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Chapter 6

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“Why,” said the Dodo, “the best way to explain it is to do it.”
- Lewis Carroll, Alice in Wonderland.

Questions marked with [R] are routine, [H] harder and [M] MATLAB. You should make sure that you can do the easier questions before you tackle the more difficult questions.

Problems 6.1

1. [R] Solve (if possible) the following equations for $x \in \mathbb{N}$, $x \in \mathbb{Z}$, $x \in \mathbb{Q}$ and $x \in \mathbb{R}$.
   a) $x + 25 = 0$, $3x - 9 = 0$, $3x + 9 = 0$, $3x + 10 = 0$.
   b) $x^2 + 4x - 5 = 0$, $2x^2 - 13x + 15 = 0$, $x^2 - x - 1 = 0$, $x^2 + 3x + 4 = 0$.
   c) $\sin(\pi x/3) = 0$, $\sin(x/3) = 0$.

2. [R] Is the set \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} closed under addition? Prove your answer.

3. [H] Can any finite set of integers be closed under addition? Prove your answer.

4. [R] Is the set \{-1, 1\} closed under multiplication and division?

Problems 6.3

5. [R] Let $z = 2 + 3i$, $w = -1 + 2i$. Calculate $3z$, $z^2$, $z + 2w$, $z(w + 3)$, $\frac{z}{w}$, $\frac{w}{z}$.

6. [R] Write the following expressions in $a + ib$ or “Cartesian” form:
   a) $\frac{1 + i}{1 + 2i}$, b) $\frac{2 - i}{3 + i} - \frac{3 - i}{2 + i}$.

7. [R] If $z = a + ib$, express the following in “Cartesian” form:
   a) $z^2$, b) $\frac{1}{z}$, c) $\frac{z + 1}{z - 1}$.

8. [R] Use the quadratic formula to find all complex roots of the following polynomials.
   a) $z^2 + z + 1$, b) $z^2 + 2z + 3$, c) $z^2 - 6z + 10$,
   d) $-2z^2 + 6z - 3$, e) $z^4 + 5z^2 + 4$.

9. [H] Show that \([(\sqrt[3]{3} + 1) + (\sqrt[3]{3} - 1)i]^3 = 16(1 + i)$.

10. [R] Simplify $(\sqrt[3]{3} + 4i + \sqrt[3]{3} - 4i)^2$ (where we assume $\sqrt[3]{z}$ has non negative real part).

11. [H] Simplify $\left(\frac{a + bi}{a - bi}\right)^2 - \left(\frac{a - bi}{a + bi}\right)^2$ where $a$ and $b$ are real numbers not both zero.
Problems 6.4

12. [R] Find $\text{Re}(z)$, $\text{Im}(z)$ and $\bar{z}$ for $z = -1 + i$, $2 + 3i$, $2 - 3i$, $\frac{2 - i}{1 + i}$, $\frac{1}{(1 + i)^2}$.

13. [R] Let $z = 1 + 2i$ and $w = 3 - 4i$. Calculate $z^2$ and $\frac{\bar{z}}{w}$, expressing the answers in Cartesian form.

14. [R] Given that $2z + 3w = 1 + 12i$ and $z - w = 3 - i$, find $z$ and $w$.

15. [R] By evaluating each side of the equations, check that $\bar{zw} = z \bar{w}$, and $\left(\frac{z}{w}\right) = \frac{\bar{z}}{\bar{w}}$ are satisfied by the complex numbers $z = 2 + 3i$, $w = -1 + 2i$.

16. [R] Prove that for any two complex numbers $z$ and $w$

   a) $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$
   b) $2\text{Re}(z) = z + \bar{z}$
   c) $(z - w) = \bar{z} - \bar{w}$
   d) $\left(\frac{1}{z}\right) = \frac{1}{\bar{z}}$
   e) $\bar{zw} = z \bar{w}$
   f) $\left(\frac{z}{w}\right) = \frac{\bar{z}}{\bar{w}}$.

17. [H] a) Use the properties of the complex conjugate to show that if the complex number $\alpha$ is a root of a quadratic equation $ax^2 + bx + c = 0$ with $a$, $b$, $c$ being real coefficients, then so is $\bar{\alpha}$.

   b) Write down the monic quadratic polynomial with real coefficients which has $3 - 2i$ as one of its roots.

   c) Does the result of a) generalise to higher degree polynomials?

Problems 6.5, 6.6

18. [R] Find the modulus, argument and polar form of each of the following numbers and plot them on an Argand diagram:

   a) $6 + 6i$,  b) $-4$,  c) $\sqrt{3} - i$,  d) $\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$,  e) $-7 + 3i$.

19. [R] If $z = 4 + 3i$ and $w = 2 + i$ find $|3z - 3iw|$, $\text{Im}\left((1 - i)z - 3|w|\right)$.

20. [H] If $z = 1 + i$, calculate the powers $z^j$ for $j = 1, 2, \ldots, 10$ and plot them on an Argand diagram. Is there a pattern? What is the smallest positive integer $n$ such that $z^n$ is a real number?

21. [R] Find the “$a + ib$” form of the complex numbers whose modulus and argument are
$\text{CHAPTER 6. COMPLEX NUMBERS}$

23. a) $|z| = 3$, Arg($z$) = $\frac{\pi}{3}$; b) $|z| = 3$, Arg($z$) = $\frac{5\pi}{6}$; c) $|z| = 3$, Arg($z$) = $-\frac{2\pi}{3}$; d) $|z| = 3$, Arg($z$) = $-\frac{\pi}{6}$; e) $|z| = 3$, Arg($z$) = $\frac{\pi}{8}$.

22. [R] a) Show that $z\overline{z} = |z|^2$. Hence, or otherwise, show that if $|z| = 1$, then $\overline{z} = z^{-1}$. b) Show that $|z| = |\overline{z}|$ for all $z \in \mathbb{C}$. c) If $z = r(\cos \theta + i \sin \theta)$, show that a polar form for the complex conjugate is $\overline{z} = r(\cos(-\theta) + i \sin(-\theta))$.

23. [H] Show that Re $\left( \frac{1 - z}{1 + z} \right) = 0$ for any complex $z$ with $|z| = 1$.

24. [H] Use $z\overline{z} = |z|^2$ to prove the identity $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

25. [H] Use $z\overline{z} = |z|^2$ to show that

$$|1 - \overline{zw}|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2)$$

and deduce that $|1 - \overline{zw}|^2 = |z - w|^2$ if either $z$ or $w$ lies on the unit circle.

**Problems 6.7**

26. [R] Plot the following complex numbers on an Argand diagram:

a) $2e^{\frac{i\pi}{4}}$, b) $3e^{\frac{i\pi}{6}}$, c) $e^{-\frac{2i\pi}{3}}$, d) $2e^{-\frac{i\pi}{3}}$, e) $4e^{i\pi}$.

27. [R] Let $z = (1 - i)$ and $w = 2e^{i\pi/3}$. Calculate $w^6$, $z - w$ and $\frac{w}{\overline{z}}$ and express your answers in Cartesian form.

28. [R] For $z = 3e^{-5\pi i/6}$ and $w = 1 + i$, find Re $(iw + \overline{z}^2)$.

29. [R] Solve $|e^{i\theta} - 1| = 2$ for $-\pi < \theta \leq \pi$.

30. [R] Find Arg($-1 + i$) and Arg($-\sqrt{3} + i$) and hence find the argument of $(-1 + i)(-\sqrt{3} + i)$ and the argument of $\frac{-1 + i}{-\sqrt{3} + i}$. (Make sure you get the argument in the interval $(-\pi, \pi]$.)

31. [H] Let $z = (1 + \sqrt{3}i)$ and $w = (1 + i)$. Find Arg$z$ and Arg$w$ and hence Arg$(zw)$. Evaluate $zw$ and hence show that $\cos \left( \frac{7\pi}{12} \right) = \frac{1 - \sqrt{3}}{2\sqrt{2}}$. Find a similar expression for $\sin \left( \frac{7\pi}{12} \right)$.

32. [R] Find polar forms for $z = 1 + i\sqrt{3}$ and $w = 1 - i$, and hence find first the polar forms and then the “$a + ib$” forms of $zw$, $z^9$, and $\left( \frac{z}{w} \right)^{12}$.
PROBLEMS FOR CHAPTER 6

33. [R] Find the polar, and hence also the Cartesian form for:
   a) \((\sqrt{3} + i)^5\)
   b) \(\left(\frac{-1 + i}{\sqrt{2}}\right)^{1002}\)
   c) \(\left(\frac{1 + \sqrt{3}i}{2}\right)^{-8}\).

34. [H] Find the square roots (in Cartesian Form) of
   a) \(21 - 20i\)
   b) \(-16 + 30i\)
   c) \(24 + 70i\).

35. [H] a) Explain why multiplying a complex number \(z\) by \(e^{i\theta}\) rotates the point represented by \(z\) anticlockwise about the origin, through an angle \(\theta\).
   b) The point represented by the complex number \(1 + i\) is rotated anticlockwise about the origin through an angle of \(\frac{\pi}{6}\). Find its image in polar and Cartesian form.
   c) Find the complex number (in Cartesian form) obtained by rotating \(6 - 7i\) anticlockwise about the origin through an angle \(\frac{3\pi}{4}\).

36. [H] If \(z = re^{i\theta}, 0 \leq \theta \leq \frac{\pi}{2}\) show that
   a) \(|(1 - i)z^2| = \sqrt{2}r^2\)
   b) \(\text{Arg}((1 - i)z^2) = 2\theta - \frac{\pi}{4}\)
   c) \(\left|\frac{1 + i\sqrt{3}}{z}\right| = \frac{2}{r}\)
   d) \(\text{Arg}\left(\frac{1 + i\sqrt{3}}{z}\right) = \frac{\pi}{3} - \theta\).

37. [R] Find the roots (in Cartesian Form) of
   a) \(z^2 - 3z + (3 - i) = 0\)
   b) \(z^2 - (7 - i)z + (14 - 5i) = 0\)
   c) \(z^2 + (4 - i)z + (1 + 13i) = 0\).

38. [R] Find the seventh roots of \(-1\) and plot the roots on an Argand diagram.

39. [R] Find the sixth roots of \(i\) and plot the roots on an Argand diagram.

40. [R] Find the fifth roots of \(16 - 16i\sqrt{3}\) and plot the roots on an Argand diagram.

41. [H] Find all \(z \in \mathbb{C}\) satisfying \((z - 6 + i)^3 = -27\).

42. [H] Show that if \(\omega\) is an \(n\)th root of unity (\(\omega \neq 1\) and \(n > 1\)) then
   \[\omega + \omega^2 + \cdots + \omega^n = 0\]
   Hint: Sum the geometric progression.

43. [H] Show that the set \(\{z \in \mathbb{C} : |z| \leq 1\}\) is closed under multiplication. Is the set closed under division (zero excluded)? Is the set closed under addition or subtraction?

44. [H] Use the properties of complex conjugates to show that if \(a, b \in \mathbb{R}\) and \(|z| = 1\), then
   \[|a + bz| = |az + b|\]
   Hint: You might find the results of Question 22 useful.
45. \[H\] Suppose \(a\) and \(b\) are real numbers (not both zero) and \(w = \frac{az + bz^{-1}}{bz + az^{-1}}\). Show that if \(|z| = 1\), then \(|w| = 1\).

Hint: You might find the results of Question 22 useful.

46. \[H\] Let \(z, w\) be complex numbers.

a) Using polar forms, show that \(|\text{Re}(zw)| \leq |z||w|\).

b) Use the result in (a) to show that \(|z + w| \leq |z| + |w|\), and interpret the result geometrically. Hint: Write \(|z + w|^2 = (z + w)(\overline{z} + \overline{w})\) and expand.

47. \[H\] Let \(z = \frac{i(1 + is)}{1 - is}\) where \(s \in \mathbb{R}\).

a) Show that \(\text{Arg}(z) = \begin{cases} \frac{\pi}{2} + 2\tan^{-1}s \quad &\text{for } s \leq 1 \\ -\frac{3\pi}{2} + 2\tan^{-1}s \quad &\text{for } s > 1. \end{cases}\)

b) Describe geometrically what happens to \(z\) as \(s\) increases from \(-\infty\) to \(\infty\).

48. \[H\] Suppose \(\theta, \phi \neq \frac{\pi}{2}(2k + 1)\) where \(k\) is an integer. Use the fact that \(z = \frac{1 + z}{1 + z^{-1}}\)

a) to find the real and imaginary parts of \(\frac{1 + \cos 2\theta + i \sin 2\theta}{1 + \cos 2\theta - i \sin 2\theta}\).

b) to show that if \(n\) is a positive integer then \(\left(\frac{1 + \sin \phi + i \cos \phi}{1 + \sin \phi - i \cos \phi}\right)^n = \cos n\left(\frac{\pi}{2} - \phi\right) + i \sin n\left(\frac{\pi}{2} - \phi\right)\).

49. \[H\] For \(n > 1\), let \(\omega_1, \omega_2, ..., \omega_n\) be the \(n\) distinct \(n\)th roots of 1 and let \(A_k\) be the point on the Argand diagram which represents \(\omega_k\). Let \(P\) represent any point \(z\) on the unit circle, and let \(PA_k\) denote the distance from \(P\) to \(A_k\).

a) Prove that \((PA_k)^2 = (z - \omega_k)(\overline{z} - \overline{\omega_k})\).

b) Deduce that \(\sum_{k=1}^{n} (PA_k)^2 = 2n\).

c) Now let \(P\) represent the point \(x\) on the real axis, \(-1 < x < 1\), prove that \(\prod_{k=1}^{n} PA_k = 1 - x^n\).
Problems 6.8

50. [R] Using De Moivre’s theorem and the binomial theorem, prove the identity
\[ \cos 3\theta = 4\cos^3 \theta - 3\cos \theta. \]

51. [R] 
   a) Use De Moivre’s Theorem to express \( \cos 6\theta \) and \( \sin 6\theta \) in terms \( \cos \theta \) and \( \sin \theta \).
   b) Write \( \cos 6\theta \) in terms of \( \cos \theta \) only.

52. [R] Express \( \cos 7\theta \) and \( \sin 7\theta \) in terms of powers of \( \cos \theta \) and \( \sin \theta \).

53. [R] 
   a) Derive a formula for \( \cos \theta \) in terms of \( e^{i\theta} \) and \( e^{-i\theta} \).
   b) Deduce a formula for \( \cos^6 \theta \) in terms of \( \cos k\theta \), \( 1 \leq k \leq 6 \).
   c) Show that \( \int_0^{\frac{\pi}{2}} \cos^6 \theta \, d\theta = \frac{5\pi}{32} \).

54. [R] Express \( \sin^5 \theta \) and \( \cos^4 \theta \) in terms of sines or cosines of multiples of \( \theta \), and hence find their integrals.

55. [H] 
   a) Use De Moivre’s Theorem to express \( \cos 5\theta \) as a polynomial \( p(x) \) in \( x = \cos \theta \).
   b) Put \( \theta = 36^\circ = \frac{\pi}{5} \) and show that \( x = \cos \frac{\pi}{5} \) is a root of \( P(x) = 16x^5 - 20x^3 + 5x + 1 \).
   c) Check that \( P(x) = (x + 1)(4x^2 - 2x - 1)^2 \).
   d) What are the 5 roots of \( P(x) \)? Give full reasons for your answer.
   e) Deduce that \( \cos \frac{\pi}{5} + \cos \frac{3\pi}{5} + \cos \frac{7\pi}{5} + \cos \frac{9\pi}{5} = 1 \) and \( \cos \frac{\pi}{5} \cos \frac{3\pi}{5} \cos \frac{7\pi}{5} \cos \frac{9\pi}{5} = \frac{1}{16} \).

56. [H] Let \( \omega_1, \omega_2, \ldots, \omega_n \) be the \( n \) distinct \( n \)th roots of unity \( (n \geq 1) \). Show that if \( k \) is an integer then
\[ \omega_1^k + \omega_2^k + \cdots + \omega_n^k \]
equals 0 or \( n \). Find the values of \( k \) for which the sum is \( n \).
Hint: Write the roots in polar form and sum the resulting geometric progression. See Example ?? of Section 6.8.

57. [H] Show that if \( \theta \) is not a multiple of \( 2\pi \), then the imaginary part of
\[ \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \]
is equal to \( \frac{\sin \left(\frac{1}{2}(n+1)\theta\right)}{\sin \frac{1}{2}\theta} \).
Hint. See Example ?? of Section 6.8.

58. [H] Find the sum of
\[ \sin \theta + \sin 2\theta + \cdots + \sin n\theta. \]
Hint. See previous exercise.
59. [H] a) Calculate the sum of the series
\[ S = e^{i\theta} - \frac{e^{3i\theta}}{3^2} + \frac{e^{5i\theta}}{3^4} - \frac{e^{7i\theta}}{3^6} + \cdots \]

b) Hence show that
\[ \sin \theta - \frac{\sin 3\theta}{3^2} + \frac{\sin 5\theta}{3^4} - \frac{\sin 7\theta}{3^6} + \cdots = \frac{72\sin \theta}{82 + 18\cos 2\theta} \]

Problems 6.9

60. [R] Sketch the set of points on the complex plane corresponding to each of the following:

a) \(|z - i| \leq 2\),

b) \(|z - i| \leq 2\) or \(-\frac{\pi}{3} \leq \text{Arg}(z - i) \leq \frac{2\pi}{3}\),

c) \(|z| \geq 2\) and \(|\text{Im}(z)| \leq 3\),

d) \(\text{Re}(z) \geq \text{Im}(z)\),

e) \(|z - i| = |z + i|\),

f) \(|z - 1 - i| < 1\) and \(-\frac{\pi}{4} < \text{Arg}(z - 1 - i) \leq \frac{\pi}{2}\),

g) \(|z - i| = 2|z + i|\),

h) [H] \(|z - i| + |z + i| = 6\).

61. [R] Sketch the following on two carefully labelled Argand diagrams.

a) \(S_1 = \{z : \text{Re}(z) \geq 3\, \text{Im}(z)\ \text{and}\ |z - (3 + i)| > 2\}\),

b) \(S_2 = \{z : |z - i| < |z + i|\ \text{and}\ -\frac{\pi}{6} \leq \text{Arg}(z - i) \leq \frac{\pi}{6}\}\).

62. [R] Let \(S = \{z \in \mathbb{C} : \text{Im}(z) > -4\ \text{and}\ |z - 1 - i| \geq 3\}\).

a) Sketch \(S\) on a carefully labelled Argand diagram.

b) Does 2 + 4i belong to \(S\)?

63. [R] Let \(z\) be a complex number. Prove that \(|z - \text{Re}(z)| \leq |z - x|\) for all real numbers \(x\). Draw a sketch to illustrate the result.

64. [H] Let \(z, w\) be complex numbers.

a) Sketch the subset of the complex plane defined by \(w = e^{ia}\) for \(-\pi < \alpha \leq \pi\).

b) Given that \(\text{Arg}(z) = \theta\), prove that \(|z - e^{i\theta}| \leq |z - e^{i\alpha}|\) for all \(\alpha \in \mathbb{R}\).

c) Give a geometric interpretation of the result in part b).

Problems 6.10

65. [R] Use the remainder theorem to find the following remainders when.

a) \(2 + 3z - z^2 + 6z^3\) is divided by \(z - 5\),

b) \(1 - 6z + 5z^2 - 8z^3 + 2z^4\) is divided by \(z + 2\),

b) \(3z + 2z^2 + z^3\) is divided by \(z - 1 - i\).
66. \[\text{R}\] Use the remainder theorem and the factor theorem to show that \(z - 2\) is a factor of \(p(z) = 30 - 17z - 3z^2 + 2z^3\). Then divide \(p\) by \(z - 2\) and hence find all linear factors of \(p\).

67. \[\text{R}\] Use the method of the previous question to show that \(z - 1\) and \(z + 2\) are factors of \(p(z) = -8 - 6z + 7z^2 + 6z^3 + z^4\). Then find all linear factors of \(p\).

68. \[\text{R}\] Find all linear factors of
   a) \(z^5 + i\),
   b) \(z^6 + 8\).

69. \[\text{R}\] a) Factorise \(x^8 - 1\) into real linear and real quadratic factors.
   b) Repeat for \(x^6 + 8\).

70. \[\text{R}\] Factorise \(z^4 + 4\) over the rational numbers.

71. \[\text{R}\] Factorise the polynomial \(z^4 + i\) into complex linear factors.

72. \[\text{R}\] a) Solve the equation \(z^6 = -1\) where \(z \in \mathbb{C}\).
   b) Plot your solutions from part a) as points in the Argand diagram.
   c) Write \(z^6 + 1\) as a product of complex linear factors.
   d) Write \(z^6 + 1\) as a product of real quadratic factors.

73. \[\text{H}\] Let \(p(z) = z^6 + z^4 + z^2 + 1\).
   a) By using the identity, \((z^2 - 1)p(z) = z^8 - 1\), find all 6 complex roots of \(p(z)\) in polar form.
   b) Hence factorise \(p(z)\) into complex linear factors.
   c) Factorise \(p(z)\) into a product of 3 real irreducible quadratic polynomials.

74. \[\text{H}\] Let \(p(z) = 1 + z + z^2 + z^3 + z^4\).
   a) Solve \(z^5 - 1 = 0\) and hence factorise \(p(z)\) into linear factors.
   b) Find all linear and quadratic factors with real coefficients for \(p(z)\).
   c) Divide the equation \(p(z) = 0\) by \(z^2\). Let \(x = z + \frac{1}{2}\) and deduce that \(x^2 + x - 1 = 0\).
   d) Deduce that
   \[
   \cos \frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4} \quad \text{and} \quad \cos \frac{4\pi}{5} = \frac{-1 - \sqrt{5}}{4}
   \]

75. \[\text{H}\] Consider \(f(t) = t^6 + t^5 - t^4 - 5t^3 - 6t^2 - 6t - 4\). Given that \(-1 + i\) is a root of \(f\) and that \(f\) also has two real integer roots,
   a) factorise \(f\) into complex linear factors,
   b) factorise \(f\) into linear and quadratic factors with real coefficients.
76. [H] Let \( f(z) = z^5 - 2z^4 + 2z^3 - 5z^2 + 10z - 10 \). Given that \( 1 + i \) is a root, find all solutions to \( f(z) = 0 \).

77. [H] a) By considering \( z^9 - 1 \) as a difference of two cubes, write

\[
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8
\]

as a product of two real factors one of which is a quadratic.

b) Solve \( z^9 - 1 = 0 \) and hence write down the six solutions of \( z^6 + z^3 + 1 = 0 \) by \( z^3 \), deduce that

\[
\cos \frac{2\pi}{9} + \cos \frac{4\pi}{9} + \cos \frac{8\pi}{9} = 0.
\]

c) By letting \( y = z + \frac{1}{z} \) and dividing \( z^6 + z^3 + 1 = 0 \) by \( z^3 \), deduce that

\[
\cos \frac{2\pi}{9} + \cos \frac{4\pi}{9} + \cos \frac{8\pi}{9} = 0.
\]

d) Show that if \( |z| > 3 \), then

\[
|3z - z^3 + 5z^4| < |z|^6.
\]

e) Hence or otherwise show that for \( j = 1, \ldots, 6 \), \( |\alpha_j| \leq 3 \).

78. [H] Let \( p(z) = 3z - z^3 + 5z^4 + z^6 \). You are told that the six roots of \( p \), say \( \alpha_1, \ldots, \alpha_6 \), are distinct.

a) Prove that at least two of these roots are real.

b) Show that

\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0.
\]

c) Hence or otherwise, show that there is at least one root with positive real part, and at least one root with negative real part.

d) Show that if \( |z| > 3 \), then

\[
|3z - z^3 + 5z^4| < |z|^6.
\]

e) Hence or otherwise show that for \( j = 1, \ldots, 6 \), \( |\alpha_j| \leq 3 \).

79. [H] Cardan (approximately 1545) gave a formula for the roots of the cubic equation

\[
x^3 + ax + b = 0
\]

in the form \( x = u - v \), where

\[
u = \left\{ \frac{-b}{2} + \sqrt{\left( \frac{b}{2} \right)^2 + \frac{a^3}{27}} \right\}^{\frac{1}{3}}
\]

and

\[
v = \left\{ \frac{b}{2} + \sqrt{\left( \frac{b}{2} \right)^2 + \frac{a^3}{27}} \right\}^{\frac{1}{3}}
\]

and where the cube roots \( u \) and \( v \) must be selected to satisfy \( uv = \frac{a}{3} \). It can be shown that there are three pairs of values of \( u \) and \( v \) which satisfy the above conditions.

If \( a \neq 0 \) a simpler way of writing Cardan’s formula is that the three roots of the cubic are of the form

\[
x = u - \frac{a}{3u},
\]

where \( u \) satisfies

\[
u^3 = \frac{b}{2} + \sqrt{\left( \frac{b}{2} \right)^2 + \frac{a^3}{27}}.
\]
a) Use the simpler version of Cardan’s formula to find all three roots of $x^3 - 6x + 4 = 0$. (Note that complex numbers are used in the calculation even though all three roots are real).

b) Use the fact that one root of the cubic $x^3 - 6x + 4$ is 2 to factor the cubic as $(x - 2)q(x)$ where $q(x)$ is a quadratic. Hence find all roots of the cubic. Hence deduce that

$$\cos \frac{5\pi}{12} = \frac{-1 + \sqrt{3}}{2\sqrt{2}} \quad \text{and} \quad \cos \frac{11\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}}.$$ 

80. [H]  

a) Show that $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$.

b) Make the substitution $x = k \cos \theta$ in the equation $x^3 - ax - b = 0$ and show that the left hand side will become $\cos 3\theta$ if we choose $k$ such that $k^2 = \frac{4}{a}$ (assume $a > 0$).

c) With the above choice of $k$ show that the cubic equation can then be written as

$$\cos 3\theta = \frac{4b}{k^3} \quad \text{provided} \quad -1 \leq \frac{4b}{k^3} \leq 1.$$ 

d) Use the method outlined above to find the three real roots of $x^3 - 6x - 4 = 0$.

81. [H] 

Consider the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the all coefficients $a_0, a_1, \ldots, a_n$ are integers. Show that, if $r/s$ is a rational root of $p$ for which the integers $r$ and $s$ have no common factors, then $r$ is a divisor of $a_0$ and $s$ is a divisor of $a_n$. Hence find all rational roots of

a) $-5 + 3x - x^2 + 3x^3$,  
  b) $5 - 22x + x^2 + 28x^3$,  
  c) $4 - x - 100x^2 + 25x^3$.

82. [H] Let $M, N$ be positive integers. If $x^M (1 - x)^N$ is divided by $(1 + x^2)$, and the remainder is $ax + b$, show that $a = (\sqrt{2})^N \sin \frac{(2M - N)\pi}{4}$ and $b = (\sqrt{2})^N \cos \frac{(2M - N)\pi}{4}$.

Problems 6.11

83. [R] Are the solutions $x_n$ to the following discrete time systems stable ($|x_n| \to 0$ as $n \to \infty$), or do they blow up ($|x_n| \to \infty$ as $n \to \infty$), or does something else happen?

a) $x_{n+1} + x_n + \frac{5}{4} x_{n-1} = 0$.  
  b) $2x_{n+1} + 3x_n + 2x_{n-1} = 0$.  
  c) $x_{n+1} + x_n + 2x_{n-1} = 0$.

84. [R] Are the solutions $x(t)$ to the following continuous time systems stable ($|x(t)| \to 0$ as $t \to \infty$), or do they blow up ($|x(t)| \to \infty$ as $t \to \infty$), or does something else happen?

a) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = 0$.  
  b) $3\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0$.  
  c) $\frac{d^2x}{dt^2} + x = 0$.  

11
Problems 6.12

85. [R] Prove by mathematical induction that for all positive integers $n$,
\[ 1.2 + 2.3 + \cdots + n(n + 1) = \frac{1}{3} n(n + 1)(n + 2). \]

86. [R] Prove that, for all integers $n \geq 1$,
\[ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n + 1)(2n + 1). \]

87. [R] Prove that, for all integers $n \geq 1$,
\[ 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} n^2(n + 1)^2. \]

88. [H] Prove that, for all integers $n \geq 1$,
\[ 1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{30} n(n + 1)(6n^3 + 9n^2 + n - 1). \]

89. [H] Prove, by induction, that the sum to $k$ terms of
\[ 1^2 - 3^2 + 5^2 - 7^2 + 9^2 - 11^2 + \cdots \]

is $-8n^2$ if $k = 2n$ and $8n^2 + 8n + 1$ if $k = 2n + 1$.

90. [H] A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is defined recursively by
\[ a_1 = 1, \quad a_{n+1} = \sum_{j=1}^{n} \frac{a_j}{2j} \quad \text{for } n \geq 1. \]

Use the second Principle of Induction to prove that $a_n \leq 1$ for all $n \geq 1$.

91. [H] Suppose we draw $n$ lines in the plane with no three lines concurrent and no two lines parallel. Let $s_n$ denote the number of regions into which these lines divide the plane. For example, $s_1 = 2, s_2 = 4, s_3 = 7, \ldots$. Prove that $s_{n+1} = s_n + (n + 1)$. Deduce by induction that $s_n = \frac{1}{2} n(n + 1) + 1$. 
Problems 6.14

92. [M] Use MATLAB to evaluate \((5 + i)^4 (239 - i)\), and to check numerically that

\[
\frac{\pi}{4} = 4 \cot^{-1} 5 - \cot^{-1} 239.
\]  

Then use de Moivre’s Theorem to show (1).

**HINT:** Use `format long` to display more significant digits, and remember that floating point calculations are done with an accuracy determined by the relative machine precision given by the MATLAB command `eps`.

93. [M] Let \(z = -\sqrt{3} + i\) and \(w = z^2\).

a) Evaluate \(u = z^{123}\) by hand and using MATLAB.

b) Evaluate \(v = w^{123}\) by hand and using MATLAB.

c) Does \(u = 2^{123}v\) in MATLAB? Remember that MATLAB does floating point arithmetic which only has around 15 decimal digits of accuracy.

“I can do Addition,” she said, “if you give me time
—- but I can’t do Subtraction under ANY circumstances!”

Lewis Carroll, Through the Looking Glass.
Chapter 7

VECTOR SPACES

7.1 Definitions and examples of vector spaces
7.2 Vector arithmetic
7.3 Subspaces of $\mathbb{R}^n$
7.4 Linear combinations and spans
7.5 Linear independence
7.6 Basis and dimension
7.7 Coordinate vectors
7.8 Further important examples of vector spaces
7.9 Data fitting and polynomial interpolation
7.10 Appendix: A brief review of set and function notation
7.11 Vector spaces and Matlab
Problems for Chapter 7

You should try to solve some of the questions in Sections 7.4 to 7.8 with MATLAB.

Problems 7.1

1. [R] Show that the set

\[ S = \{ x \in \mathbb{R}^3 : x_1 \leq 0, \quad x_2 \geq 0 \}, \]

with the usual rules for addition and multiplication by a scalar in \( \mathbb{R}^3 \) is not a vector space by showing that at least one of the vector-space axioms is not satisfied. Give a geometric interpretation of this result.

2. [R] Show that the system \( S \) with the usual rules for addition and multiplication by a scalar in \( \mathbb{R}^3 \), and where

\[ S = \{ x \in \mathbb{R}^3 : 2x_1 + 3x_2^3 - 4x_3^2 = 0 \}, \]

is not a vector space by showing that at least one of the vector-space axioms is not satisfied.

3. [R] Let \( S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : (a - b)c = 0 \right\} \).

   a) Write down two non-zero elements of \( S \).
   b) Show that \( S \) is not closed under vector addition.

4. [H] The set \( \mathbb{C}^n \) is a vector space over \( \mathbb{C} \) (see Example ?? of Section 7.1). Check that axioms 1, 2, 6, 9 are satisfied by this system.

5. [H] Let \( M_{mn}(\mathbb{C}) \) be the set of all \( m \times n \) matrices with complex entries with addition the usual rule for addition of complex matrices, and multiplication by a scalar the usual rule for multiplication of a complex matrix by a complex scalar. Prove that the vector space \( M_{mn}(\mathbb{C}) \) satisfies axioms 1, 3, 6 and 10.

6. [H] Prove that the system \( (\mathbb{C}^n, +, *, \mathbb{R}) \) with “natural” definitions of + and * is a vector space, whereas the system \( (\mathbb{R}^n, +, *, \mathbb{C}) \) with “natural” definitions of + and * is not a vector space.

7. [H] Consider the system \( (\mathbb{R}^2, \oplus, \odot, \mathbb{R}) \) in which the usual operations of “addition” and “multiplication by a scalar” are replaced by the new definitions:

\[
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \oplus \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 - 3b_2 \end{pmatrix}
\]

\[
\lambda \odot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4\lambda a_1 \\ \lambda a_2 \end{pmatrix}.
\]

Give a list of the vector-space axioms satisfied by this system, and a list of any which are not satisfied. Is this system a vector space?
Problems 7.2

8. [H] Prove that the following properties are true for every vector space.
   a) \( 2\mathbf{v} = \mathbf{v} + \mathbf{v} \).
   b) \( n\mathbf{v} = \mathbf{v} + \cdots + \mathbf{v} \), where there are \( n \) terms on the right.

9. [H] Prove parts 2, 4 and 5 of Proposition ?? of Section 7.2.

10. [R] Let \( A, B, P \) be points in \( \mathbb{R}^3 \) with position vectors
    \[
    \mathbf{a} = \begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.
    \]
    Let \( Q \) be the point on \( AB \) such that \( AQ = \frac{2}{3} AB \).
    a) Find \( \mathbf{q} \), the position vector of \( Q \).
    b) Find the parametric vector equation of the line that passes through \( P \) and \( Q \).

11. [R] Consider the three points \( A(1, 1, 1), \ B(2, 0, 3) \) and \( C(3, -1, 1) \).
    a) Find \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \).
    b) Find a parametric vector form of the line through \( A \) and \( B \).
    c) Find a parametric vector form of the plane through \( A, B \) and \( C \).
    d) Find \( \overrightarrow{AB} \times \overrightarrow{AC} \).
    e) Find a point-normal form of the plane through \( A, B \) and \( C \).
    f) Find a Cartesian equation of the plane through \( A, B \) and \( C \).

12. [R] Given the vectors \( \mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \) and \( \mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \), find \( |\mathbf{p}|, |\mathbf{q}|, \mathbf{p} \cdot \mathbf{q} \), then the cosine of the angle between \( \mathbf{p} \) and \( \mathbf{q} \).

13. [R] Consider the equation
    \[
    \det \begin{pmatrix} x - 1 & y - 2 & z + 1 \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{pmatrix} = 0
    \]
    a) Show that the equation represents the Cartesian equation of a plane.
    b) Write the equation in point-normal form.

14. [R] For the points \( P(1, 2, 0), \ Q(1, 3, -1) \) and \( R(2, 1, 1) \), find \( \overrightarrow{PQ} \times \overrightarrow{PR} \) and the area of the triangle with vertices \( P, Q \) and \( R \).
15. Suppose that \( A \) is the point \((2, -1, 3)\) and \( \Pi \) is the plane

\[
x = \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}
\]

for \( \lambda, \mu \in \mathbb{R} \).

(a) Find a vector \( n \) which is normal to \( \Pi \).

(b) Find the projection of \( \overrightarrow{OA} \) on the direction \( n \).

(c) Hence find the shortest distance of \( A \) from \( \Pi \).

Problems 7.3

16. Suppose that \( v \) is a vector in \( \mathbb{R}^n \). Show that the line segment defined by

\[
S = \{ x \in \mathbb{R}^n : x = \lambda v, \quad 0 \leq \lambda \leq 10 \}
\]

is not a subspace of \( \mathbb{R}^n \).

17. Show that the set

\[
S = \{ x \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 6 \},
\]

is not a subspace of \( \mathbb{R}^3 \). Give a geometric interpretation of this result.

18. Let \( S \) the set

\[
S = \{ x \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 0 \},
\]

(a) Find three distinct members of \( S \).

(b) Show that \( S \) is a subspace of \( \mathbb{R}^3 \).

(c) Give a geometric interpretation of this latter result.

19. Show that

\[
T = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : -1 \leq x + y + z \leq 1 \right\}
\]

is not a vector subspace of \( \mathbb{R}^3 \).

20. Show that the set

\[
S = \{ x \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 4x_1 - 2x_2 + 3x_3 = 0 \}
\]

is a subspace of \( \mathbb{R}^3 \).

21. Show that the set

\[
S = \{ x \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 0 \quad \text{or} \quad 4x_1 - 2x_2 + 3x_3 = 0 \}
\]

is not a subspace of \( \mathbb{R}^3 \).
22. [R] Show that the set

\[ S = \{ \mathbf{b} \in \mathbb{R}^2 : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^3 \}, \]

where

\[ A = \begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{pmatrix}, \]

is a subspace of \( \mathbb{R}^2 \). Explain why each column of the matrix belongs to the set \( S \).

23. [R] For each of the following subsets of \( \mathbb{R}^3 \), either prove that the given subset is a subspace of \( \mathbb{R}^3 \) or explain why it is not a subspace.

   a) \( S = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + 2x_3 \geq 0 \right\} \).

   b) \( T = \left\{ \sum_{i=1}^{4} \lambda_i \mathbf{v}_i : \lambda_i \in \mathbb{R}, 1 \leq i \leq 4 \right\} \), where \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \) are given fixed vectors in \( \mathbb{R}^3 \).

   c) \( U = \{ A\mathbf{x} : \mathbf{x} \in \mathbb{R}^5 \} \), where \( A \) is a fixed \( 3 \times 5 \) matrix.

24. [H] Suppose that \( \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \) and \( \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \). Show, by the Subspace Theorem that the set

\[ S = \{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda \mathbf{u} + \mu \mathbf{v}, \text{ for } \lambda, \mu \in \mathbb{R} \} \]

is a subspace of \( \mathbb{R}^3 \).

25. [H] Prove that the set \( S \) in Example ?? on page ?? is closed under multiplication by a scalar.

26. [H] Let \( \mathbf{a} \) and \( \mathbf{b} \) be two fixed non-zero vectors in \( \mathbb{R}^5 \). Show that

\[ W = \{ \mathbf{x} \in \mathbb{R}^5 : \mathbf{x} \cdot \mathbf{a} = \mathbf{x} \cdot \mathbf{b} = 0 \} \]

is a subspace of \( \mathbb{R}^5 \).

   If \( \mathbf{a} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) and \( \mathbf{b} = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \), describe \( W \).

27. [R] Show that the set \( S = \{ p \in \mathbb{P}_2 : p(0) = 1 \} \) is NOT a subspace of \( \mathbb{P}_2 \).

28. [R] Show that the set

\[ S = \{ p \in \mathbb{P}_3 : p''(x) = 0 \text{ for all } x \in \mathbb{R} \} \]

is a subspace of \( \mathbb{P}_3 \).
29. \[ R \] Is the set 
\[ S = \{ p \in \mathbb{P}_3 : p'(x) + x + 1 = 0 \quad \text{for all} \quad x \in \mathbb{R} \} \]
a subspace of \( \mathbb{P}_3 \)?

30. \[ R \] Consider the set 
\[ S = \{ p \in \mathbb{P}_3 : (x + 1)p'(x) - 3p(x) = 0 \quad \text{for all} \quad x \in \mathbb{R} \}. \]
a) Show that \( S \) is a subspace of \( \mathbb{P}_3 \), (the set of all real polynomials of degree \( \leq 3 \)).
b) Find a polynomial in \( S \) where not all the coefficients are zero.

31. \[ H \] By constructing a counterexample, show that the union of two subspaces is not, in general, a subspace.

32. \[ H \] Let \( W_1 \) and \( W_2 \) be two subspaces of a vector space \( V \) over the field \( \mathbb{F} \). Prove that the intersection of \( W_1 \) and \( W_2 \) (i.e., the set \( W_1 \cap W_2 \)) is a subspace of \( V \).

33. \[ H \] Let \( V \) be a vector space over the field \( \mathbb{F} \).
   a) Let \( \{ W_k : 1 \leq k \leq m \} \) be \( m \) subspaces of \( V \), and let \( W \) be the intersection of these \( m \) subspaces. Prove that \( W \) is a subspace of \( V \).
   b) Let \( S \) be any set of vectors in \( V \), and let \( W \) be the intersection of all subspaces of \( V \) which contain \( S \) (that is, \( x \in W \) if and only if \( x \) lies in every subspace which contains \( S \)). Prove that \( W \) is the set of finite linear combinations of vectors from \( S \).

Problems 7.4

34. \[ R \] Let \( a = \begin{pmatrix} 10 \\ 1 \\ 4 \end{pmatrix} \), \( v_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \), \( v_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \) and \( v_3 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} \).
   a) Does \( a \in \text{span}(v_1, v_2, v_3) \)? If so, express \( a \) as a linear combination of \( v_1, v_2 \) and \( v_3 \).
   b) Do the vectors \( v_1, v_2, v_3 \) span \( \mathbb{R}^3 \)? If not, find condition(s) on \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \) for \( b \) to belong to \( \text{span}(v_1, v_2, v_3) \) and interpret your answer geometrically.

35. \[ R \] Repeat the preceding question using 
\[ a = \begin{pmatrix} 9 \\ -2 \\ -4 \end{pmatrix} \), \( v_1 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \), \( v_2 = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} \) and \( v_3 = \begin{pmatrix} 15 \\ -4 \\ -6 \end{pmatrix} \).

36. \[ R \] Repeat using \( a = \begin{pmatrix} 1 \\ 1 \\ -9 \\ 1 \end{pmatrix} \), \( v_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} \), \( v_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} \), \( v_3 = \begin{pmatrix} -1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \) and \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \). [Replace \( \mathbb{R}^3 \) by \( \mathbb{R}^4 \), of course.]
37. [R] Is the vector \( \mathbf{b} = \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix} \) \( \in \) span \( (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \), where
\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]

38. [R] Is the set of vectors \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \) a spanning set for \( \mathbb{R}^3 \)?

39. [R] Does \( \mathbf{v} \) belong to the column space of \( A \), \( \text{col}(A) \), where
\[
\mathbf{v} = \begin{pmatrix} 2 \\ -5 \\ 19 \\ -13 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 2 & -3 & -5 \\ 1 & 2 & 7 \end{pmatrix}.
\]

If so, write \( \mathbf{v} \) as a linear combination of the columns of \( A \).

40. [R] Is the polynomial \( p(x) = 1 + x + x^2 \) \( \in \) span \( (1-x+2x^2, -1+x^2, -2-x+5x^2) \)?

41. [R] Is \( S = \{1+x, 1-x^2, x+2x^2\} \) a spanning set for \( \mathbb{P}_2 \)?

42. [H] Prove Proposition ?? of Section 7.4.

43. [H] Use the vector-space axioms to prove that we do not need to use brackets when writing down the linear combination
\[
\sum_{k=1}^{n} \lambda_k \mathbf{v}_k = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n.
\]
That is, prove that the result of the operations in independent of the order in which the additions are performed.

**Problems 7.5**

44. [R] Is the set of vectors \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \) linearly independent? Are these three vectors coplanar?

45. [R] Is the set \( S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \), where \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} \), a linearly independent set? Are these three vectors coplanar?

46. [R] Can a set of linearly independent vectors contain a zero vector? Explain your answer.
47. [R] Given the set $S = \{v_1, v_2, v_3\}$, where 
\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix},
\end{align*}
\]
do the following.

a) Show that $S$ is a linearly dependent set.

b) Show that at least one of the vectors in $S$ can be written as a linear combination of the others, and find the corresponding linear combination.

c) Find all possible ways of writing the vector \[
\begin{pmatrix} 8 \\ 9 \\ 5 \end{pmatrix}
\]
as a linear combination of the vectors in the set.

d) Find a linearly independent subset of $S$ with the same span as $S$, and then show that \[
\begin{pmatrix} 8 \\ 9 \\ 5 \end{pmatrix}
\]
can be written as a unique linear combination of this subset.

e) Give a geometric interpretation of span $(S)$.

48. [R] Repeat the previous question for the set of four vectors $S = \{v_1, v_2, v_3, v_4\}$, where 
\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \\
v_4 &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.
\end{align*}
\]

49. [R] Is \{1 - x + 2x^2, -1 + x^2, -2 - x + 5x^2\} a linearly independent subset of $\mathbb{P}_2$? If the set is not linearly independent express one of the polynomials as a linear combination of the others.

50. [H] (For discussion). Let the set $S = \{v_1, \ldots, v_n\}$ be a linearly independent subset of a vector space $V$. You are standing at the origin of $V$ and set off in the direction of $v_1$. After a certain length of time, you turn and head in direction $v_2$ — then in direction $v_3$ and so on. Is it possible for you to return to the origin? (Note: You may walk any distance that you like along any of the directions, but you are not allowed to retrace your steps).

51. [H] What would happen in the previous question if the set $S$ were a linearly dependent set?

52. [H] Assume that $m \leq n$ and that $S = \{v_1, \ldots, v_m\}$ is a set of mutually orthogonal, non-zero vectors in $\mathbb{R}^n$, that is, the dot products satisfy (see Section 5.3.1) 
\[
\begin{align*}
v_i \cdot v_j &= 0 \quad \text{for } i \neq j; \\
v_i \cdot v_i &= 0 \quad \text{for } 1 \leq i \leq m.
\end{align*}
\]
Show that $S$ is a linearly independent set.
Problems 7.6

53. [R] Is the set \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \), where \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} \), a basis for \( \mathbb{R}^3 \)?

54. [R] Find a basis for, and the dimension of, \( W = \text{span} (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \), where
\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}.
\]

55. [R] Without doing any calculation, explain why
\[
\begin{bmatrix}
1 \\ 3 \\
2 \\ 0 \\
-1 \\ 4 \\
3 \\ 0
\end{bmatrix}
\]

is not a spanning set for \( \mathbb{R}^4 \).

Similarly, without doing any calculation, explain why
\[
\begin{bmatrix}
1 \\ 3 \\
2 \\ 0 \\
-1 \\ 4 \\
1 \\ 0 \\
3 \\ 1 \\
1 \\ 1
\end{bmatrix}
\]

is a linearly dependent set.

56. [R] Which of the following statements are true and which are false? Explain your answer.

a) Any set of 6 vectors in \( \mathbb{R}^5 \) is linearly dependent.

b) Some sets of 6 vectors in \( \mathbb{R}^5 \) are linearly independent.

c) Any set of 6 vectors in \( \mathbb{R}^5 \) is a spanning set for \( \mathbb{R}^5 \).

d) Some sets of 6 vectors in \( \mathbb{R}^5 \) span \( \mathbb{R}^5 \).

e) Same as in (a) – (d), with 6 replaced by 4.

f) Any set of 5 vectors in \( \mathbb{R}^5 \) is a basis for \( \mathbb{R}^5 \).

g) Some sets of 5 vectors in \( \mathbb{R}^5 \) are bases for \( \mathbb{R}^5 \).

h) Any set of vectors which spans \( \mathbb{R}^5 \) is linearly independent.

i) Any set of 5 vectors which spans \( \mathbb{R}^5 \) is linearly independent.

j) Any 5 linearly independent vectors in \( \mathbb{R}^5 \) form a basis for \( \mathbb{R}^5 \).

57. [R] Let \( V \) be a finite dimensional real vector space, and let \( S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) be a finite set of vectors in \( V \). Suppose also that the dimension of \( V \) is \( \ell \). State, with brief reasons, the relationship, if any, between \( n \) and \( \ell \) if

a) \( S \) is linearly independent.

b) \( S \) is linearly dependent.

c) \( S \) spans \( V \).

d) \( S \) is a basis for \( V \).

58. [H] Explain why it is impossible to have a set of \( m \) mutually orthogonal, non-zero vectors in \( \mathbb{R}^n \) with \( m > n \).
59. Consider the plane $P$ in $\mathbb{R}^3$ whose equation is \[ x + y + z = 0. \]
   a) Prove that $P$ is a subspace of $\mathbb{R}^3$.
   b) Find a basis for $P$. Give reasons for your answer.

60. Find a basis for, and the dimension of, the column space of the matrix
   \[
   A = \begin{pmatrix}
   1 & 1 & -1 & -2 & 1 \\
   0 & 0 & 1 & 4 & -1 \\
   0 & 0 & 0 & 2 & 2 \\
   0 & 0 & 0 & 0 & 0 
   \end{pmatrix}.
   \]

61. Find a basis for, and the dimension of, $\text{col}(A)$, where
   \[
   A = \begin{pmatrix}
   1 & 1 & 0 & 2 & 1 \\
   0 & 0 & -1 & -2 & 2 \\
   -1 & -1 & 1 & 4 & -1 \\
   1 & 1 & 0 & 4 & 2 
   \end{pmatrix}.
   \]

62. Show that the columns of the matrix $A$ given below are not a spanning set for $\mathbb{R}^4$. Then find a basis for $\mathbb{R}^4$ which contains as many of the columns of $A$ as possible.
   \[
   A = \begin{pmatrix}
   1 & 3 & 3 & -7 & 5 \\
   2 & 6 & 5 & -8 & 1 \\
   3 & 9 & 5 & -3 & -2 \\
   4 & 12 & 5 & 2 & -5 
   \end{pmatrix}.
   \]

63. Consider the set $T = \{v_1, v_2, v_3, v_4, x\}$ where
   \[
   v_1 = \begin{pmatrix}
   1 \\
   2 \\
   -1 \\
   0 
   \end{pmatrix}, \quad v_2 = \begin{pmatrix}
   3 \\
   6 \\
   -3 \\
   0 
   \end{pmatrix}, \quad v_3 = \begin{pmatrix}
   2 \\
   1 \\
   -1 \\
   4 
   \end{pmatrix}, \quad v_4 = \begin{pmatrix}
   -1 \\
   -5 \\
   2 \\
   4 
   \end{pmatrix}, \quad x = \begin{pmatrix}
   6 \\
   -3 \\
   -1 \\
   20 
   \end{pmatrix}.
   \]
   a) Find a basis $B$ for $\text{span}(v_1, v_2, v_3, v_4, x)$.
   b) Explain why $x$ belongs to $\text{span}(v_1, v_2, v_3, v_4)$. Write $x$ as a linear combination of $B$.
   c) Suppose the matrix $A$ has columns $v_1, v_2, v_3, v_4$. What is the dimension of the column space of $A$?

64. Show that the set \{1 – $x^2 + x^3$, $x + 2x^2$, $2 + x - x^2 + 2x^3$, $2x - x^2 + x^3$\} forms a basis for $\mathbb{P}_3$.

65. Consider the set of polynomials $S = \{p_1, p_2, p_3, p_4\}$ in $\mathbb{P}_2$, where
   \[
   p_1(z) = 1 + z - z^2, \quad p_2(z) = 2 - z, \quad p_3(z) = 5 - 4z + z^2, \quad p_4(z) = z^2.
   \]
   Show that $S$ is a linearly dependent spanning set for $\mathbb{P}_2$, and then find a subset of $S$ which is a basis for $\mathbb{P}_2$. 

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66. [H] You are given that \( V \) is a vector space and that \( S = \{v_1, v_2, v_3\} \) is a subset of \( V \). Suppose that \( w \in \text{span}(S) \). Prove that the set 
\[ \{v_1, v_2, v_3, w\} \]
is a linearly dependent set.

67. [H] Prove that the only subspaces of \( \mathbb{R}^4 \) are 
   i) \( \{0\} \), where \( 0 \) is the zero vector,
   ii) lines through the origin,
   iii) planes through the origin,
   iv) subspaces of the form \( \text{span} (S) \), where \( S \) is any set of three linearly independent vectors in \( \mathbb{R}^4 \), and
   v) \( \mathbb{R}^4 \) itself.

Problems 7.7

68. [R] Show that the columns of the matrix 
\[ A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ 5 & 3 & 0 & -1 \end{pmatrix} \]
are a basis for \( \mathbb{R}^4 \). Then find the coordinate vector of \( \mathbf{v} = \begin{pmatrix} -2 \\ -6 \\ -4 \\ -2 \end{pmatrix} \) with respect to the ordered basis given by the columns of \( A \).

69. [R] A vector \( \mathbf{v} \in \mathbb{R}^4 \) has the coordinate vector \( \begin{pmatrix} 1 \\ 6 \\ -1 \\ -4 \end{pmatrix} \) with respect to the ordered basis formed by the columns of the matrix \( A \) of the previous question. Find \( \mathbf{v} \).

70. [R] Find the vector \( \mathbf{v} \) that has coordinate vector \( \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \) with respect to the ordered basis 
\[ \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 9 \end{pmatrix} \right\} \] of \( \mathbb{R}^3 \).

71. [R] Find the coordinates of the following vectors with respect to the given ordered bases.
a) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ with respect to } \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.

b) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ with respect to } \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -2 \\ 0 \\ 0 \\ -1 \end{pmatrix}.

72. [R] With respect to the basis \( B = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 4 \\ 6 \\ -2 \\ 3 \\ -3 \end{pmatrix} \) of \( \mathbb{R}^3 \),

a) find the vector \( \mathbf{v} \) with coordinate vector \([\mathbf{v}]_B = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} \); 

b) find the coordinate vector of \( \mathbf{w} = \begin{pmatrix} 7 \\ -3 \\ 11 \end{pmatrix} \).

73. [H] Consider the set \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \), where \( \mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \), \( \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \), \( \mathbf{v}_3 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \).

Without solving systems of linear equations, do the following.

a) Show that \( S \) is an orthonormal set of vectors in \( \mathbb{R}^3 \).

b) Show that \( S \) is a basis for \( \mathbb{R}^3 \).

c) Find the coordinate vector of \( \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \) with respect to the ordered basis \( S \).

\text{Hint. See Example ?? of Section 7.6.}

74. [H] Let \( S = \{ \mathbf{u}_1, \ldots, \mathbf{u}_n \} \) be an orthonormal set of \( n \) vectors in \( \mathbb{R}^n \). Prove that \( S \) is a basis for \( \mathbb{R}^n \), and then show that the coordinate vector for any \( \mathbf{v} \in \mathbb{R}^n \) is given by

\[ [\mathbf{v}]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \text{ where } x_j = \mathbf{u}_j \cdot \mathbf{v}. \]

Problems 7.8

75. [R] Let \( M_{22} \) be the vector space of all \( 2 \times 2 \) matrices with real entries (see Example ?? of Section 7.1). Let \( S \) be the set

\[ S = \{ A \in M_{22} : a_{11} + a_{22} = 5 \}. \]
a) Find three matrices in $S$.

b) Is $S$ a subspace of $M_{22}$? Give a reason.

76. [R] Let $T$ be the set
$$T = \{ A \in M_{22} : a_{11} + a_{22} = 0 \}.$$

a) Find three matrices in $T$.

b) Is $T$ a subspace of $M_{22}$? Give a reason.

77. [R] Show that the four matrices
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
form a basis for $M_{22}(\mathbb{R})$. This set is called the standard basis for $M_{22}(\mathbb{R})$.

78. [H] Show that the four matrices of the previous question also form a basis for the vector space $M_{22}(\mathbb{C})$ of all $2 \times 2$ matrices with complex entries.

Hint: Can you see why your proof of the previous question will also be valid for complex numbers?

79. [H] Show that the set of four matrices
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
form a basis for $M_{22}(\mathbb{C})$. These matrices are called the Pauli spin matrices, and they are important in quantum physics and chemistry.

80. [H] Find the coordinates of the following vectors with respect to the given ordered bases.

a) The matrix
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
with respect to the standard basis for $M_{22}$ given in Question 77. Note that the results are the same for both real and complex numbers

b) Repeat part (b) for the basis of Pauli spin matrices given in Question 79. In this case the entries of $A$ should be regarded as complex numbers.

81. [R] Let
$$R = \left\{ \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix} \right\}.$$

a) Express
$$\begin{pmatrix} -4 & 2 \\ -1 & -3 \end{pmatrix}$$
as a linear combination of elements of $R$.

b) Does $R$ span $M_{22}(\mathbb{R})$, the space of all $2 \times 2$ matrices? Give a brief reason for your answer.
82. [H] Complete the proof of Proposition 7.8 of Section 7.8 that the system \((\mathcal{R}[X], +, *, \mathbb{R})\) is a vector space.

83. [H] Let \(C[X]\) be the set of all complex-valued functions with domain \(X\). Show that the system \((C[X], +, *, \mathbb{C})\), where + and * are the usual rules for addition and multiplication by a scalar of functions, satisfies vector-space axioms 2, 4, 7, and 9. This system is a vector space over the complex field \(\mathbb{C}\).

84. [H] Show that the set

\[
S = \left\{ y \in \mathcal{R}[\mathbb{R}] : \frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 4y = 0 \right\}
\]

is a subspace of the vector space \(\mathcal{R}[\mathbb{R}]\) of all real-valued functions with domain \(\mathbb{R}\).

85. [H] Is the set

\[
S = \left\{ y \in \mathcal{R}[\mathbb{R}] : \frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 4y = 5 \right\}
\]
a subspace of \(\mathcal{R}[\mathbb{R}]\)? Prove your answer.

86. [H] Let \(C^{(k)}[\mathbb{R}]\) be the set of all real-valued functions with domain \(\mathbb{R}\) for which the first \(k\) derivatives exist and are continuous. Prove that \(C^{(k)}[\mathbb{R}]\) is a subspace of the vector space \(\mathcal{R}[\mathbb{R}]\) of all real-valued functions with domain \(\mathbb{R}\).

87. [H] Show that the vector spaces \(C^{(k)}[\mathbb{R}]\) defined in the previous question have the property that, if \(m \geq n\), then \(C^{(m)}[\mathbb{R}]\) is a subspace of \(C^{(n)}[\mathbb{R}]\).

88. [H] Let \(S\) be the subset of \(\mathcal{R}([-\pi, \pi])\) defined by

\[
S = \left\{ f \in \mathcal{R}([-\pi, \pi]) : \int_{-\pi}^{\pi} \cos(x + t)f(t)dt = 0 \quad \text{for all} \quad x \in [-\pi, \pi] \right\}.
\]

Prove that \(S\) is a subspace of the vector space \(\mathcal{R}([-\pi, \pi])\).

89. [H] This question generalises the results of Question 52 to real-valued functions.

Let \(S = \{f_1, \ldots, f_n\}\) be a set of real-valued functions defined on an interval \([a, b]\) with the properties that

\[
\int_a^b f_i(x)f_j(x)dx = 0 \quad \text{for} \quad i \neq j; \quad 1 \leq i, j \leq n
\]

\[
\int_a^b f_i^2(x)dx \neq 0 \quad \text{for} \quad 1 \leq i \leq n.
\]

Prove that \(S\) is a linearly independent set.

Note. A set of functions with these properties is said to be mutually orthogonal on the interval \([a, b]\).
90. [H] Show that the set

\[ S = \{ p \in \mathbb{P}_n(\mathbb{R}) : 5p'(6) + 3p(6) = 0 \} , \quad \text{where} \quad p'(x) = \frac{dp}{dx}, \]

is a subspace of the vector space \( \mathbb{P}_n(\mathbb{R}) \) of all real polynomials of degree less than or equal to \( n \).

91. [H] Is the set

\[ S = \{ p \in \mathbb{P}_n(\mathbb{R}) : 5p'(6) + 3p(6) = 8 \} , \quad \text{where} \quad p'(x) = \frac{dp}{dx}, \]
a subspace of \( \mathbb{P}_n(\mathbb{R}) \)? Prove your answer.

92. [H] Let \( \mathbb{P} \) be the set of all polynomials over the complex-number field \( \mathbb{C} \). Show that \( \mathbb{P} \) is a subspace of the vector space \( \mathbb{C}[\mathbb{C}] \) of all complex-valued functions with domain \( \mathbb{C} \).

93. [R] Is the polynomial \( p \in \text{span}(p_1, p_2, p_3) \), where the polynomials are defined by

\[ p(z) = -6 + 2z + 3z^2, \quad p_1(z) = 1 + 2z + 3z^2, \quad p_2(z) = -4 - z + 9z^2, \quad p_3(z) = -5 - z + 12z^2. \]

94. [R] Find conditions on the coefficients of the polynomial \( p \in \mathbb{P}_2 \) for \( p \) to be a linear combination of the three polynomials \( p_1, p_2, p_3 \), where the polynomials are given by

\[ p_1(z) = 2z + 3z^2, \quad p_2(z) = 5 - 2z - 3z^2, \quad p_3(z) = 15 - 4z - 6z^2. \]

95. [R] Are the polynomials \( p_1, p_2, p_3 \) in the previous two questions spanning sets for \( \mathbb{P}_2 \)?

96. [R] Is the set of polynomials \( S = \{ p_1, p_2, p_3 \} \) in \( \mathbb{P}_2 \), where

\[ p_1(z) = 1 + z - z^2, \quad p_2(z) = 2 - z, \quad p_3(z) = 5 - 4z + z^2, \]
a linearly independent set? If not, express one of the polynomials as a linear combination of the others.

97. [R] Show that

\[
\begin{align*}
p_1(z) & = -2 + 5z - 4z^2 + 15z^3 - 5z^4 + z^5, \\
p_2(z) & = 3z + 4z^2 - 3z^3 + 6z^5, \\
p_3(z) & = 2 + 3z^2 - 4z^3 + 10z^4 - 5z^5, \\
p_4(z) & = 3 + 14z^2 - 5z^3 + 6z^4 - 3z^5, \\
p_5(z) & = 3 + 8z + 17z^2 + 3z^3 + 11z^4 - z^5, \\
p_6(z) & = -3 + 11z - 7z^2 + 10z^3 - z^4 + 11z^5,
\end{align*}
\]

are not a spanning set for \( \mathbb{P}_5 \), and then construct a basis for \( \mathbb{P}_5 \) containing as many of the given polynomials as possible.

**Hint.** Check using MATLAB.
98. [R] Find the coordinate vector for \( p(x) = 1 + 2x + x^2 \) with respect to the ordered basis \( \{ 1 + x, 1 - x^2, x + 2x^2 \} \) of \( \mathbb{P}_2 \).

99. [R] Find the coordinate vector of \( 1 + 2z + 3z^2 \) with respect to the ordered basis of \( \mathbb{P}_2 \) given by \( \left\{ \frac{1}{8}z(z-2), 1 - \frac{1}{4}z^2, \frac{1}{8}z(z+2) \right\} \).

Note. This question and the one that follows do not require Gaussian Elimination.

100. [H] Find the coordinate vector of \( a_0 + a_1z + a_2z^2 \) with respect to the ordered basis of \( \mathbb{P}_2 \) given by \( \left\{ \frac{1}{2}z(z-1), 1 - z^2, \frac{1}{2}z(z+1) \right\} \).

101. [H] Let \( S = \{ p_1, \ldots, p_n \} \) be a set of \( n \) polynomials in \( \mathbb{P}_{n-1}(\mathbb{R}) \) with the property that

\[
\int_a^b p_i(x)p_j(x)dx = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \quad 1 \leq i, j \leq n.
\]

A set of polynomials with this property is called an orthonormal set of polynomials on the interval \([a, b]\).

Prove that \( S \) is a basis for \( \mathbb{P}_{n-1}(\mathbb{R}) \), and then show that the coordinate vector for any \( p \in \mathbb{P}_{n-1}(\mathbb{R}) \) is given by

\[
[p]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{where} \quad x_i = \int_a^b p_i(x)p(x)dx.
\]

Problems 7.9

102. [R] For the data in Table ?? on page ??

a) find the interpolating polynomial of degree 3 using the first 4 data points and the Lagrange basis. Use this to predict the interest rate at 1 year. Is this interpolation or extrapolation? Does it agree with the data value given for 1 year?

b) Use the MATLAB script `polint.m` available from the MATH1251 web page to fit a polynomial of degree 4 using all 5 data points and predict the rate for 2 years.

103. [H] Let the \( m \geq n \) points \( t_i \) for \( i = 1, \ldots, m \) be distinct, and define the matrix \( A \in M_{mn} \) by \( A_{ij} = \phi_j(t_i) \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \).

a) Show that if \( A \) has linearly independent columns then the functions \( \phi_j(t) \) for \( j = 1, \ldots, n \) are linearly independent.
b) Give an example of two distinct points and two linearly independent functions for which the matrix $A$ does not have linearly independent columns. Hint: Try low order polynomials at 0 and 1.

104. [H] a) Show that the Vandermonde determinant

$$
\begin{vmatrix}
1 & t_1 & \cdots & t_1^{n-1} \\
1 & t_2 & \cdots & t_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & t_n & \cdots & t_n^{n-1}
\end{vmatrix}
= \prod_{i<j}(t_j - t_i)
$$

for $n = 2, 3, 4$. (This formula holds for all integers $n > 1$).

b) What is the significance of this result for polynomial interpolation at the points $t_1, t_2, \ldots, t_n$?
Chapter 8

LINEAR TRANSFORMATIONS

8.1 Introduction to linear maps
8.2 Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices
8.3 Geometric examples of linear transformations
8.4 Subspaces associated with linear maps
8.5 Further applications and examples of linear maps
8.6 Representation of linear maps by matrices
8.7 Matrix arithmetic and linear maps
8.8 One-to-one, onto and invertible linear maps and matrices
8.9 Proof of the Rank-Nullity Theorem
8.10 One-to-one, onto and inverses for functions
8.11 Linear transformations and Matlab
Problems for Chapter 8

Problems 8.1

1. [R] Explain why the function $S : [-1, 1] \to \mathbb{R}$ defined by $S(x) = 5x$ for $x \in [-1, 1]$ is not a linear map. Then show that the function $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 5x$ for $x \in \mathbb{R}$ is a linear map.

2. [R] For the following examples, determine whether $T$ is a linear map by using Definition ??.

   a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

      \[ T(x) = \begin{pmatrix} 3x_1 - x_2 \\ 2x_1 + 4x_2 \\ -3x_1 - 3x_2 \\ x_2 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2. \]

   b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

      \[ T(x) = \begin{pmatrix} -2x_1 + 5x_3 \\ 6x_1 - 8x_2 + 2x_4 \\ -2x_1 + 4x_2 - 3x_3 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4. \]

   c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

      \[ T(x) = \begin{pmatrix} 3x_1 + 4 \\ -2x_1 + 3x_2 - x_3 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3. \]

   d) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined for $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ by

      \[ T(x) = x_1 \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}. \]

   e) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

      \[ T(x) = \begin{pmatrix} 3x_2^2 - x_3 \\ x_1 - 4x_2 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3. \]
3. **[H]** Consider the complex numbers as a real vector space. Specify the “natural” domain and codomain for each of the following functions of a complex number and determine if the function is a linear function with $\mathbb{F} = \mathbb{R}$.

   a) $T(z) = \text{Re}(z)$,  
   b) $T(z) = \text{Im}(z)$,  
   c) $T(z) = |z|$,  
   d) $T(z) = \text{Arg}(z)$,  
   e) $T(z) = \bar{z}$.

4. **[R]** Show that the sine function $T: \mathbb{R} \rightarrow \mathbb{R}$, defined by

   $$T(x) = \sin(x) \quad \text{for} \quad x \in \mathbb{R},$$

   satisfies parts 1 and 2 of Proposition ?? of Section 8.1 but that it is not a linear map.

5. **[H]** Use proof by induction to prove Theorem ?? of Section 8.1.

6. **[R]** If $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent in a real vector space $V$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, is there a linear map $T: W \rightarrow \mathbb{R}^2$ where $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ such that

   $$T(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, T(\mathbf{v}_2) = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 4 \end{pmatrix}, T(\mathbf{v}_3) = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$?

7. **[R]** A linear map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ has function values given by

   $$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 4 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \\ 6 \end{pmatrix}.$$ Find $T\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

8. **[R]** Show that any function $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ with function values given by $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$,

   $$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 4 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \\ 6 \end{pmatrix},$$

   and $T\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 1 \end{pmatrix}$ is not a linear map.

9. **[R]** A linear function $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ has function values given by

   $$T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$ Write $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ as a linear combination of the vectors in the basis $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Hence find $T\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

   **HINT.** Use Theorem ?? of Section 8.1.
10. [H] Given that \( T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \), \( T \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \) and \( T \begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \), show that \( T \) is not a linear map.

Problems 8.2

11. [R] For any function in Question 2 which is a linear map, find a matrix \( A \) such that \( T(x) = Ax \) for \( x \) in the domain by using the results of the Matrix Representation Theorem of Section 8.2.

12. [R] For any function in Question 2 which is a linear map, write a system of linear equations for \( T(x) \) for \( x \) in the domain, and hence find a matrix \( A \) such that \( T(x) = Ax \). Check that the matrices you obtain in this question are the same as the matrices that you obtained in the previous question.

Problems 8.3

13. [R] For each of the following \( 2 \times 2 \) matrices, draw a picture to show \( A e_1, A e_2, Ab \), where \( e_1 \) and \( e_2 \) are the standard basis vectors in \( \mathbb{R}^2 \) and where \( b = 2e_1 + 3e_2. 

\[ \text{a) } \begin{pmatrix} 2 & 0 \\ 0 & 0.7 \end{pmatrix}, \text{ b) } \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \text{ c) } \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}, \text{ d) } \begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix}, \text{ e) } \begin{pmatrix} 4 & -4 \\ 3 & -4 \end{pmatrix}. \]

14. [R] Draw the image of the star in Figure ??(a) on page ?? under each of the transformations defined by the matrices in Question 13.

15. [R] Let \( T \) be the rotation in the plane \( \mathbb{R}^2 \) through angle \( \frac{\pi}{3} \) in the anti-clockwise direction. Find the matrix which represents the linear transformation \( T \).

16. [H] Let \( x \) be the position vector of a point \( X \) in \( \mathbb{R}^2 \), and let \( x' \) be the position vector of the point \( X' \) which is the reflection of \( X \) in the \( x_2 \)-axis. (That is, assume that a mirror is placed along the \( x_2 \)-axis and that \( X' \) is the reflection of \( X \) in the mirror.) Show that the function \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T(x) = x' \) is a linear map. Find a matrix \( A \) which transforms \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) into \( x' = \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \).

17. [H] Let \( x \) be the position vector of a point \( X \) in \( \mathbb{R}^3 \) and let \( x' \) be the position vector of the point \( X' \) which is the reflection of \( X \) in the \( (x_1, x_2) \)-plane. Show that the function \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( T(x) = x' \) is a linear map. Find a matrix \( A \) which transforms the position vector \( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \) of \( X \) into the position vector \( \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \) of \( X' \).
18. [H] Let \( \mathbf{p} \) be the position vector of a point \( P \) in \( \mathbb{R}^n \) and let \( \mathbf{q} \) be the position vector of the point \( Q \) which is the reflection of \( P \) in the line

\[
\mathbf{x} = \lambda \mathbf{d}; \quad \lambda \in \mathbb{R}.
\]

Show that the function \( T : \mathbb{R}^n \to \mathbb{R}^n \) defined by \( T(\mathbf{p}) = \mathbf{q} \) is a linear map. Find a matrix \( A \) which transforms \( \mathbf{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \) into \( \mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \).

19. [R] Let \( \mathbf{b} \) be a fixed vector in \( \mathbb{R}^3 \). Is the function \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by

\[
T(\mathbf{x}) = \mathbf{b} \times \mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3,
\]

where \( \mathbf{b} \times \mathbf{x} \) is the cross product, a linear map? Prove your answer. Find a matrix \( A \) which transforms the vector \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \) into its function value \( T(\mathbf{x}) = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \).

20. [H] In Example ?? of Section 8.3 we have stated that if \( \mathbf{b} \) is a fixed non-zero vector in \( \mathbb{R}^n \) then the projection function \( T : \mathbb{R}^n \to \mathbb{R}^n \) of vectors onto \( \mathbf{b} \), which is defined by

\[
T(\mathbf{a}) = \text{proj}_\mathbf{b}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||^2} \mathbf{b} \quad \text{for} \quad \mathbf{a} \in \mathbb{R}^n,
\]

is a linear map. Prove this statement. If \( n = 3 \) and \( \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \), find a matrix \( A \) which transforms the vector \( \mathbf{a} \) into its projection \( T(\mathbf{a}) \).

21. [H] If \( \mathbf{a} \) is regarded as a fixed non-zero vector, is the function \( S : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
S(\mathbf{b}) = \begin{cases} \text{proj}_\mathbf{a} \mathbf{b} & \text{for} \quad \mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \\ \mathbf{0} & \mathbf{b} = \mathbf{0} \end{cases}
\]

a linear map? Prove your answer.

22. [H] Let \( A_\phi \), \( A_\theta \) and \( A_{\phi+\theta} \) be the matrices for rotations in the plane by angles \( \phi \), \( \theta \) and \( \phi + \theta \) respectively (see Example ?? of Section 8.3). Prove that

\[
A_\theta A_\phi = A_{\phi+\theta}.
\]

What is this saying geometrically?

23. [H] Let \( B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \) be an ordered orthonormal basis (Cartesian coordinate system) for a three-dimensional geometric vector space. Let \( \mathbf{a} \) be a three-dimensional geometric vector and let \( \mathbf{a}' = R_\alpha(\mathbf{a}) \) be the vector obtained by rotating \( \mathbf{a} \) anticlockwise by an angle \( \alpha \).
about an axis parallel to \( j \). If the coordinate vectors of \( a \) and \( a' \) are 
\[
[a]_B = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}
\]
and
\[
[a']_B = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix},
\]
find the rule \( R_\alpha \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} \) and show that it defines a linear map from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \). Also find the matrix \( A \) such that \( a' = Aa \).

### Problems 8.4

24. [R] Show that the set \[ \left\{ \begin{pmatrix} \lambda \\ -2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\} \] is the kernel of the matrix \( \begin{pmatrix} 3 & 1 & -1 \\ 8 & 3 & -2 \end{pmatrix} \).

25. [R] Find the kernel and the nullity of each of the following matrices.

   a) \[
   A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -2 & 3 \\ 4 & 1 & -1 \end{pmatrix},
   \]
   b) \[
   B = \begin{pmatrix} 0 & 5 & 15 \\ 2 & -2 & -4 \\ 3 & -3 & -6 \end{pmatrix},
   \]
   c) \[
   C = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix}.
   \]

   Where possible, give a geometric interpretation of the kernels.

26. [R] Find a basis for the kernel, and the nullity, of each of the following matrices.

   a) \[
   D = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & -1 & -4 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix},
   \]
   b) \[
   E = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 5 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
   \]

27. [H] Let \( W = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T : x_1 + x_2 + x_3 + x_4 = x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \right\} \). Find a matrix \( A \) such that \( W = \ker A \).

28. [R] Find \( \ker(T) \) and \( \nullity(T) \) for the linear functions of Question 2.

29. [R] Find \( \ker(T) \) and \( \nullity(T) \) for the linear functions of Questions 16 through 20. Give a geometric interpretation of the kernels.

30. [H] Suppose that \( b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \).

   a) Prove that the mapping \( T : \mathbb{R}^3 \to \mathbb{R}^3 \), given by \( T(x) = b \times x \) for all \( x \in \mathbb{R}^3 \), is a linear mapping.
b) Find the dimension of the kernel of this mapping.

31. [R] For each given vector \( \mathbf{b} \) and matrix \( A \), determine if \( \mathbf{b} \in \text{im}(A) \).

a) \( \mathbf{b} = \begin{pmatrix} 11 \\ 10 \\ 4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 3 \\ 4 & 1 & -1 \end{pmatrix}. \)

b) \( \mathbf{b} = \begin{pmatrix} 9 \\ -2 \\ -4 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 5 & 15 \\ 2 & -2 & -4 \\ 3 & -3 & -6 \end{pmatrix}. \)

c) \( \mathbf{b} = \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix}. \)

32. [R] Find conditions on \( b_1, b_2, b_3 \) for the vector \( \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3 \) for \( \mathbf{b} \) to belong to im(\( A \)) for the matrices in the preceding question.

33. [R] Find a basis for the image, and the rank, of each of the matrices in Questions 25 and 26.

34. [R] By comparing the answers to Questions 25, 26 and 33, verify the conclusion of the Rank-Nullity Theorem.

35. [R] Find a basis for the image, and the rank, of each of the matrices

\[
A = \begin{pmatrix} 1 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ -1 & -1 & 1 & 4 & -1 \\ 1 & 1 & 0 & 4 & 2 \end{pmatrix}. \]

36. [R] Find a basis for \( \mathbb{R}^3 \) which contains a basis of im(\( C \)), where

\[
C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -4 & 6 & -2 \\ -1 & 2 & -3 & 1 \end{pmatrix}. \]

37. [R] Find a basis for \( \mathbb{R}^4 \) which contains a basis of im(\( D \)), where

\[
D = \begin{pmatrix} 1 & 3 & 3 & -1 & 7 \\ 2 & 6 & 3 & 1 & 8 \\ 3 & 9 & 3 & 4 & 7 \\ 4 & 12 & 0 & 8 & 4 \end{pmatrix}. \]

38. [R] A linear map \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) has the property that

\[
T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}. \]
a) Write down the matrix representation of $T$ with respect to the standard basis (in both domain and co-domain).

b) Find a basis for the image of $T$ and find the rank of $T$.

c) State the dimension of the kernel of $T$.

d) Does the vector $\begin{pmatrix} 8 \\ -3 \\ 1 \\ 2 \end{pmatrix}$ belong to the image of $T$? Give reasons.

39. [H] Let $A \in M_{mn}(\mathbb{R})$. Show that the following statements are equivalent, that is, show that if any statement is true then all are true, whereas if any statement is false then all are false.

a) For all $x$ and $y$ in $\mathbb{R}^n$, $Ax = Ay$ if and only if $x = y$.

b) $\ker(A) = \{0\}$.

c) $\text{nullity}(A) = 0$.

d) $\text{rank}(A) = n$.

e) $\text{im}(A) = \mathbb{R}^n$.

f) The columns of $A$ form a basis for $\mathbb{R}^n$.

40. [H] Let $A \in M_{mn}(\mathbb{R})$, and let $\{e_j : 1 \leq j \leq m\}$ be the set of $m$ standard basis vectors of $\mathbb{R}^m$. If $A$ is of rank $r$, explain why at most $r$ of the $m$ equations $Ax_j = e_j$ can have solutions.

41. [H] Let $A$ and $e_j$ be as in the previous question. If $\text{nullity}(A) = \nu$, explain why at least $m - n + \nu$ of the $m$ equations $Ax_j = e_j$ do not have solutions.

42. [H] Let $A \in M_{mn}(\mathbb{R})$, $\text{rank}(A) = n$, and $e_j$ be the standard basis vectors for $\mathbb{R}^n$. Prove that each of the $n$ equations $Ax_j = e_j$, $1 \leq j \leq n$, has a unique solution.

43. [H] Let $T : V \rightarrow V$ be a linear map and assume that $\dim(V) = n$. Show that the following statements are equivalent.

a) $T(v) = T(w)$ if and only if $v = w$ for all $v, w \in V$.

b) $\ker(T) = \{0\}$.

c) $\text{nullity}(T) = 0$.

d) $\text{rank}(T) = n$.

e) $\text{im}(T) = V$.

Problems 8.5

44. [R] Show that the function $T : \mathbb{R}^4 \rightarrow M_{22}(\mathbb{R})$ defined by

$$T(a) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{for} \quad a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$$

is a linear map.
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45. [R] Show that the function \( T : \mathbb{R}^4 \to M_{22}(\mathbb{R}) \) defined by
\[
T(a_1, a_2, a_3, a_4) = \begin{pmatrix}
3a_1 - 2a_4 & a_4 + 2a_3 \\
-5a_2 + 3a_3 & a_1
\end{pmatrix}
\]
is a linear map.

46. [R] Is the function \( T : M_{23}(\mathbb{R}) \to \mathbb{R}^6 \) defined by
\[
T(A) = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{pmatrix}^T 
\]
for \( A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{pmatrix} \in M_{23}(\mathbb{R}) \)
a linear map?

47. [R] Show that the function \( T : \mathbb{P}_2 \to \mathbb{C}^3 \) defined by
\[
T(a_0 + a_1 z + a_2 z^2) = \begin{pmatrix}
a_0 \\
a_1 \\
a_2
\end{pmatrix}
\]
is a linear map. Note that \( T \) maps a polynomial in \( \mathbb{P}_2 \) into its coordinate vector with respect to the standard basis \( \{1, z, z^2\} \).

48. [R] Show that the function \( T : \mathbb{C}^4 \to \mathbb{P}_4 \) defined by \( T(\mathbf{a}) = p \) for \( \mathbf{a} = \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix} \in \mathbb{C}^4 \), where
\[
p(z) = (a_1 - 3a_2) + (2a_3 - 3a_4)z + a_2 z^2 + (3a_1 - a_2 + 2a_3 + 4a_4)z^4 \quad \text{for all} \quad z \in \mathbb{C},
\]
is a linear map.

49. [H] Show that the function \( T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R}) \) defined by
\[
T(p) = 4p' + 3p, \quad \text{where} \quad p'(x) = \frac{dp}{dx}
\]
is a linear map.

50. [H] Is the function \( T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R}) \) defined by \( T(p) = q \), where
\[
q(x) = 4xp'(x) - 8p(x) \quad \text{for} \quad x \in \mathbb{R},
\]
a linear map? Prove your answer.

51. [H] Show that the function \( T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R}) \) defined by \( T(p) = q \), where
\[
q(x) = \int_0^x p(t) dt \quad \text{for} \quad x \in \mathbb{R},
\]
is a linear map.
52. Let $V$ be the subset of the vector space $\mathcal{R}[\mathbb{R}]$ of all real-valued functions on $\mathbb{R}$ defined by

$$V = \left\{ f \in \mathcal{R}[\mathbb{R}] : \int_0^x f(t) \, dt \text{ exists for all } x \in \mathbb{R} \right\}.$$ 

Show that $V$ is a subspace of $\mathcal{R}[\mathbb{R}]$, and then show that the rule $T : V \to \mathcal{R}[\mathbb{R}]$ defined by $T(f) = g$, where

$$g(x) = \int_0^x f(t) \, dt \quad \text{for} \quad f \in V \quad \text{and} \quad x \in \mathbb{R},$$

is a linear map.

53. A function $S : \mathbb{R} \to \mathbb{Z}$ is defined by $S(x) = y$, where $y$ is the integer obtained on rounding $x$ to the nearest integer. Is $S$ a linear map? A function $T : \mathbb{R} \to \mathbb{R}$ is defined by $T(x) = y$, where $y$ is the integer obtained on rounding $x$ to the nearest integer. Is $T$ a linear map?

54. Let $y$ be a real-valued function with domain $\mathbb{R}$ such that $y$ and its first two derivatives $y'$ and $y''$ exist, and such that the Laplace transforms (see Example ?? of Section 8.5) of $y$, $y'$ and $y''$ also exist on the interval $(0, \infty)$. Given that $y(0) = 1$ and $y'(0) = 2$ and that $y$ satisfies the differential equation

$$y''(x) + 4y'(x) + 3y = e^{-3x},$$

find an explicit formula for the Laplace transform $y_L(s)$ of $y$ in terms of $s$.

**Hint.** Take the Laplace transform of the differential equation and use integration by parts to find formulae for the Laplace transforms of $y'$ and $y''$ in terms of $y_L(s)$.

55. Consider the mapping $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p(x)) = \begin{pmatrix} a - b \\ c - d \end{pmatrix} \quad \text{where} \quad p(x) = a + bx + cx^2 + dx^3.$$ 

a) Prove that $T$ is linear.

b) Show that $p(x) = 3x^3 + 3x^2 - 2x - 2$ is in the kernel of $T$.

56. Consider the function $T : \mathbb{R}^4 \to \mathbb{P}_1$ defined by

$$T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a - 2b) + (c + d)x.$$ 

a) Find $T \begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}$.

b) Show $T$ is a linear transformation.
c) Write down a non-zero vector in $\mathbb{R}^4$ which lies in $\ker(T)$.

57. [H] A linear map $T : \mathbb{C}^3 \to \mathbb{P}_3$ has function values given by

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 + (2 + i)z - 3z^3, \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (4 - 3i)z + z^2, \quad T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -2 \quad \text{for} \quad z \in \mathbb{C}.$$ 

Find $T \begin{pmatrix} i \\ 2 \\ -1 \end{pmatrix}$ and $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

58. [H] Let $\mathbb{P}_n$ be the real vector space of polynomials of degree less than or equal to $n$, and take its standard basis to be

$$\{1, x, x^2, \ldots, x^n\}.$$ 

For $p(x) \in \mathbb{P}_3$, let $(T(p))(x) \in \mathbb{P}_4$ be defined by

$$(T(p))(x) = \int_0^x p(t)dt.$$ 

a) Show that $T$ is a linear transformation from $\mathbb{P}_3$ to $\mathbb{P}_4$.

b) Calculate the matrix $A$ of this linear transformation with respect to the standard bases of $\mathbb{P}_3$ and $\mathbb{P}_4$.

c) Find a basis for the image of $T$, $\text{im}(T)$.

d) Find a basis for the kernel of $T$, $\ker(T)$.

59. [R] A car manufacturer produces a station wagon, a four-wheel drive, a hatchback and a sedan model. Each model is made from steel, plastics, rubber and glass, and it also requires a number of hours of labour to produce. The requirements per car of these inputs for each model are as shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Steel (tonnes)</th>
<th>Plastics (tonnes)</th>
<th>Rubber (tonnes)</th>
<th>Glass (tonnes)</th>
<th>Labour (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Station wagon</td>
<td>1</td>
<td>0.5</td>
<td>0.1</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>4-wheel drive</td>
<td>1.5</td>
<td>0.6</td>
<td>0.2</td>
<td>0.15</td>
<td>1.5</td>
</tr>
<tr>
<td>hatchback</td>
<td>0.8</td>
<td>0.7</td>
<td>0.2</td>
<td>0.2</td>
<td>1.1</td>
</tr>
<tr>
<td>sedan</td>
<td>0.9</td>
<td>0.6</td>
<td>0.25</td>
<td>0.3</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Construct a matrix which can be used to express the factory input as a linear function of the factory output.

Problems 8.6

60. [R] Let $\text{id}_{\mathbb{R}^2}$ be the identity map for $\mathbb{R}^2$. Find a matrix representation of this map with respect to standard bases in domain and codomain.
61. [H] Let \( \text{id}_{\mathbb{R}^2} \) be the identity map for \( \mathbb{R}^2 \). Find a matrix representation of this map with respect to the domain basis \( \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} \right\} \) and the codomain basis \( \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} \right\} \).

62. [H] A linear mapping \( G : \mathbb{P}_2 \to \mathbb{P}_2 \) has matrix representation

\[
A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}
\]

with respect to the standard basis \( \{1, x, x^2\} \) in both domain and co-domain. Find \( G(p) \), where \( p(x) = -3 + x + 5x^2 \).

63. [H] For each of the linear maps in Questions 48 to 51, find a matrix which represents the linear map for standard bases in the domain and codomain.

64. [H] Using your results of the previous question or otherwise, find the kernel, nullity, image and rank of the linear maps in Questions 48 to 51.

65. [H] For the linear map \( T : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_n(\mathbb{R}) \) defined by \( T(p) = q \), where

\[
q(x) = x^2 \frac{d^2p}{dx^2} - 3x \frac{dp}{dx} + 3p(x) \quad \text{for} \quad x \in \mathbb{R},
\]

find a matrix which represents \( T \) with respect to standard bases in domain and codomain. Hence, or otherwise, find the kernel and nullity of \( T \).

66. [H] Let \( V \) be a vector space and let \( B = \{u_1, u_2, u_3\} \) be an orthonormal basis for \( V \). Let \( a \in V \) be a vector whose coordinate vector with respect to \( B \) is \( \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \). Let \( \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} \) be the coordinate vector of \( a \) with respect to the basis \( B' = \{u'_1, u'_2, u'_3\} \) given by

\[
\begin{align*}
u'_1 &= \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3, \\
u'_2 &= -\frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_3, \\
u'_3 &= -u_2 
\end{align*}
\]

Show that \( B' \) is an orthonormal basis, then show that the rule \( T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} \) is a linear map from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \), and find a matrix representation for this function with respect to standard bases for \( \mathbb{R}^3 \) in domain and codomain. Finally, show that the matrix you have constructed is a matrix representation of the identity map \( \text{id}_V : V \to V \) with respect to the basis \( B \) in domain and \( B' \) in codomain.
67. Suppose $T$ is the linear transformation with matrix with respect to standard bases given by
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & -4 & 6 & -2 \\
-1 & 2 & -3 & -1
\end{pmatrix}.
\]
Find the matrix of $T$ with respect to the bases
\[
B_1 = \begin{Bmatrix}
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{Bmatrix}
\]
in the domain, and
\[
B_2 = \begin{Bmatrix}
\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{Bmatrix}
\]
in the codomain. Can you explain where these bases have come from?

68. Suppose $T$ is the linear transformation with matrix with respect to standard bases given by
\[
\begin{pmatrix}
3 & 0 & 0 \\
0 & -4 & -1 \\
0 & 6 & 3
\end{pmatrix}.
\]
Find the matrix of $T$ with respect to the basis
\[
B = \begin{Bmatrix}
\begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{Bmatrix}
\]
in both domain and codomain. Can you explain what is going on here?

Problems 8.7

69. Let $T : V \rightarrow W$ and $S : V \rightarrow W$ be linear maps. Let $B_V$ be a basis for $V$ and $B_W$ be a basis for $W$ and let $A$ and $B$ be the matrices representing $T$ and $S$ with respect to the bases $B_V$ and $B_W$. Prove that $A + B$ is the matrix representing the sum function $T + S$ with respect to the bases $B_V$ and $B_W$.

70. Let $T : U \rightarrow V$ and $S : V \rightarrow W$ be linear maps. Let $B_U$, $B_V$ and $B_W$ be bases for $U$, $V$ and $W$ respectively. Let $A$ be the matrix representing $T$ with respect to bases $B_U$ and $B_V$ and let $B$ be the matrix representing $S$ with respect to bases $B_V$ and $B_W$. Prove that the matrix product $BA$ is the matrix which represents the composition function $S \circ T : U \rightarrow W$ with respect to the bases $B_U$ and $B_W$. 

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Problems 8.8

71. [H] Let $V = C[\mathbb{R}]$, the vector space of all continuous real-valued functions on $\mathbb{R}$. Let

$$B = \{e^x, (x - 1)e^x, (x - 1)(x - 2)e^x\} \subseteq V$$

a) Prove that $B$ is linearly independent.

b) Let $W = \text{span} (B)$, and let $D : W \to W$ denote the linear transformation

$$D(f) = f',$$

where $f'$ is the derivative of $f$.

i) Find the matrix for $D$ with respect to the ordered basis $B$ of $W$.  

ii) Find the matrix for the linear transformation $T = D \circ D$.

iii) Hence or otherwise, prove that for every $g \in W$, there exists $f \in W$ such that

$$f'' = g.$$

72. [H] For a field $\mathbb{F}$, define $T : M_{22}(\mathbb{F}) \to \mathbb{F}^3$ by

$$T \left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{22} \\ a_{12} - a_{21} \\ 3a_{11} + a_{12} \end{bmatrix}.$$ 

a) Show that $T$ is a linear transformation.

b) Find the kernel of $T$ and the nullity of $T$.

c) Find the rank of $T$.

d) Is $T$ one-to-one (injective)? Give a brief reason for your answer.

e) Find the matrix of $T$ with respect to the standard bases of $M_{22}(\mathbb{F})$ and $\mathbb{F}^3$. 
Chapter 9

EIGENVALUES AND EIGENVECTORS

9.1 Definitions and examples
9.2 Eigenvectors, bases, and diagonalisation
9.3 Eigenvalues of matrices with special structure
9.4 Applications of eigenvalues and eigenvectors
9.5 Markov systems
9.6 Eigenvalues and Matlab
Problems for Chapter 9

Problems 9.1

1. [R] Let
   \[ A = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}, \]
   and let \( e_1 \) and \( e_2 \) be the standard basis vectors for \( \mathbb{R}^2 \).
   
   a) Write down the eigenvalues and eigenvectors of \( A, B \) and \( C \).
   
   b) Draw a sketch of \( e_1, Ae_1, e_2, Ae_2 \). Then, for some vector \( x \) which is not parallel to either \( e_1 \) or \( e_2 \), draw a sketch of \( x \) and \( Ax \).
   
   c) Repeat part (b) for the matrix \( B \). Comment on any differences you observe between the results for \( A \) and \( B \).
   
   d) Repeat part (b) for the matrix \( C \). Again comment on any differences you observe between the results for \( A \) and \( C \).
   
   e) For \( x \neq 0 \), prove algebraically that \( Ax \) is parallel to \( x \) if and only if \( x \) is parallel to either \( e_1 \) or \( e_2 \), that \( Bx \) is parallel to \( x \) for all \( x \) and that \( Cx \) is parallel to \( e_1 \) for all \( x \).

2. [R] Show that the vector \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is an eigenvector of the matrix \( \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix} \) and find the corresponding eigenvalue.

3. [H] Let \( A \) be a fixed 3 × 3 matrix and define a linear map \( T : M_{33} \to M_{33} \) by \( T(X) = AX \). If \( \lambda \) is a real eigenvalue of \( T \) corresponding to an invertible eigenvector \( X \), find \( \lambda \) in terms of \( \text{det}(A) \).

4. [H] Let \( T \) be the linear map which reflects vectors in \( \mathbb{R}^2 \) about the line \( y = x \).
   
   a) Explain why \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) are eigenvectors of \( T \) and give their corresponding eigenvalues.
   
   b) Find the matrix \( A \) such that \( Tx = Ax \) for all \( x \in \mathbb{R}^2 \).

5. [R] Find the eigenvalues and eigenvectors for
   
   a) \( A = \begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix} \), \quad b) \( A = \begin{pmatrix} -5 & 2 \\ -6 & 3 \end{pmatrix} \).

6. [H] For each of the matrices in the preceding question find two independent eigenvectors \( v_1 \) and \( v_2 \). On one diagram sketch the lines
   
   \( \ell_1 = \{ x : x = \mu v_1, \mu \in \mathbb{R} \} \)
   
   \( \ell_2 = \{ x : x = \mu v_2, \mu \in \mathbb{R} \} \)
and the parallelogram

\[ P = \{ \mathbf{x} : \mathbf{x} = \mu_1 \mathbf{v}_1 + \mu_2 \mathbf{v}_2, \quad 0 \leq \mu_1 \leq 1, \ 0 \leq \mu_2 \leq 1 \}. \]

Then identify and sketch (on a separate diagram)

\[ \{ \mathbf{y} : \mathbf{y} = A \mathbf{x}, \ \mathbf{x} \in \ell_1 \} \]
\[ \{ \mathbf{y} : \mathbf{y} = A \mathbf{x}, \ \mathbf{x} \in \ell_2 \} \]
\[ \{ \mathbf{y} : \mathbf{y} = A \mathbf{x}, \ \mathbf{x} \in P \}. \]

Describe the linear mapping \( T \mathbf{x} = A \mathbf{x} \) geometrically.

7. \([\text{R}]\) Find the eigenvalues and eigenvectors of the following matrices. In each case, note if the eigenvalues are real, occur in complex conjugate pairs, or are general complex numbers. Also note if the eigenvectors form a basis for \( \mathbb{C}^2 \).

a) \( \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \), b) \( \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \), c) \( \begin{pmatrix} 3 & 5 \\ 0 & -6 \end{pmatrix} \).

d) \( \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix} \), e) \( \begin{pmatrix} 4 & 2i \\ 2i & 6 \end{pmatrix} \), f) \( \begin{pmatrix} 4 & -2i \\ 2i & 6 \end{pmatrix} \).

8. \([\text{H}]\) Show that the eigenvalues of a square row-echelon form matrix \( U \) are equal to the diagonal elements of the matrix. (A square row-echelon form matrix is also called an upper triangular matrix).

9. \([\text{R}]\) Find the eigenvalues and eigenvectors of the row-echelon matrix

\[ U = \begin{pmatrix} 2 & -4 & 1 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}. \]

10. \([\text{R}]\) Find the eigenvalues and eigenvectors of the following matrices.

a) \( A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \), b) \( B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 6 & 3 \end{pmatrix} \).

**Problems 9.2**

11. \([\text{R}]\) For each of the matrices in Questions 7, 9 and 10, decide if the matrix is diagonalisable, and if it is find an invertible matrix \( M \) and a diagonal matrix \( D \) such that \( D = M^{-1}AM \).

12. \([\text{H}]\) Show that if \( \lambda \) is an eigenvalue of \( A \) then \( \lambda \) is also an eigenvalue of the matrix \( A' = B^{-1}AB \), where \( B \) is any invertible matrix. Also show that if \( \mathbf{v} \) is an eigenvector of \( A \) for eigenvalue \( \lambda \) then \( B^{-1} \mathbf{v} \) is an eigenvector of \( A' \) for eigenvalue \( \lambda \).
13. [H] Show that if $\lambda$ is an eigenvalue of $A$ then $\lambda$ is also an eigenvalue of $A^T$.

**HINT:** Use the characteristic equation and the properties of determinants.

14. [H] Let $A$ be an $n \times n$ matrix. Let $T_A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the linear transformation defined by

$$T_A(x) = Ax \quad \text{for} \quad x \in \mathbb{C}^n.$$ 

Let the columns of an $n \times n$ matrix $B$ be an ordered basis for $\mathbb{C}^n$. Show that the matrix representing $T_A$ with respect to the basis formed by the columns of $B$ is $B^{-1}AB$.

**HINT.** The method used in Example ?? of Section 8.6 might be helpful.

**NOTE.** Modern methods of finding eigenvalues search for a change of basis which makes $A' = B^{-1}AB$ into an upper triangular matrix. As shown in Question 8 the eigenvalues are then the diagonal elements of the upper triangular matrix. The actual algorithms for finding the change of basis are complicated.

15. [H] Let $T : V \rightarrow V$ be linear. Show that if $B$ is any basis for $V$ and $A$ is the matrix representing $T$ with respect to the basis $B$ in both domain and codomain of $T$ then the eigenvalues of $T$ and $A$ are the same. What is the relation between the eigenvectors of $T$ and $A$?

### Problems 9.3

16. [R] Find an orthogonal matrix $Q$ which diagonalises \( \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \).

17. [R] Show that \( \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \) are eigenvectors of

$$A = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -2 \\ 2 & -2 & 3 \end{pmatrix}$$

and hence find an orthogonal matrix $Q$ which diagonalises $A$.

### Problems 9.4

18. [R] Let $A = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$. Diagonalise $A$ and hence find $A^5$.

19. [R] Let $A = \begin{pmatrix} 0 & 3 \\ 8 & 2 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of $A$. 

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b) Find matrices $P$ and $D$ such that

$$A = PDP^{-1}.$$  

c) Write down an expression for $A^n$ in terms of $P$ and $D$. Hence evaluate $A^nP$.

20. [R] A first-order linear difference equation (often called a first-order linear recurrence relation) is an equation of the form

$$x(k + 1) = Ax(k), \quad \text{where} \quad k = 0, 1, 2, \ldots ,$$

and where $A$ is a fixed matrix.

The solution of this equation is

$$x(k) = A^k x(0),$$

as you can check by direct substitution. For the diagonalisable matrices of Questions 7, 9 and 10, find $A^k$ and hence evaluate $x(k)$.

21. [R] For each of the diagonalisable matrices of Questions 5, 9 and 10 find general solutions of the differential equations

$$\frac{dy}{dt} = Ay.$$  

22. [H] Solve

$$\frac{dy}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} y.$$  

23. [R] a) Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

b) Hence solve the system of differential equations:

$$\begin{cases} \frac{dx_1}{dt} = 2x_1 + 3x_2, \\ \frac{dx_2}{dt} = x_1 + 4x_2. \end{cases}$$  

24. [R] Solve the following systems of differential equations, given that $x(0) = y(0) = 100$.

a) $$\begin{cases} \frac{dx}{dt} = 5x - 8y, \\ \frac{dy}{dt} = x - y \end{cases}$$  

b) $$\begin{cases} \frac{dx}{dt} = 3x - 15y, \\ \frac{dy}{dt} = x - 5y \end{cases}$$  

25. [R] Solve the following second-order linear differential equations with constant coefficients by the “calculus method” and by the matrix method and compare your answers.

a) $$5\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 0.$$  

b) \( \frac{d^2y}{dt^2} - 16y = 0 \).

26. [R] What happens if you try to solve the second-order equation

\[
\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0
\]

by the matrix method?

27. [H] Consider the second-order linear differential equation

\[
a \frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0,
\]

where \( a, b, c \in \mathbb{R} \) and \( a \neq 0 \).

a) Assume that the solutions to the characteristic equation \( a\lambda^2 + b\lambda + c = 0 \) for this second-order differential equation are distinct. By making the substitutions \( y_1 = y \) and \( y_2 = \frac{dy_1}{dt} \), convert the differential equation into a system of first-order linear differential equations

\[
\frac{dy}{dt} = Ay, \quad \text{where} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

b) Using matrix methods, show that the general solution of this system is

\[
y = \alpha_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + \alpha_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.
\]

Compare this solution with that obtained using the usual “calculus method” of solving the original second-order linear differential equation.

28. [H] A radioactive isotope \( A \) decays at the rate of 2% per century into a second radioactive isotope \( B \), which in turn decays at a rate of 1% per century into a stable isotope \( C \).

a) Find a system of linear differential equations to describe the decay process. If we start with pure \( A \), what are the proportions of \( A \), \( B \), and \( C \) after 500 years, after 1000 years, and after 1000000 years?

First solve this problem using matrix methods, and then try to solve the problem directly by solving the original two differential equations in the right order.

b) Explain how the problem would be different if the rates of decay of \( A \) and \( B \) were both 2% per century.

29. [H] Algebraically consider the behaviour of \( x(k) \) as \( k \to \infty \), where

\[
x(k+1) = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} x(k)
\]

and \( x(k) \) is a probability vector.
30. \textbf{[H]} There are 3 mathematics lecturers $A$, $B$ and $C$ who are teaching parallel streams in algebra to a total of 900 students. At the first lecture equal numbers go to each lecture group. After each lecture a certain percentage of the students in each group decide to stay with the same lecturer while the remaining percentage divide evenly among the other two lecturers for the next lecture. If 98% of $A$’s students stay with $A$ each time, 96% of $B$’s students stay with $B$ and 94% of $C$’s students stay with $C$, find the numbers of students in each group in the 12th lecture and in the 24th lecture. Make the assumption that no students stop attending lectures.

**HINT.** Set up a model as a difference equation of the type given in Question 20. You may use \textsc{Matlab} to find all eigenvalues and eigenvectors. Alternatively, if you wish to solve the problem by hand calculations, you will need to know that one of the eigenvalues is 1.

**NOTE.** This problem is an example of a Markov chain process. Markov chain processes are important in many areas of mathematics and its applications, such as statistics, psychology, finance, economics, operations research, queuing theory, inventory theory, diffusion processes, theory of epidemics etc.

31. \textbf{[H]} Repeat the previous question on the following assumptions. After each lecture, 1% of each group stop attending lectures altogether, and the remaining percentage either stay with the same lecturer or divide equally among the other two lecturers for the next lecture. If 97% of $A$’s students stay with $A$ each time, 95% of $B$’s students stay with $B$ and 93% of $C$’s students stay with $C$, find the numbers of students in each group in the 12th lecture and in the 24th lecture. Also find the total number of students attending lectures in the 12th lecture and in the 24th lecture.

32. \textbf{[H]} Consider a modified version of the population dynamics model of Example ?? of Section 8.5, in which all females are assumed to die at age 74 instead of at age 89, as in the model given. Use eigenvalues and eigenvectors to solve this modified model, given that there are one million females in each age group at January 1, 1970. What happens to the population for large values of $k$?

**NOTE.** You will need to use \textsc{Matlab} to find the eigenvalues and eigenvectors of the matrix $A$, and you may also use \textsc{Matlab}, if you wish, to carry out all other matrix manipulations required to solve the problem.

### Problems 9.5

33. \textbf{[H]} Let $A$ be a $2 \times 2$ matrix with the property that all its entries are non-negative and both its columns sum to 1. Show that $\lambda_1 = 1$ is always an eigenvalue for $A$, and that if $\lambda_2$ is another eigenvalue of $A$ then $-1 \leq \lambda_2 \leq 1$.

34. \textbf{[H]} Show that the matrix of the original population dynamics model of Example ?? of Section 8.5 is not diagonalisable.

**HINT.** Use \textsc{Matlab} to find the eigenvalues, and then show that the eigenvalue $\lambda = 0$ has multiplicity 2 and that $\dim(\ker(A)) = 1$, i.e., a basis for the kernel of $A - 0I$ consists of
one vector.

Note. The original population dynamics model can be solved by a generalisation of the eigenvalue-eigenvector methods which makes use of “Jordan forms”, and is covered in our second year linear algebra subjects.
ANSWERS TO SELECTED PROBLEMS

Chapter 6

1. | $x \in \mathbb{N}$ | $x \in \mathbb{Z}$ | $x \in \mathbb{Q}$ | $x \in \mathbb{R}$ |
---|---|---|---|---|
| a) | - | -25 | -25 | -25 |
| | 3 | 3 | 3 | 3 |
| | - | -3 | -3 | -3 |
| | - | - | $-\frac{10}{3}$ | $-\frac{10}{3}$ |
| b) | 1 | 1, -5 | 1, -5 | 1, -5 |
| | 5 | 5 | $5, \frac{3}{2}$ | $5, \frac{3}{2}$ |
| | - | - | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| c) | $3j, j \in \mathbb{N}$ | $3j, j \in \mathbb{Z}$ | $3j, j \in \mathbb{Z}$ | $3j, j \in \mathbb{Z}$ |
| | 0 | 0 | 0 | 3$\pi$, $k \in \mathbb{Z}$ |

2. No.

3. Yes. The set \{0\} and the empty set $\emptyset = \{\}$.

4. Yes.

5. $3z = 6 + 9i$, $z^2 = -5 + 12i$, $z + 2w = 7i$, $z(w + 3) = -2 + 10i$, $\frac{z}{w} = \frac{1}{5}(4 - 7i)$, $w = \frac{1}{13}(4 + 7i)$.

6. a) $\frac{1}{5}(3 - i)$, b) $-\frac{1}{2}(1 - i)$.

7. a) $a^2 - b^2 + 2abi$, b) $\frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$, c) $\frac{1}{(a - 1)^2 + b^2} (a^2 - 1 + b^2 - 2ib)$.
8. a) \( \frac{1}{3} (-1 \pm \sqrt{3}i) \), b) \(-1 \pm \sqrt{2}i\), c) \(3 \pm i\), d) \( \frac{1}{2} (3 \pm \sqrt{3})\), e) \( \pm i, \pm 2i\).

10. 16

11. \( \frac{8abi(a^2 - b^2)}{(a^2 + b^2)^2} \)

12. 

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<thead>
<tr>
<th>( z )</th>
<th>( \text{Re}(z) )</th>
<th>( \text{Im}(z) )</th>
<th>( \overline{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1 + i)</td>
<td>(-1)</td>
<td>1</td>
<td>(-1 - i)</td>
</tr>
<tr>
<td>(2 + 3i)</td>
<td>2</td>
<td>3</td>
<td>(2 - 3i)</td>
</tr>
<tr>
<td>(2 - 3i)</td>
<td>2</td>
<td>(-3)</td>
<td>(2 + 3i)</td>
</tr>
<tr>
<td>(\frac{2-i}{1+i})</td>
<td>(\frac{1}{2})</td>
<td>(-\frac{3}{2})</td>
<td>(1+3i)</td>
</tr>
<tr>
<td>(\frac{1}{(1+i)^2})</td>
<td>0</td>
<td>(-\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
</tbody>
</table>

13. \(-3 + 4i, \quad \frac{11}{25} - \frac{2}{25}i\).

14. \(z = 2 + 3i, \quad w = -1 + 2i\).

17. b) \(z^2 - 6z + 13\)

18. 

| \( z \) | \(|a|\) | \(\text{Arg}(z)\) | Polar Form |
|-------|------|---------|------------|
| \(6 + 6i\) | \(6\sqrt{2}\) | \(\frac{\pi}{4}\) | \(6\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)\) |
| \(-4\) | 4 | \(\pi\) | \(4 \cos \pi + i \sin \pi\) |
| \(\sqrt{3} - i\) | 2 | \(-\frac{\pi}{6}\) | \(2 \left( \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right)\) |
| \(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\) | 1 | \(-\frac{3\pi}{4}\) | \(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4}\) |
| \(-7 + 3i\) | \(\sqrt{34}\) | \(\alpha\) | \(\sqrt{34} \left( \cos \alpha + i \sin \alpha \right)\) |

Here \(\alpha = \pi - \tan^{-1} \frac{3}{7}\).

19. \(\sqrt{234}, \quad -1\).

20. \(n = 4\)

21. a) \(\frac{3}{2} (1 + \sqrt{3}i)\), b) \(\frac{3}{2} (-\sqrt{3} + i)\), c) \(-\frac{3}{2} (1 + \sqrt{3}i)\), d) \(\frac{3}{2} (\sqrt{3} - i)\), e) \(\frac{3}{2} \left( \sqrt{2 + \sqrt{2}} + i \sqrt{2 - \sqrt{2}} \right)\) (Double angle formula used).

27. \(64, \quad -(1 + \sqrt{3})i, \quad \frac{1 + \sqrt{3}}{2} + \sqrt{3} - 1 \cdot \frac{i}{2}\).

28. \(\frac{7}{2}\).
ANSWERS

29. $\pi$.

30. $\text{Arg} (-1 + i) = \frac{3\pi}{4}$; $\text{Arg} (-\sqrt{3} + i) = \frac{5\pi}{6}$;

$\text{Arg} ((-1 + i) (-\sqrt{3} + i)) = -\frac{5\pi}{12}$; $\text{Arg} \left( \frac{-1 + i}{-\sqrt{3} + i} \right) = -\frac{\pi}{12}$.

31. $\sin \frac{7\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$.

32. $zw = 2\sqrt{2}e^{i\pi/12} = 2\sqrt{2} \left[ \cos \left( \frac{\pi}{12} \right) + i \sin \left( \frac{\pi}{12} \right) \right]$; $z^0 = -512$; $\left( \frac{z}{w} \right)^{12} = 64e^{i\pi} = -64$.

33. a) $16 (-\sqrt{3} + i)$, b) $-i$, c) $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

34. a) $\pm (5 - 2i)$, b) $\pm (3 + 5i)$, c) $\pm (7 + 5i)$.

35. b) $\sqrt{2}e^{i\pi/12} = \frac{1}{2} ((\sqrt{3} - 1) + i (\sqrt{3} + 1))$, c) $\frac{1}{\sqrt{2}}(1 + 13i)$.

37. a) $2 + i$, $1 - i$; b) $4 + i$, $3 - 2i$; c) $1 - 2i$, $-5 + 3i$.

38. $e^{i\pi/7}$, $e^{3i\pi/7}$, $e^{5i\pi/7}$, $e^{i\pi}$, $e^{-i\pi/7}$, $e^{-3i\pi/7}$, $e^{-5i\pi/7}$.

39. $e^{in\pi/12}$ for $n = -11, -7, -3, 1, 5, 9$.

40. $2e^{in\pi/15}$ for $n = -13, -7, -1, 5, 11$.

48. a) Real part = $\cos(2\theta)$. Imaginary part = $\sin(2\theta)$.

49. a) $\cos 6\theta = \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta$

$\sin 6\theta = 6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta$

b) $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$

50. $\sin 7\theta = 7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta$

$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta$.

51. a) $\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta})$.

b) $\cos^6 \theta = \frac{1}{32} (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10)$.

52. $\sin^5 \theta = \frac{1}{16} (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$

$\int \sin^5 \theta d\theta = \frac{1}{16} \left( -\frac{1}{5} \cos 5\theta + \frac{5}{3} \cos 3\theta - 10 \cos \theta \right) + C$,
\[ \cos^4 \theta = \frac{1}{8} [3 + 4 \cos(2\theta) + \cos(4\theta)] \]
\[ \int \cos^4 \theta \, d\theta = \frac{1}{8} \left[ 3\theta + 2\sin(2\theta) + \frac{1}{4}\sin(4\theta) \right] + C. \]

55. a) \( \cos 5\theta = 16x^5 - 20x^3 + 5x \)
   d) \(-1, \cos \frac{\pi}{5}, \cos \frac{3\pi}{5}, \cos \frac{7\pi}{5}, \cos \frac{9\pi}{5} \).

56. The sum is \( n \) when \( k \) is an integer multiple of \( n \) and 0 otherwise.

58. \[ \frac{\sin \left( \frac{1}{2} (n + 1)\theta \right) \sin \left( \frac{1}{2} n\theta \right)}{\sin \frac{1}{2} \theta} \]

59. a) \[ \frac{9e^{i\theta}}{9 + e^{2i\theta}}. \]

60. a) \( |z - i| \leq 2 \)
   b) \( |z - i| \leq 2 \) or \( -\frac{\pi}{3} \leq \text{Arg}(z - i) \leq \frac{2\pi}{3} \)
   c) \( |z| \geq 2 \) and \( |\text{Im}(z)| \leq 3 \)
   d) \( y \leq x \)
e) The real axis

f) $|z - 1 - i| < 1 \& -\frac{\pi}{4} < \text{Arg}(z - 1 - i) \leq \frac{\pi}{2}$

![Diagram of the real axis]

![Diagram of a circle centered at 1 + i]

![Diagram of a circle and an ellipse]

h) Ellipse: $\left(\frac{x}{2\sqrt{2}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

61. a) $\text{Re}(z) \geq 3 \text{Im}(z)$ and $|z - (3 + i)| > 2$

![Graph illustrating the circle and the ellipse with labeled points and equations]
b) $|z - i| < |z + i|$ and \(-\frac{\pi}{6} \leq \text{Arg}(z - i) \leq \frac{\pi}{6}\)

\[\text{Im}(z) > 0\]

62. a) $\text{Im}(z) > -4$ and $|z - 1 - i| \geq 3$

b) Yes.

63. $|z - x| \geq |z - \text{Re}(z)|$
64. a) \( w = e^{i\alpha}, \ -\pi < \alpha \leq \pi \)  

\[ |z - e^{i\alpha}| \geq |z - e^{i\theta}|, \quad \theta = \text{Arg}(z) \]

65. a) 742, b) 129, c) 1 + 9i.

66. \( p(z) = (z - 2)(2z - 5)(z + 3) \).

67. \( p(z) = (z - 1)(z + 1)(z + 2)(z + 4) \).

68. a) \( (z - e^{-\frac{5\pi}{6}})(z - e^{\frac{3\pi}{4}})(z - e^{\frac{2\pi}{3}})(z - e^{-\frac{\pi}{4}})(z - e^{-\frac{2\pi}{3}})(z - e^{-\frac{5\pi}{6}}) \)

b) \( (z - \sqrt{2}e^{\frac{\pi}{4}})(z - \sqrt{2}e^{-\frac{3\pi}{4}})(z - \sqrt{2}e^{-\frac{\pi}{4}})(z - \sqrt{2}e^{-\frac{3\pi}{4}})(z - \sqrt{2}e^{-\frac{5\pi}{6}})(z - \sqrt{2}e^{-\frac{7\pi}{6}}) \)

69. a) \((x - 1)(x + 1)(x^2 + 1)(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) \)

b) \((x^2 + 2)(x^2 + \sqrt{6}x + 2)(x^2 - \sqrt{6}x + 2) \)

70. \((z^2 + 2z + 2)(z^2 - 2z + 2) \)

71. \((z - e^{-i\pi/8})(z - e^{i3\pi/8})(z - e^{i17\pi/8})(z - e^{i11\pi/8}) \)

72. a) \( e^{-\frac{5\pi}{6}}, \quad e^{\frac{\pi}{4}}, \quad e^{-\frac{\pi}{6}}, \quad e^{\frac{\pi}{6}}, \quad e^{\frac{4\pi}{6}}, \quad e^{\frac{5\pi}{6}} \)

b) Note that the solutions are evenly spaced around the unit circle centred on 0.

c) \( (z - e^{-\frac{5\pi}{6}})(z - e^{\frac{\pi}{6}})(z - e^{-\frac{\pi}{6}})(z - e^{\frac{\pi}{6}})(z - e^{-\frac{5\pi}{6}})(z - e^{\frac{5\pi}{6}}) \)

d) \((z^2 + 1)(z^2 + \sqrt{3}z + 1)(z^2 - \sqrt{3}z + 1) \).
73. a) $e^{i\pi/4}, e^{i\pi/2}, e^{i3\pi/4}, e^{-i\pi/4}, e^{-i\pi/2}, e^{-3\pi/4}$.  
b) $(z - e^{i\pi/4})(z - e^{-i\pi/4})(z - e^{i3\pi/4})(z - e^{-i3\pi/4})(z - e^{i\pi/2})(z - e^{-i\pi/2})$.  
c) $(z^2 - \sqrt{2}z + 1)(x^2 + \sqrt{2}z + 1)(z^2 + 1)$.

74. a) $(z - e^{2i\pi/5})(z - e^{-2i\pi/5})(z - e^{4i\pi/5})(z - e^{-4i\pi/5})$.  
b) $\left(z^2 - 2z \cos \left(\frac{2\pi}{5}\right) + 1\right)\left(z^2 - 2z \cos \left(\frac{4\pi}{9}\right) + 1\right)$.  

75. a) $(t + 1 - i)(t + 1 + i)(t - 2)(t + 1)(t + i)(t - i)$.  
b) $(t^2 + 2t + 2)(t - 2)(t + 1)(t^2 + 1)$.

76. $1 + i, 1 - i, \frac{3\sqrt{5}}{2}(-1 + i\sqrt{3}), \frac{3\sqrt{5}}{2}(-1 - i\sqrt{3})$.

77. a) $(z^2 + z + 1)(z^6 + z^3 + 1)$.  
b) $e^{\pm2i\pi/9}, e^{\pm4i\pi/9}, e^{\pm8i\pi/9}$.

79. a) One of the roots is $(-2 + 2i)^{1/3} + (-2 - 2i)^{1/3}$.

80. d) $-2, 2\sqrt{2}\cos \frac{\pi}{12}, 2\sqrt{2}\cos \frac{7\pi}{12}$.

81. a) 1;  
b) $-1, \frac{5}{7}, \frac{1}{4}$;  
c) 4, $\pm\frac{1}{5}$.

83. a) Unstable.  
b) Neither decays nor grows.  
c) Unstable.

84. a) Stable.  
b) Unstable.  
c) Neither decays nor grows.

93. a) $2^{123}i$.  
b) $i$.  
c) No.

Chapter 7

3. a) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$.

7. Axioms 1, 4, 5, 6, 10 are satisfied, others are not. It is not a vector space.

8. a) $2\mathbf{v} = (1 + 1)\mathbf{v} = 1\mathbf{v} + 1\mathbf{v} = \mathbf{v} + \mathbf{v}$. Identify the axioms that have been used.  
b) Use induction.

9. For part 5: If $\lambda\mathbf{v} = \mu\mathbf{v}$ then $\lambda\mathbf{v} - \mu\mathbf{v} = \mathbf{0}$, so $(\lambda - \mu)\mathbf{v} = \mathbf{0}$. But $\mathbf{v} \neq \mathbf{0}$ so, by part 4 of the proposition, $\lambda - \mu = 0$. So $\lambda = \mu$. 

62
10. a) \[
\begin{pmatrix}
3 \\
-4 \\
1
\end{pmatrix}
\]. b) \[
\mathbf{x} = \begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix} + \lambda \begin{pmatrix}
2 \\
-3 \\
-1
\end{pmatrix}
\] for \( \lambda \in \mathbb{R} \).

11. a) \[
\begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix}, \quad \begin{pmatrix}
2 \\
-2 \\
0
\end{pmatrix}
\]. b) \[
\mathbf{x} = \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} + \lambda \begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix}
\] for \( \lambda \in \mathbb{R} \).

c) \[
\mathbf{x} = \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} + \lambda \begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix} + \mu \begin{pmatrix}
2 \\
-3 \\
-1
\end{pmatrix}
\] for \( \lambda, \mu \in \mathbb{R} \). d) \[
\begin{pmatrix}
4 \\
4 \\
0
\end{pmatrix}
\]. e) \[
\begin{pmatrix}
4 \\
0
\end{pmatrix} \cdot \left( \mathbf{x} - \begin{pmatrix}
1 \\
1
\end{pmatrix} \right) = 0.
\] f) \( x + y - 2 = 0 \).

12. \( \sqrt{3}, \sqrt{6}, 4, \frac{2\sqrt{2}}{3} \).

13. a) \( 2x + 4y - z = 11 \). b) \[
\begin{pmatrix}
2 \\
-1 \\
-1
\end{pmatrix}, \quad \begin{pmatrix}
4 \\
0 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
2 \\
-2
\end{pmatrix}
\] are in \( \mathcal{S} \).

14. \[
\begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad \frac{\sqrt{2}}{2}
\].

15. a) \[
\begin{pmatrix}
-1 \\
-2
\end{pmatrix}
\]. b) \[
\begin{pmatrix}
-1 \\
-1
\end{pmatrix}
\]. c) \[
\begin{pmatrix}
1 \\
3
\end{pmatrix}
\].

17. The position vectors of the points on a plane which does not pass through the origin do not form a vector subspace.

18. a) For example, \[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
2 \\
0 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
2 \\
-2
\end{pmatrix}
\] are in \( \mathcal{S} \).

c) The position vectors of the points on a plane which passes through the origin form a vector subspace.

22. Column 1 belongs to \( \mathcal{S} \) as \[
A \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
2 \\
4
\end{pmatrix}.
\] Similar arguments apply for columns 2 and 3.

23. a) No. b) Yes. c) Yes.
26. $W = \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \\ \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$, a copy of $\mathbb{R}^3$.

29. No, $S$ is not a subspace because the zero polynomial is not in it.

30. b) $x^3 + 3x^2 + 3x + 1$.

31. HINT: Suppose that $S_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$ and $S_2 = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R} \right\}$. Show that $S_1$ and $S_2$ are subspaces of $\mathbb{R}^2$ but $S_1 \cup S_2$ is not.

34. a) Yes, $a = 2v_1 - 3v_2 + v_3$.  
   b) Yes.

35. a) No.  
   b) No, $3b_2 - 2b_3 = 0$.  
   span($v_1, v_2, v_3$) is a plane in $\mathbb{R}^3$.

36. a) Yes, $a = v_1 - v_2 - 2v_3$  
   b) No, $b \in$ span($v_1, v_2, v_3$) if and only if $b_1 - 2b_2 + b_3 = 0$.  
   The span is a 3-dimensional subspace in $\mathbb{R}^4$.

37. Yes, $b = -10v_1 + 12v_2 + 16v_3$.

38. No, span($v_1, v_2, v_3$) is plane in $\mathbb{R}^3$ given by $5b_1 - 4b_2 + b_3 = 0$.

39. Yes. $v = 3v_1 - v_2 - 2v_3$, where $v_1, v_2, v_3$ are the columns of $A$.

40. No.

41. Yes.

44. Linearly dependent, coplanar.

45. Linearly independent, not coplanar.

46. Set containing $0$ is linearly dependent as coefficient of zero vector can be varied to make linear combination non-unique.

47. b) $v_3 = v_1 + v_2$.  
   e) span($S$) is plane through origin parallel to $v_1$ and $v_2$.

48. b) $v_4 = -2v_1 + 2v_2 + v_3$  
   e) span($S$) = $\mathbb{R}^3$.

49. No. $-2 - x + 5x^2 = (1 - x + 2x^2) + 3(-1 + x^2)$.

50. No, it is impossible to return to origin.
51. Yes, it is possible to return to origin.

53. Yes, it is a basis.

54. A basis for $W$ is \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \). $\text{Dim}(W) = 2$.

56. a) True. b) False. c) False. d) True. e) False, True, False, False.
   f) False. g) True. h) False. i) True. j) True.

57. a) $n \leq \ell$ b) No relation. c) $n \geq \ell$. d) $n = \ell$.

59. b) \( \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \).

60. A basis for $\text{col}(A)$ is \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \). $\text{Dim}(\text{col}(A)) = 3$.

61. A basis for $\text{col}(A)$ is \( \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \\ 4 \end{pmatrix} \right\} \). $\text{Dim}(\text{col}(A)) = 3$.

62. A basis is \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 2 \\ 4 \\ 5 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ -8 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \).

63. a) $B = \{v_1, v_3\}$. b) $x = -4v_1 + 5v_3$. c) $\text{dim}(\text{col}(A)) = 2$.

65. Bases are $\{p_1, p_2, p_4\}$ or $\{p_1, p_3, p_4\}$ or $\{p_2, p_3, p_4\}$. $\{p_1, p_2, p_3\}$ is not a basis; why not?

68. Coordinate vector is \( \begin{pmatrix} -2 \\ 4 \\ 12 \\ 4 \end{pmatrix} \).

69. \( v = \begin{pmatrix} 18 \\ 7 \\ 11 \\ 19 \end{pmatrix} \).
70. \( \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix} \).

71. a) \( \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \), b) \( \begin{pmatrix} -a_1 - a_2 + 2a_3 \\ a_2 \\ -a_1 + a_3 \end{pmatrix} \).

72. a) \( \begin{pmatrix} 12 \\ -8 \\ 21 \end{pmatrix} \); b) \( \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \).

73. c) Coordinates are \( \lambda_1 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_1 = -2\sqrt{2}, \quad \lambda_2 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_2 = 2\sqrt{3}, \quad \lambda_3 = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \mathbf{v}_3 = \sqrt{6}. \) Coordinate vector is \( \begin{pmatrix} -2\sqrt{2} \\ 2\sqrt{3} \\ \sqrt{6} \end{pmatrix} \).

75. a) For example, \( \begin{pmatrix} 10 \\ 0 \end{pmatrix} \), \( \begin{pmatrix} 12 \\ 3 \end{pmatrix} \), \( \begin{pmatrix} -10 \end{pmatrix} \) are in \( S \).

b) \( S \) is not a subspace because the \( 2 \times 2 \) zero matrix is not in it.

76. a) For example, \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \), \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) are in \( S \).

b) Use the Subspace Theorem to prove that \( S \) is a subspace.

80. a) \( \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \), b) \( \begin{pmatrix} \frac{1}{2}(a_{11} + a_{22}) \\ \frac{1}{2}(a_{12} + a_{21}) \\ \frac{1}{2}(-a_{12} + a_{21}) \\ \frac{1}{2}(a_{11} - a_{22}) \end{pmatrix} \).

81. a) \( \begin{pmatrix} -4 \\ 2 \\ -3 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). b) No.

85. Not a subspace.

91. Not a subspace.

93. No.

94. Let \( p \in \mathbb{P}_2 \) be \( p(z) = a_0 + a_1 z + a_2 z^2 \). Then the condition is \( -\frac{3}{2}a_1 + a_2 = 0 \). (An equivalent condition is \( p'(-\frac{1}{3}) = 0 \).)

95. No for Question 93. No for Question 94.
96. No. \( p_3 = -p_1 + 3p_2 \).

97. A basis is \( \{ p_1, p_2, p_3, p_4, 1, z \} \).

98. \[
\begin{pmatrix}
2 \\
-1 \\
0
\end{pmatrix}
\]

99. \[
\begin{pmatrix}
9 \\
17
\end{pmatrix}
\]

100. \[
\begin{pmatrix}
a_0 - a_1 + a_2 \\ a_0 \\ a_0 + a_1 + a_2
\end{pmatrix}
\]

Chapter 8

1. \( S \) is not a linear map as the domain \([-1, 1]\) is not a vector space.

2. a) Linear. b) Linear. c) Not Linear. d) Linear. e) Not Linear.

3. a) Domain \( \mathbb{C} \), codomain \( \mathbb{R} \), linear. b) Domain \( \mathbb{C} \), codomain \( \mathbb{R} \), linear.
   c) Domain \( \mathbb{C} \), codomain \( \mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \} \), not linear.
   d) Domain \( \mathbb{C} - \{0\} \), codomain \( (-\pi, \pi] \), not linear. e) Domain \( \mathbb{C} \), codomain \( \mathbb{C} \), linear.

6. No.

7. \( T \begin{pmatrix}
2 \\
-1 \\
4
\end{pmatrix} = \begin{pmatrix}
21 \\
4 \\
-15 \\
28
\end{pmatrix} \) and \( T \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
x_1 - 3x_2 + 4x_3 \\
2x_1 \\
3x_1 + x_2 - 5x_3 \\
4x_1 + 4x_2 + 6x_3
\end{pmatrix} \).

9. \[
\begin{pmatrix}
0 \\
2 \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} + \begin{pmatrix}
-2 \\
1 \\
-4
\end{pmatrix} + \begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix}, \text{ and so } T \begin{pmatrix}
0 \\
2 \\
1
\end{pmatrix} = \begin{pmatrix}
-1 \\
1
\end{pmatrix}.
\]

11. a) \[
\begin{pmatrix}
3 & -1 \\
2 & 4 \\
-3 & -3 \\
0 & 1
\end{pmatrix}
\]
   b) \[
\begin{pmatrix}
-2 & 0 & 5 & 0 \\
6 & -8 & 0 & 2
\end{pmatrix}
\]
   d) \[
\begin{pmatrix}
1 & 0 & -2 & -4 \\
3 & -4 & -3 & 1
\end{pmatrix}
\]

12. Same as for 11.
13. a) \( A\mathbf{e}_1 = 2\mathbf{e}_1, \ A\mathbf{e}_2 = 0.7\mathbf{e}_2, \ A\mathbf{b} = 4\mathbf{e}_1 + 2.1\mathbf{e}_2. \) (\( \mathbf{e}_1 \) is stretched to twice its length, \( \mathbf{e}_2 \) is compressed to 0.7 of its length and \( \mathbf{b} \) is stretched and rotated.)

d) Notice that \( A\mathbf{b} = 3\mathbf{b} \). This means \( \mathbf{b} \) is stretched to three times its length.

e) Notice that \( A\mathbf{b} = -2\mathbf{b} \). This means the direction of \( \mathbf{b} \) is reversed and it is stretched to twice its length.

15. \[ \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} . \]

16. \( \mathbf{x}' = T(\mathbf{x}) = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \)

17. \( \mathbf{x}' = T(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \)

18. \( \mathbf{q} = A\mathbf{p} \), where diagonal entries of \( A \) are \( a_{ii} = -1 + 2d_i^2/|\mathbf{d}|^2 \) and off-diagonal entries of \( A \) are \( a_{ij} = 2d_i d_j/|\mathbf{d}|^2 \).

19. \( T \) is linear. For \( T(\mathbf{x}) = A\mathbf{x} \), the matrix is

\[
A = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} .
\]

20. \( T(\mathbf{a}) = A\mathbf{a} \), where \( A = \frac{1}{5} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{pmatrix} \).

21. \( S \) is not linear.

23. \[ \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = R_\alpha \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \cos \alpha + a_3 \sin \alpha \\ a_2 \\ -a_1 \sin \alpha + a_3 \cos \alpha \end{pmatrix} . \]

25. a) \( \ker(A) = \{ \mathbf{0} \} \), \( \text{nullity}(A) = 0 \), no basis.

b) Kernel: \( \left\{ \lambda \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\} \), \( \text{nullity}(B) = 1 \).

c) Kernel: \( \left\{ \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} : \lambda \in \mathbb{R} \right\} \), \( \text{nullity}(C) = 1 \).
26. a) \[
\begin{pmatrix}
1 \\
-3 \\
1 \\
0
\end{pmatrix}, \quad \text{nullity}(D) = 1.
\]
b) \(\{0\}\), \(\text{nullity}(E) = 0\).

27. For example \(A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\).

28. a) \(\ker(A) = \{0\}\), \(\text{nullity}(A) = 0\).

b) \(\ker(A) = \left\{ \lambda \begin{pmatrix} 5 \\ 4 \\ 2 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}\), \(\text{nullity}(A) = 1\).

d) \(\ker(A) = \left\{ \lambda \begin{pmatrix} 6 \\ 4 \\ 1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}\), \(\text{nullity}(A) = 1\).

29. For Questions 16, 17 and 18, \(\ker(T) = \{0\}\) and \(\text{nullity}(T) = 0\).

For Question 19, \(\ker(T) = \{x \in \mathbb{R}^3 : x = \lambda b \text{ for } \lambda \in \mathbb{R}\}\). Nullity \((T) = 1\). Kernel is set of all vectors parallel to \(b\).

For Question 20, \(\ker(T) = \{x \in \mathbb{R}^n : b \cdot x = 0\}\). Nullity \((T) = n-1\) (Why?). Kernel is set of all vectors orthogonal to \(b\).

30. b) 1

31. a) \(b \in \text{im}(A)\) as \(A \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 10 \\ 4 \end{pmatrix}\).

b) \(b\) is not in \(\text{im}(A)\) as \(Ax = \begin{pmatrix} 9 \\ -2 \\ -4 \end{pmatrix}\) has no solution.

c) \(b \in \text{im}(A)\), since, for example, \(x = \begin{pmatrix} -10 \\ 12 \\ 16 \\ 0 \end{pmatrix}\) is a solution of \(Ax = b\).

32. a) No conditions, \(\text{im}(A) = \mathbb{R}^3\).

b) \(3b_2 - 2b_3 = 0\).

c) No conditions, \(\text{im}(A) = \mathbb{R}^3\).

33. rank \((A) = 3\). Columns 1,2,3 of \(A\) form a basis for \(\text{im}(A)\).

rank \((B) = 2\). Columns 1,2 of \(B\) form a basis for \(\text{im}(B)\).

rank \((C) = 3\). Columns 1,2,3 of \(C\) form a basis for \(\text{im}(C)\).
rank($D$) = 3. Columns 1,2,4 of $D$ form a basis for im($D$).
rank($E$) = 3. Columns 1,2,3 of $E$ form a basis for im($E$).

35. rank($A$) = 3. Columns 1,3,4 of $A$ form a basis for im($A$).
rank($B$) = 3. Columns 1,3,4 of $B$ form a basis for im($B$).

36. One possible answer is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

37. One possible answer is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

38. a) $\begin{pmatrix} 3 & 4 & -1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. b) $\left\{ \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. 3. c) 1. d) No.

46. $T$ is linear.

50. $T$ is a linear function.

53. $S$ is not linear as $\mathbb{Z}$ is not a vector space.
$T$ is not linear as, for example, $T(1.5) + T(1.5) = 2 + 2 = 4$ while $T(1.5 + 1.5) = T(3) = 3$.

54. $y_L(s) = \frac{s^2 + 9s + 19}{(s+3)^2(s+1)}$.

56. a) $7 - 2x$ c) $\begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

57. $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (2 + i) + (7 - 4i)z + 2z^2 - 3iz^3,$
$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1 - 2x_3) + [(2 + i)x_1 + (4 - 3i)x_2]z + x_2z^2 - 3x_1z^3.$

58. b) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \end{pmatrix}$. c) $\{x, x^2, x^3, x^4\}$. d) The empty set.
59. Let input vector be \( \mathbf{b} = (b_1 \ b_2 \ b_3 \ b_4 \ b_5)^T \), where \( b_1, b_2, b_3, b_4 \) and \( b_5 \) are the amounts of steel, plastics, rubber, glass and labour used respectively. Let the output vector be \( \mathbf{x} = (x_1 \ x_2 \ x_3 \ x_4)^T \), where \( x_1, x_2, x_3, x_4 \) are the numbers of station wagons, 4-wheel drives, hatchbacks and sedans made. Then the factory is represented by the linear map \( T_A : \mathbb{R}^4 \to \mathbb{R}^5 \), where \( T_A(\mathbf{x}) = \mathbf{b} = A\mathbf{x} \) with

\[
A = \begin{pmatrix}
1 & 1.5 & 0.8 & 0.9 \\
0.5 & 0.6 & 0.7 & 0.6 \\
0.1 & 0.2 & 0.2 & 0.25 \\
0.2 & 0.15 & 0.3 & 0.3 \\
1 & 1.5 & 1.1 & 0.9
\end{pmatrix}.
\]

60. The matrix is
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

61. The matrix is
\[
\begin{pmatrix}
-1 & -3 \\
2 & 4
\end{pmatrix}.
\]

62. \(-6 + 4x\)

\[
\begin{pmatrix}
1 & -3 & 0 & 0 \\
0 & 0 & 2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
3 & -1 & 2 & 4
\end{pmatrix}.
\]

63. For 48, \( \begin{pmatrix} 3 & 4 & 0 & 0 \\ 0 & 3 & 8 & 0 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 3 \end{pmatrix} \).

64. 48: \( \text{im}(T) = \{ p \in \mathbb{P}_4(\mathbb{R}) : p(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^3 + \lambda_3 z^4 \text{ for } \lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \} \), (note \( z^2 \) is not in \( \text{im}(T) \)) \( \text{rank}(T) = 4 \), \( \text{ker}(T) = \{0\} \), \( \text{nullity}(T) = 0 \).

49: \( \text{im}(T) = \mathbb{P}_3(\mathbb{R}) \), \( \text{rank}(T) = 4 \), \( \text{ker}(T) = \{0\} \), \( \text{nullity}(T) = 0 \).

50: \( \text{im}(T) = \{ p \in \mathbb{P}_3(\mathbb{R}) : p(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^3 \text{ for } \lambda_0, \lambda_1, \lambda_2 \in \mathbb{R} \} \), \( \text{rank}(T) = 3 \), \( \text{ker}(T) = \{ p \in \mathbb{P}_3(\mathbb{R}) : p(x) = \lambda x^2 \text{ for } \lambda \in \mathbb{R} \} \), \( \text{nullity}(T) = 1 \).

51: \( \text{im}(T) = \{ p \in \mathbb{P}_4(\mathbb{R}) : p(x) = \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 \text{ for } \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \} \), \( \text{rank}(T) = 4 \), \( \text{ker}(T) = \{0\} \), \( \text{nullity}(T) = 0 \).

65. The matrix \( A \) is diagonal with diagonal elements \( a_{kk} = (k - 1)(k - 2) - 3(k - 1) + 3 \) for \( 1 \leq k \leq n + 1 \). The kernel is \( \alpha_1 x + \alpha_2 x^3 \) and the nullity is 2. Note that the kernel is the solution of the homogeneous differential equation.
66. The matrix is \[
\begin{pmatrix}
  \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
  -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
   0 & -1 & 0
\end{pmatrix}.
\]

67. \[
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 0
\end{pmatrix}.
\]

68. \[
\begin{pmatrix}
  2 & 0 & 0 \\
  0 & -3 & 0 \\
   0 & 0 & 3
\end{pmatrix}.
\]

71. b) i) \[
\begin{pmatrix}
  1 & 1 & -1 \\
  0 & 1 & 2 \\
   0 & 0 & 1
\end{pmatrix},
\]
ii) \[
\begin{pmatrix}
  1 & 2 & 0 \\
  0 & 1 & 4 \\
   0 & 0 & 1
\end{pmatrix}.
\]

72. b) \[\lambda \begin{pmatrix} 1 & -3 \\ -3 & 0 \end{pmatrix}, \lambda \in \mathbb{F};\]
c) 3. d) No. e) \[
\begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & 1 & -1 & 0 \\
   3 & 1 & 0 & 0
\end{pmatrix}.
\]

Chapter 9

1. a) In each case, the eigenvalues are the diagonal entries and the respective eigenvectors are \(te_1\) and \(te_2\) \((t \neq 0)\).
   For interpretations of (b), (c) and (d), see part (e) of question.

2. Eigenvalue is 2.

3. \[\lambda = (\det A)^{1/3}.\]

4. b) \[
\begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}.
\]

5. a) \[\lambda = 2\) with eigenvectors \(\left\{ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \neq 0 \right\}\) and \[\lambda = 3\) with eigenvectors \(\left\{ t \begin{pmatrix} 2 \\ 3 \end{pmatrix} : t \neq 0 \right\}\).
   b) \[\lambda = -3\) with eigenvectors \(\left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \neq 0 \right\}\) and \[\lambda = 1\) with eigenvectors \(\left\{ t \begin{pmatrix} 1 \\ 3 \end{pmatrix} : t \neq 0 \right\}\).

7. a) \[\lambda = 3\) with eigenvectors \(\left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \neq 0 \right\}\) and
   \[\lambda = -1\) with eigenvectors \(\left\{ t \begin{pmatrix} -1 \\ 1 \end{pmatrix} : t \neq 0 \right\}\).
b) Only one eigenvalue $\lambda = 2$ with multiplicity 2 and eigenvectors $\{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix} : t \neq 0 \}$. 

c) $\lambda = 3$ with eigenvectors $\{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix} : t \neq 0 \}$ and 
$\lambda = -6$ with eigenvectors $\{ t \begin{pmatrix} -5 \\ 9 \end{pmatrix} : t \neq 0 \}$. 

d) $\lambda = 1 \pm i$ with eigenvectors $\{ t \begin{pmatrix} -1 \pm i \\ 1 \end{pmatrix} : t \neq 0 \}$. 

e) $\lambda = 5 \pm i\sqrt{3}$ with eigenvectors $\{ t \begin{pmatrix} \pm \frac{\sqrt{3}}{2} \pm \frac{1}{2}i \\ 1 \end{pmatrix} : t \neq 0 \}$. 

f) $\lambda = 5 \pm \sqrt{5}$ with eigenvectors $\{ t \begin{pmatrix} \frac{1}{2}(1 \pm \sqrt{5})i \\ 1 \end{pmatrix} : t \neq 0 \}$. 

9. The eigenvalues are the diagonal entries, 2, -2, 3, 5. Corresponding eigenvectors are 

\[ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 25 \\ -3 \\ 21 \end{pmatrix}. \]

10. a) -1, 4, 6; \( \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) . b) 2, -3, 3; \( \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

11. In each of the following answers, the diagonal entries in $D$ and the columns in $M$ may be rearranged in the same way and the answer is still correct. Also, any column in $M$ may be multiplied by a scalar and the new $M$ is still correct.

For Question 7:

a) $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

b) The matrix is not diagonalisable.

c) $D = \begin{pmatrix} 3 & 0 \\ 0 & -6 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -5 \\ 0 & 9 \end{pmatrix}$.

d) $D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \quad M = \begin{pmatrix} 1+i & 1-i \\ 0 & 1 \end{pmatrix}$.

e) $D = \begin{pmatrix} 5+i\sqrt{3} & 0 \\ 0 & 5-i\sqrt{3} \end{pmatrix}, \quad M = \begin{pmatrix} \frac{\sqrt{3}}{2} + \frac{i}{2} & -\frac{\sqrt{3}}{2} + \frac{i}{2} \\ 1 & 1 \end{pmatrix}$.

f) $D = \begin{pmatrix} 5+i\sqrt{5} & 0 \\ 0 & 5-i\sqrt{5} \end{pmatrix}, \quad M = \begin{pmatrix} \frac{1}{2}(1-i\sqrt{5})i & \frac{1}{2}(1+i\sqrt{5})i \\ 1 & 1 \end{pmatrix}$. 

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For Question 9:

\[ D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 1 & 1 & 25 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 5 & 21 \\ 0 & 0 & 0 & 14 \end{pmatrix}. \]

For Question 10:

a) \[ D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad M = \begin{pmatrix} -3 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

b) \[ D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 6 & 1 & 0 \end{pmatrix}. \]

15. If \( \mathbf{v} \) is an eigenvector of \( T \) then the coordinate vector \([\mathbf{v}]_B\) of \( \mathbf{v} \) with respect to the basis \( B \) is the corresponding eigenvector for the matrix \( A \).

16. \( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \).

17. \( \begin{pmatrix} -2 \\ \sqrt{5} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{30} \\ \sqrt{6} \\ \sqrt{5} \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{30} \\ \sqrt{6} \\ \sqrt{5} \end{pmatrix}. \)

18. \( A^5 = \begin{pmatrix} -78 & 330 \\ 55 & -133 \end{pmatrix} \).

19. a) \( 6, \quad t \left(\frac{1}{2}\right), \quad t \neq 0, \ t \in \mathbb{R}; \quad -4, \quad \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \quad t \neq 0, \ t \in \mathbb{R}. \)

b) \( P = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}. \)

c) \( \begin{pmatrix} 6^n & (-3)(-4)^n \\ 2 \times 6^n & 4(-4)^n \end{pmatrix} \).

20. When \( A \) is diagonalisable, \( A^k = MD^kM^{-1} \) and \( \mathbf{x}(k) = A^k\mathbf{x}(0) \). As a check, if you put \( k = 0 \) in the answers below you should get \( A^0 = I \), whereas if you put \( k = 1 \) you should get \( A^1 = A \).

For Question 7:

a) \( A^k = \frac{1}{2} \begin{pmatrix} 3^k + (-1)^k & 3^k + (-1)^{k+1} \\ 3^k + (-1)^{k+1} & 3^k + (-1)^k \end{pmatrix} \).

b) The matrix is not diagonalisable.

c) \( A^k = \frac{1}{9} \begin{pmatrix} 3^{k+2} & 5(3^k - (-6)^k) \\ 0 & 9(-6)^k \end{pmatrix} \).
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d) Let \( \lambda_1 = 1 + i \) and \( \lambda_2 = 1 - i \). Then
\[
A^k = \begin{pmatrix}
\left(\frac{1}{2} + \frac{3}{2}i\right)\lambda_1^k + \left(\frac{1}{2} - \frac{3}{2}i\right)\lambda_2^k & i(\lambda_1^k - \lambda_2^k) \\
-\frac{1}{2}i(\lambda_1^k - \lambda_2^k) & \left(\frac{1}{2} - \frac{3}{2}i\right)\lambda_1^k + \left(\frac{1}{2} + \frac{3}{2}i\right)\lambda_2^k
\end{pmatrix}
\]
e) Let \( \lambda_1 = 5 + \sqrt{3}i \) and \( \lambda_2 = 5 - \sqrt{3}i \). Then
\[
A^k = \frac{1}{\sqrt{3}} \begin{pmatrix}
\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\lambda_1^k + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\lambda_2^k & \lambda_1^k - \lambda_2^k \\
\lambda_1^k - \lambda_2^k & \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\lambda_1^k + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\lambda_2^k
\end{pmatrix}
\]
f) Let \( \lambda_1 = 5 + \sqrt{5} \) and \( \lambda_2 = 5 - \sqrt{5} \). Then
\[
A^k = \frac{i}{\sqrt{5}} \begin{pmatrix}
\frac{1}{2}i(1 - \sqrt{5})\lambda_1^k - \frac{1}{2}i(1 + \sqrt{5})\lambda_2^k & -\lambda_1^k + \lambda_2^k \\
\lambda_1^k - \lambda_2^k & -\frac{1}{2}i(1 + \sqrt{5})\lambda_1^k + \frac{1}{2}i(1 - \sqrt{5})\lambda_2^k
\end{pmatrix}
\]

For Question 9:
\[
A^k = \begin{pmatrix}
2^k & -2^k + (-2)^k & -\frac{1}{5}(2)^k + \frac{1}{5}(3)^k & -2(2^k) + \frac{18}{35}(2)^k - \frac{3}{10}(3)^k + \frac{25}{14}(5)^k \\
0 & (-2)^k & -\frac{1}{5}(2)^k + \frac{1}{5}(3)^k & \frac{18}{35}(2)^k - \frac{3}{10}(3)^k - \frac{3}{14}(5)^k \\
0 & 0 & 3^k & -\frac{3}{2}(3)^k + \frac{3}{2}(5)^k \\
0 & 0 & 0 & 5^k
\end{pmatrix}
\]

For Question 10:

a) \( A^k = \frac{1}{3} \begin{pmatrix}
3(-1)^k + 2(4)^k & 3(-1)^{k+1} + 3(4)^k \\
2(-1)^{k+1} + 2(4)^k & 2(-1)^k + 3(4)^k
\end{pmatrix}
\]
b) \( A^k = \frac{1}{3} \begin{pmatrix}
(5(3)^k) & 0 \\
0 & (6(2)^k - 6(-3)^k)
\end{pmatrix}
\]

21. \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) are arbitrary real numbers.

For Question 5:

a) \( y(t) = \alpha_1e^{3t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \alpha_2e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

b) \( y(t) = \alpha_1e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \alpha_2e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

For Question 9:
\[
y(t) = \alpha_1e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3e^{3t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_4e^{5t} \begin{pmatrix} 25 \\ -3 \\ 21 \\ 14 \end{pmatrix}.
\]
For Question 10:

a) \( y(t) = \alpha_1 e^{-\frac{2}{3}t} \begin{pmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + \alpha_2 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 e^{6t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \)

b) \( y(t) = \alpha_1 e^{2t} \begin{pmatrix} 0 \\ \frac{1}{6} \\ 1 \end{pmatrix} + \alpha_2 e^{-3t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \alpha_3 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \)

22. \( e^{2t} \left( \alpha_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \alpha_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \right) \)

23. a) 1, \( \lambda \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \lambda \neq 0; \quad 5, \quad \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda \neq 0. \)

b) \( \begin{cases} x_1 = 3\alpha e^t + \beta e^{5t}, \\ x_2 = -\alpha e^t + \beta e^{5t}. \end{cases} \)

24. a) \( x = 300e^t - 200e^{3t}, \quad y = 150e^t - 50e^{3t}. \)

b) \( x = -500 + 600e^{-2t}, \quad y = -100 + 200e^{-2t}. \)

25. The solutions by the two methods are:

a) \( y(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = \alpha_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 e^{t/5} \begin{pmatrix} 1 \\ \frac{1}{5} \end{pmatrix}, \quad \text{(matrix method)} \)

\( y(t) = \alpha_1 e^t + \alpha_2 e^{t/5}. \quad \text{(calculus method)} \)

b) \( y(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = \alpha_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad \text{(matrix method)} \)

\( y(t) = \alpha_1 e^{4t} + \alpha_2 e^{-4t}. \quad \text{(calculus method)} \)

26. The matrix method given in notes is not applicable as the matrix is not diagonalisable.

27. a) \( A = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{a} & \frac{1}{b} \end{pmatrix}. \)

28. a) \[
\begin{array}{c|ccc}
\text{time} & \text{0.9048; 0.0928; 0.0024} & \text{0.8187; 0.1722; 0.0091} & \text{1.4 \times 10^{-87}; 7 \times 10^{-44}; 1} \\
\hline
500 \text{ years} & & & \\
1000 \text{ years} & & & \\
1000000 \text{ years} & & & \\
\end{array}
\]

b) The associated matrix is not diagonalisable.

29. \( x(k) \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)

30. In the 12th: \( \begin{pmatrix} 0.98 & 0.02 & 0.03 \\ 0.01 & 0.96 & 0.03 \\ 0.01 & 0.02 & 0.94 \end{pmatrix}^{11} \begin{pmatrix} 300 \\ 300 \end{pmatrix} \approx \begin{pmatrix} 378 \\ 293 \end{pmatrix} \),
In the 24th: \[
\begin{pmatrix}
0.98 & 0.02 & 0.03 \\
0.01 & 0.96 & 0.03 \\
0.01 & 0.02 & 0.94
\end{pmatrix}^{23} \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix} \approx \begin{pmatrix} 426 \\ 280 \\ 194 \end{pmatrix}.
\]

31. In the 12th: \[
\begin{pmatrix} 339 \\ 262 \\ 205 \end{pmatrix} \text{ total} = 806; \quad \text{In the 24th: } \begin{pmatrix} 339 \\ 222 \\ 154 \end{pmatrix} \text{ total} = 715.
\]

32. The population settles to the proportions \(1.156 : 1.124 : 1.116 : 1.086 : 1\) but eventually dies out.
PAST CLASS TESTS

In the years up to 2007 there were 3 algebra class tests per session. From semester 2 2008 there will be only 2 algebra class tests per semester so the pre-2008 tests included here do not have the same coverage of material as the class tests for 2008 and onwards. The 2014 Information booklet for MATH1251 lists the material available for examination in the current schedule of class tests, as does page (??) of these notes. Also there have been some changes to the syllabus for 2008 and beyond and some parts of the questions in the following pre-2008 class tests are no longer examinable. Thus the following pre-2008 tests should only be taken as a guide to the level of difficulty to be expected in class test questions for 2008 and onwards.

Sample class tests from 2008 and onwards are included after all the pre-2008 tests and these correspond to the current course syllabus and class test schedule. However, the content of the 2014 class tests is defined in the current 2014 Information booklet for MATH1251.
Note: The use of a calculator is NOT permitted in this test

QUESTIONS  (Time allowed: 20 minutes)

1. (3 marks)
   Find the real and imaginary parts of \((-\sqrt{3} + i)^{200}\).

2. (4 marks)
   Factorise \(z^5 + 100000\) into real linear and quadratic factors.

3. (3 marks)
   Prove that if \(\text{arg } z = \theta\) then
   \[
   (z/\overline{z})^2 - (\overline{z}/z)^2 = 2i \sin 4\theta .
   \]
   Be sure to set out your argument clearly and logically.
1. (4 marks)

Let \( \vec{a} \) and \( \vec{b} \) be two fixed vectors in \( \mathbb{R}^3 \); as usual, denote by \( \vec{u} \cdot \vec{v} \) the dot product of two vectors in \( \mathbb{R}^3 \). Prove that

\[
S = \{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{a} = \vec{x} \cdot \vec{b} \}
\]

is a subspace of \( \mathbb{R}^3 \). Note. To obtain full marks for this question you must give a clearly written, carefully explained and logically precise answer.

2. (1 mark)

Show that

\[
S = \{ \vec{x} \in \mathbb{R}^2 \mid x_1 = x_2^2 \}
\]

is not a subspace of \( \mathbb{R}^2 \).

3. (2 marks)

Let

\[
\begin{align*}
A_1 &= \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} -1 & -3 \\ 3 & 0 \end{pmatrix} \quad \text{and} \\
A_3 &= \begin{pmatrix} 2 & -1 \\ -5 & 1 \end{pmatrix}.
\end{align*}
\]

Is \( \{ A_1, A_2, A_3 \} \) an independent set? Is it a spanning set for \( M_{2,2} \)? Give reasons for your answers.

4. (3 marks)

Prove that

\[
B = \{ 1 + t, t + t^2, t^2 + t^3 \}
\]

is a basis for

\[
V = \{ p \in \mathbb{P}_3 \mid p(-1) = 0 \}.
\]
1. (3 marks)
   Prove that the map \( T : \mathbb{P}_3 \to \mathbb{P}_6 \) defined by
   \[
   T(p(x)) = p(x^2)
   \]
   is a linear transformation. **Note.** To obtain full marks for this question you must give a clearly written, carefully explained and logically precise answer.

2. (1 mark)
   Show that the map \( T : M_{2,2} \to \mathbb{R} \) defined by
   \[
   T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc
   \]
   is not a linear transformation.

3. (3 marks)
   Find a basis for the image of the matrix
   \[
   A = \begin{pmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 3 & 7 \\ -1 & 2 & -9 & -5 \end{pmatrix},
   \]
   and a basis for \( \mathbb{R}^3 \) containing this first basis.

4. (3 marks)
   Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be a linear transformation; suppose that \( \vec{a}, \vec{b} \) are vectors in \( \mathbb{R}^2 \) such that
   \[
   T(\vec{a}) = (1, 0, 0) \text{ and } T(\vec{b}) = (0, 1, 0). \]
   Prove that the equation \( T(\vec{x}) = (0, 0, 1) \) has no solution.
Note: The use of a calculator is NOT permitted in this test

QUESTIONs  (Time allowed: 20 minutes)

1. (3 marks)
   Solve the equation $z^2 - (4 - i)z + (5 + i) = 0$.

2. (3 marks)
   Find all the fourth roots of $8 + 8\sqrt{3}i$. You may leave your answers in polar form.

3. (2 marks)
   Draw a neat and accurate sketch of the region of the complex plane defined by
   
   \[ 0 \leq \text{Arg} \left((z - i)^3\right) \leq \frac{\pi}{2}. \]

4. (2 marks)
   Are the solutions of the following stable or unstable? Give reasons.
   
   (i) The discrete time system $5x_{n+1} - 2x_n + 4x_{n-1} = 0$.

   (ii) The continuous time system $5\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 4x = 0$. 
This sheet must be filled in and stapled to the front of your answers.

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<th>Initials</th>
<th>Student Number</th>
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<th>Mark</th>
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Note: The use of a calculator is NOT permitted in this test

**QUESTIONS**  *(Time allowed: 20 minutes)*

1. *(1 mark)*  
   Find a real number \(a\) such that  
   \[\Re\left(\frac{1 + 2i}{a + 3i}\right) = 0.\]

2. *(3 marks)*  
   Find the real and imaginary parts of \((-1 + i)^{77}\).

3. *(2 marks)*  
   Does the discrete time system  
   \[5x_{n+1} - 19x_n - 22x_{n-1} + 4x_{n-2} = 0\]  
   have unstable solutions? Give reasons for your answer.

4. *(4 marks)*  
   Factorise \(z^6 + 1\) into real linear and quadratic factors.
QUESTIONS  \hspace{1cm}  (Time allowed: 20 minutes)

1.  (2 marks)
Let $\vec{u}, \vec{v}$ and $\vec{w}$ be elements of a vector space $V$. Prove that if $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$ then $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v}\}$. Be sure to set out your argument clearly and logically.

2.  (4 marks)
   (i) Find a basis for the column space of the matrix
   \[
   A = \begin{pmatrix}
   1 & 2 & -1 & 1 \\
   -2 & -1 & 3 & 0 \\
   3 & 9 & -2 & 5
   \end{pmatrix}.
   \]
   (ii) Find a basis for $\mathbb{R}^3$ which includes the basis you found in (a).

3.  (2 marks)
Let
   \[
   \vec{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}.
   \]
Find the coordinate vector of $\vec{w}$ with respect to the ordered basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for $\mathbb{R}^3$.

4.  (2 marks)
Let $A$ be a fixed $2 \times 3$ matrix and $B$ a fixed $4 \times 5$ matrix. Prove that the function $T : M_{3,4} \rightarrow M_{2,5}$ defined by
   \[
   T(X) = AXB
   \]
is a linear transformation.
QUESTIONS  \hspace*{1cm} (Time allowed: 20 minutes)

1.  \hspace*{1cm} (4 marks)
\hspace*{1cm} Let \( p_1(t) = 1 - 3t + 2t^2 \), \( p_2(t) = -1 + 4t - 3t^2 \), \( p_3(t) = 2 - 5t + 3t^2 \) and \( p_4(t) = 1 - t - 3t^2 \).
\hspace*{1cm} (i) \hspace*{1cm} Is \( \{p_1, p_2, p_3, p_4\} \) a linearly independent set? Give reasons.
\hspace*{1cm} (ii) \hspace*{1cm} Is \( \{p_1, p_2, p_3, p_4\} \) a spanning set for \( \mathbb{P}_2 \)? Give reasons.

2.  \hspace*{1cm} (2 marks)
\hspace*{1cm} Let
\[ \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}. \]
\hspace*{1cm} Find the coordinate vector of \( \vec{w} \) with respect to the ordered basis \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) for \( \mathbb{R}^3 \).

3.  \hspace*{1cm} (2 marks)
\hspace*{1cm} Let \( \vec{a} \) be a fixed non-zero vector in \( \mathbb{R}^3 \). Prove that the function \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by
\[ T(\vec{x}) = \frac{\vec{a} \cdot \vec{x}}{\vec{a} \cdot \vec{a}} \vec{a} \]
is a linear transformation.

4.  \hspace*{1cm} (2 marks)
\hspace*{1cm} Let \( T : V \to W \) be a linear transformation. Define the kernel of \( T \) and prove from the definition that the kernel is closed under addition. (You \textbf{may not} use the fact that the kernel is a vector space.)