
</tr>
<tr>
<td>0</td>
<td>20</td>
<td>40</td>
<td>0.4949E-04</td>
<td>0.3248E-03</td>
<td>0.1343E-04</td>
<td>0.9093E-04</td>
</tr>
<tr>
<td>0</td>
<td>40</td>
<td>80</td>
<td>0.5814E-05</td>
<td>0.5303E-04</td>
<td>0.1889E-05</td>
<td>0.1114E-04</td>
</tr>
<tr>
<td>0</td>
<td>80</td>
<td>160</td>
<td>0.2391E-08</td>
<td>0.2437E-07</td>
<td>0.4065E-06</td>
<td>0.1425E-05</td>
</tr>
<tr>
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<td>20</td>
<td>0.1033E-03</td>
<td>0.4509E-03</td>
<td>0.2828E-04</td>
<td>0.1099E-03</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>40</td>
<td>0.2922E-05</td>
<td>0.1848E-04</td>
<td>0.7933E-06</td>
<td>0.3357E-05</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
<td>80</td>
<td>0.9079E-07</td>
<td>0.8163E-06</td>
<td>0.2402E-07</td>
<td>0.1068E-06</td>
</tr>
<tr>
<td>1</td>
<td>80</td>
<td>160</td>
<td>0.8743E-09</td>
<td>0.5401E-08</td>
<td>0.1234E-08</td>
<td>0.4908E-08</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>20</td>
<td>0.6982E-04</td>
<td>0.2818E-03</td>
<td>0.2356E-04</td>
<td>0.6829E-04</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>40</td>
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<td>0.2695E-05</td>
<td>0.1310E-06</td>
<td>0.4397E-06</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>80</td>
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<td>0.2924E-07</td>
<td>0.1090E-08</td>
<td>0.4640E-08</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>160</td>
<td>0.1155E-08</td>
<td>0.6304E-08</td>
<td>0.6818E-09</td>
<td>0.4100E-08</td>
</tr>
</tbody>
</table>

## Table 6: Computational time in seconds (CPU), number of conjugate gradient iterations (ITER) and errors for scattered points from MAGSAT for \( N_{\text{truncate}} = 100 \) and \( \rho_0(r) \)

<table>
<thead>
<tr>
<th>( M )</th>
<th>( q_Y )</th>
<th>( \parallel e \parallel_{L^2(\Gamma_0)} )</th>
<th>( \parallel e \parallel_{H^{1/2}(\Gamma_0)} )</th>
<th>ITER</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>3470</td>
<td>( \pi/140 )</td>
<td>6.25503E-006</td>
<td>5.01114E-005</td>
<td>2809</td>
<td>234.6</td>
</tr>
<tr>
<td>7763</td>
<td>( \pi/200 )</td>
<td>2.41695E-006</td>
<td>2.13257E-005</td>
<td>27064</td>
<td>13323.7</td>
</tr>
<tr>
<td>10443</td>
<td>( \pi/240 )</td>
<td>1.87142E-006</td>
<td>1.62031E-005</td>
<td>30931</td>
<td>17361.9</td>
</tr>
</tbody>
</table>

## Table 7: Computational time in seconds (CPU), number of iterations (ITER) for overlapping additive Schwarz preconditioner and errors for scattered points from MAGSAT for \( N_{\text{truncate}} = 100 \) and \( \rho_0(r) \)

<table>
<thead>
<tr>
<th>( M )</th>
<th>( \cos \alpha )</th>
<th>( \cos \beta )</th>
<th>( \parallel e \parallel_{L^2(\Gamma_0)} )</th>
<th>( \parallel e \parallel_{H^{1/2}(\Gamma_0)} )</th>
<th>ITER</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>3470</td>
<td>0.9</td>
<td>-0.15500000</td>
<td>6.24823E-006</td>
<td>5.02580E-005</td>
<td>73</td>
<td>4.3</td>
</tr>
<tr>
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<td>0.9999999996</td>
<td>2.41695E-006</td>
<td>2.13257E-005</td>
<td>939</td>
<td>294.1</td>
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<tr>
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<td>0.98</td>
<td>0.999999996</td>
<td>1.87142E-006</td>
<td>1.62031E-005</td>
<td>1602</td>
<td>1085.8</td>
</tr>
</tbody>
</table>

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Figure 6: Log-log plot for $H^{1/2}(\Gamma_0)$ errors using Wendland RBFs $\rho_m(r)$

References


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