The trigonometry which you learnt at school is the wrong trigonometry. Young people over the ages have been taught a theory which unnecessarily complicates the subject and leads to loss of accuracy in practical applications. Unfortunately continual repetition has cemented this approach in the minds of educators as the only one possible—as you shall see, this is a mistake.

The thinking for the last two thousand years rests on the false assumptions that distance is the best way of measuring the separation of points, and that angle is the best way of measuring the separation of lines. So in order to study triangles students must first understand circles; they learn about $\pi$, lengths of circular arcs and the transcendental circular functions such as $\cos \theta$ and $\sin \theta$ that relate arc length on a circle to $x$ and $y$ projections. They study the relations between the circular functions and their inverse functions, ponder complicated graphs, and try to remember lots of special values. Advanced students see the infinite power series that calculators use to approximate true values. This is complicated stuff, especially if you try to do it correctly.

Yet a triangle itself is seemingly quite an elementary object. Why should the theory of a triangle be so complicated? Until now there has been no reasonable alternative. So educators have resigned themselves to the difficulties, and each year millions of students memorize the formulas, pass the tests (or not), and then promptly forget the unpleasant experience. And mathematicians wonder why the general public regards their beautiful subject with distaste bordering on hostility.

In this article, I am going to explain to you the right approach—called rational trigonometry. It is based on the idea that algebra is more basic than analysis, and that the true measurements in elementary geometry should be quadratic rather than linear in nature. Because this is a short paper, and I want to impart to you a working knowledge of the subject, the proofs are short. Giving more details is not hard, and the results are simple enough for a high school course.

Much more information can be found in ‘Divine Proportions: Rational trigonometry to Universal Geometry’ ([Wildberger]). There the theory is developed over a general field (not of characteristic two) and many useful new formulas and applications appear. This is then used to develop Euclidean geometry in a general and powerful way which fully exploits the power of the Cartesian coordinates.

This paper is an introduction to these ideas, accessible to high school students.
1 Distance and angle

Aside from being clumsy, the current treatment is also logically dubious. Currently the basic concepts of classical planar trigonometry are distance and angle. Informally, distance is what you measure with a ruler. More precisely it is given by the formula

$$|A_1, A_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

for points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$. Informally, angle is what you measure with a protractor. But what is an angle precisely? As the ancient Greeks realized, this is a tricky question.

One way is to define it in terms of arclengths of circular arcs, but this requires a prior understanding of calculus. Another common approach is to try to define an angle $\theta$ as $\arctan(y/x)$, possibly plus or minus $\pi$, but this requires a prior understanding of the $\arctan$ function. Well, $\arctan x$ is the inverse function of $\tan \theta$, which is itself the ratio $\sin \theta / \cos \theta$, and $\sin \theta$ and $\cos \theta$ are defined in terms of ratios or projections involving the angle $\theta$. And the angle $\theta$ is defined in terms of ... heh, what’s going on here?

Waffling about angles is supplemented with imprecision about the definitions or the circular (trig) functions and their properties—a subject which really belongs to the theory of special functions. The rational approach, described below, banishes hand waving—along with distance, angle and the circular functions—from the study of triangles.

2 Points and quadrance

A point $A$ is an ordered pair $[x, y]$ of numbers. The quadrance between two points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ is the number

$$Q(A_1, A_2) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2.$$  

In diagrams a quadrance is displayed beside a thin rectangle along the line between two points, or alternatively in a rectangular box, to distinguish it from distance. If three points $A_1, A_2$ and $A_3$ lie on a line, meaning that they satisfy a linear equation of the form $ax + by + c = 0$, then the three quadrances

$$Q_1 \equiv Q(A_2, A_3) \quad Q_2 \equiv Q(A_1, A_3) \quad Q_3 \equiv Q(A_1, A_2)$$

satisfy the Triple quad formula

$$(Q_1 + Q_2 + Q_3)^2 = 2 (Q_1^2 + Q_2^2 + Q_3^2).$$

Conversely if the Triple quad formula is satisfied then the three points are collinear. This is the first and most fundamental fact in the subject. It is a good exercise in algebraic manipulation once you write down what it means for three points to be collinear. The Triple quad formula can also be rewritten in the asymmetrical but still useful form $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2$. 

2
If $A_1, A_2, A_3$ and $A_4$ are four collinear points, then the relation between the quadrances

\[ Q_{12} = Q(A_1, A_2), \quad Q_{23} = Q(A_2, A_3), \quad Q_{34} = Q(A_3, A_4) \text{ and } Q_{14} = Q(A_1, A_4) \]

is more subtle. It turns out to be

\[
\left( (Q_{12} + Q_{23} + Q_{34} + Q_{14})^2 - 2 (Q_{12}^2 + Q_{23}^2 + Q_{34}^2 + Q_{14}^2) \right)^2 = 64Q_{12}Q_{23}Q_{34}Q_{14}.
\]

This Quadruple quad formula plays an important role in the study of quadrilaterals.

3 Lines and spreads

For distinct points $A_1$ and $A_2$ the unique line passing through them both is denoted $A_1A_2$. Two lines $l_1$ and $l_2$ with respective equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel precisely when

\[ a_1b_2 - a_2b_1 = 0 \]

and perpendicular precisely when

\[ a_1a_2 + b_1b_2 = 0. \]

The spread between these two lines is the number

\[ s(l_1, l_2) = \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}. \]

In diagrams a spread $s = s(l_1, l_2)$ is displayed beside a small line segment joining the two lines to distinguish it from an angle; there are four equivalent positions as shown.

The spread $s(l_1, l_2)$ is 0 precisely when the lines $l_1$ and $l_2$ are parallel, and because of Fibonacci’s identity

\[ (a_1b_2 - a_2b_1)^2 + (a_1a_2 + b_1b_2)^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2) \]

it is 1 precisely when the lines are perpendicular. Otherwise the spread is a number between 0 and 1, and can be measured in practice with a spread protractor as shown in Figure 1.

(Thanks to M. Ossmann at http://ossmann.com/protractor for this diagram.)

![Figure 1: A spread protractor](image)
The spread has a simple geometrical interpretation. Suppose $l_1$ and $l_2$ intersect at the point $A$. Choose a point $B \neq A$ on one of the lines, say $l_1$, and let $C$ be the foot of the perpendicular from $B$ to $l_2$ as in Figure 2.

![Figure 2: Spread as ratio](image)

Then the spread between $l_1$ and $l_2$ is also

$$s(l_1, l_2) \equiv s \equiv \frac{Q(B, C)}{Q(A, B)} = \frac{Q}{R}.$$  

Note that the circle plays no role in the definition. You may easily check that the spread corresponding to $30^\circ$ or $150^\circ$ is $s = 1/4$, the spread corresponding to $45^\circ$ or $135^\circ$ is $1/2$, and the spread corresponding to $60^\circ$ or $120^\circ$ is $3/4$. Spread does not distinguish between an angle and its supplement, since the spread is naturally measured between lines, not rays.

However when discussing rays it is possible for the spread of a sector to be either *acute* or *obtuse*. In Figure 3 the left diagram illustrates an acute spread of $s = 0.625$ and the right diagram an obtuse spread of $0.845$. As Euclid realized, the acuteness of the sector on the left is equivalent to the condition

$$Q(A_1, A_2) + Q(A_1, A_3) \geq Q(A_2, A_3)$$

while the obtuseness of the sector on the right is equivalent to

$$Q(A_1, A_2) + Q(A_1, A_3) \leq Q(A_2, A_3).$$

![Figure 3: Acute and obtuse sectors](image)
4 Laws of rational trigonometry

Given three distinct points $A_1, A_2$ and $A_3$, define the quadrances

$$Q_1 \equiv Q(A_2, A_3) \quad Q_2 \equiv Q(A_1, A_3) \quad Q_3 \equiv Q(A_1, A_2)$$

and the spreads

$$s_1 \equiv s(A_1A_2, A_1A_3) \quad s_2 \equiv s(A_2A_1, A_2A_3) \quad s_3 \equiv s(A_3A_1, A_3A_2)$$

as in Figure 4. Here are the main laws of planar rational trigonometry, aside from the Triple quad formula stated earlier.

**Pythagoras’ theorem** The lines $A_1A_3$ and $A_2A_3$ are perpendicular precisely when

$$Q_1 + Q_2 = Q_3.$$

**Spread law** For any triangle $A_1A_2A_3$

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}$$

**Cross law** For any triangle $A_1A_2A_3$ define the cross $c_3 \equiv 1 - s_3$. Then

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2c_3.$$  

**Triple spread formula** For any triangle $A_1A_2A_3$

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$  

The Spread law is the rational analog of the Sine law, the Cross law is the analog of the Cosine law, and the Triple spread formula is the analog of the fact that the sum of the angles is $\pi$. However the proofs we now give are independent of the usual treatment.
5 Proofs of the laws

We’ll assume that Pythagoras’ theorem is known. Consider either of the following diagrams (the proofs work for both simultaneously).

![Figure 5: Spread and Cross laws](image)

The spreads at \( A_2 \) and \( A_3 \) are respectively

\[
\begin{align*}
\hat{s}_2 &= \frac{R_1}{Q_3} \\
\hat{s}_3 &= \frac{R_1}{Q_2}
\end{align*}
\]

Solve for \( R_1 \) to get

\[ R_1 = Q_3 \hat{s}_2 = Q_2 \hat{s}_3 \]

so that

\[
\frac{\hat{s}_2}{Q_2} = \frac{\hat{s}_3}{Q_3}
\]

In a similar manner

\[
\frac{\hat{s}_1}{Q_1} = \frac{\hat{s}_2}{Q_2}
\]

This proves the Spread law. To prove the Cross law, use the same diagram. Pythagoras’ theorem shows that

\[
\begin{align*}
Q_3 &= R_1 + R_2 \\
Q_2 &= R_1 + R_3.
\end{align*}
\]

By the definition of the cross

\[
c_3 = 1 - \hat{s}_3 = 1 - R_1/Q_2 = R_3/Q_2.
\]

Solve sequentially for \( R_3, R_1 \) and then \( R_2 \) to get

\[
\begin{align*}
R_3 &= Q_2 c_3 \\
R_1 &= Q_2 (1 - c_3) \\
R_2 &= Q_3 - Q_2 (1 - c_3).
\end{align*}
\]
Since $A_2$, $A_3$ and $F$ are collinear, apply the Triple quad formula to the three quadrances $Q_1$, $R_2$ and $R_3$, yielding

$$(Q_1 + R_3 - R_2)^2 = 4Q_1R_3.$$  

Substitute the values of $R_3$ and $R_2$, to get

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2e_3.$$  

This proves the Cross law.

From the Spread law, there is a non-zero number $D$ such that

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{1}{D}. \quad (1)$$  

The Cross law can be rewritten as

$$(Q_1 + Q_2 + Q_3)^2 = 2 (Q_1^2 + Q_2^2 + Q_3^2) + 4Q_1Q_2s_3. \quad (2)$$  

Use (1) to replace $Q_1$ by $s_1D$, $Q_2$ by $s_2D$ and $Q_3$ by $s_3D$ in (2), and then divide by $D^2$. The result is the Triple spread formula

$$(s_1 + s_2 + s_3)^2 = 2 (s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$

### 6 Rational trigonometry in action

That’s it. You have now learnt the basics of rational trigonometry. Tables of trig functions and calculators can be dispensed with (although calculators are still useful for those large fractions that often come up). Knowing that

$$\sec \theta = 1 + \frac{1}{2} \theta^2 + \frac{5}{24} \theta^4 + \frac{61}{720} \theta^6 + \frac{277}{8064} \theta^8 + \frac{50521}{3628800} \theta^{10} + \cdots$$

or that $\sin (\arctan x) = x/\sqrt{1 + x^2}$ or that $\tan (\pi/5) = \sqrt{5} - 2\sqrt{5}$ is not necessary. You can delete all such formulas from your over-crowded memory banks, and still solve geometrical problems faster and with greater accuracy than your classical classmates.

With a bit of practice studying the examples in ‘Divine Proportions’, I assure you that without a calculator—i.e. just pencil and paper—you will demolish all similarly equipped competition when it comes to solving trigonometric problems quickly to high accuracy. I guess that on average you will be at least twice as fast, and often much faster. If you want, answers with rational trigonometry can be 100% accurate. The implications for computing should be obvious.

Of course some additional formulas are useful, but with the basic laws you can go far. Here are just two examples of the power of this method, which illustrate some additional points. The first treats a classical problem involving slopes, and the second one shows the rich interplay between number theory and geometry that rational trigonometry brings to the fore. For more examples, see [Wildberger].
**Problem 1** Suppose an inclined plane has a spread of $s$ with the horizontal plane. An insect climbing up the plane walks on a straight line which makes a spread of $r$ with the path of greatest slope. At what spread to the horizontal does the insect climb on this path?

![Figure 6: Path on a wedge](image)

**Solution.** As in Figure 6, let $AC$ be a line making the maximum possible spread $s$ with the horizontal, and $AD$ the path of the insect. The spread between $AC$ and $AD$ is $r$, and you need to find the spread $s(AD, DC)$. Suppose that $Q(A, C) = Q$. Then from the definition of the spread $s$,

$$Q(B, C) = sQ.$$

In the right triangle $ACD$, the spread $s(AC, AD)$ is $r$, so by Pythagoras' theorem the spread $s(DA, DC)$ is $1 - r$. The Spread law then gives

$$\frac{1}{Q(A, D)} = \frac{r}{Q(C, D)} = \frac{1 - r}{Q}.$$

So

$$Q(A, D) = \frac{Q}{1 - r}.$$

Thus in the right triangle $ADE$

$$s(AD, AE) = \frac{Q(D, E)}{Q(A, D)} = \frac{Q(B, C)}{Q(A, D)} = \frac{sQ}{Q/(1 - r)} = s(1 - r).$$

Note that this is a linear expression in both $s$ and $r$. 

**Problem 2** The triangle $A_1A_2A_3$ shown to scale in the left diagram of Figure 7 has an angle at $A_1$ of $60^\circ$, with $|A_1, A_3| = \sqrt{57}$, $A_3D$ the bisector of the angle at $A_3$ and $|D, A_3| = \sqrt{43}$. The triangle $A_1A_3D$ is acute. What is the distance $d_1 = |A_2, A_3|$ exactly? The corresponding rational formulation of the same problem is in the right diagram of the same figure. The triangle $A_1A_2A_3$
has a spread of $s_1 = 3/4$, quadrance $Q_2 \equiv 57$ and quadrance $Q(A_3, D) \equiv 43$. In this case we are to determine the quadrance $Q_1$ exactly.

![Figure 7: Classical and rational versions](image)

**Solution.** Suppose that $R \equiv Q(A_1, D)$ and $s(A_3A_1, A_3D) = s(A_3A_2, A_3D) \equiv r$ as in the Figure. The Cross law in $A_1A_3D$ gives

$$(R + 57 - 43)^2 = 4 \times R \times 57 \times (1 - 3/4)$$

or

$$R^2 - 29R + 196 = 0.$$ 

This has solutions $R = \frac{29}{2} \pm \frac{1}{2}\sqrt{57}$ which are approximately 18.27 and 10.72. But we are told that $A_1A_3D$ is acute; this implies that $R + 43 \geq 57$. So the correct solution is $R = \frac{29}{2} + \frac{1}{2}\sqrt{57}$. The Spread law in $A_1A_3D$ shows that

$$\frac{r}{R} = \frac{3/4}{43}$$

so that

$$r = \frac{87}{344} + \frac{3}{344}\sqrt{57} \approx 0.3187.$$ 

The three lines meeting at $A_3$ make spreads of $r, r$ and $s_3$. The Triple spread formula applies, and in this case reduces to

$$s_3 = 4r(1-r) = \frac{10923}{14792} + \frac{255}{14792}\sqrt{57} \approx 0.868.$$ 

Note the appearance of the *logistic map* $f(x) = 4x(1-x)$ in this context; this is an example of a *Spread polynomial*. Having $s_1 = 3/4$ and $s_3$, the Triple spread formula

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3$$

in $A_1A_2A_3$ gives for $s_2$ the quadratic equation

$$s_2^2 - s_2 \left(\frac{11265}{14792} - \frac{255}{14792}\sqrt{57}\right) + \frac{1867833}{109401632} - \frac{43605}{109401632}\sqrt{57} = 0.$$
Somewhat surprisingly, the solutions are

\[ s_2 = \frac{171}{7396} \approx 0.023 \]

or

\[ s_2 = \frac{10923}{14792} - \frac{255}{14792}\sqrt{57} \approx 0.608. \]

The second one is the relevant one, and then by the Spread law

\[
Q_1 = \frac{3}{4} \times \frac{57}{s_2} = \frac{383,735,913}{6422,528} + \frac{8958,405}{6422,528}\sqrt{57} \approx 70.279.
\]

If a distance is required, take the square root of the quadrance \( Q_1 \). This turns out to be, also somewhat surprisingly,

\[
d_1 = \frac{2451}{3584} + \frac{3655}{3584}\sqrt{57} \approx 8.383. \]

It should be remarked that the above numbers were essentially chosen at random—this is not a ‘rigged’ example. It is interesting to try this problem with classical trigonometry to compare the two approaches.

7 Conclusion

Rational trigonometry is generally superior to classical trigonometry for solving the majority of geometric problems. It cleanly separates the physical subject of uniform circular motion and the mathematical subject of triangles. For the former, the circular functions are of course useful, for the latter they are—or should be—largely irrelevant.

In scientific circles, it is reasonably common for new developments to replace rather than to augment old theories. This gives scientific research a youthful vigor that is sometimes lacking in mathematics, where tradition generally rules. Here will be a good test case. Will the merits of rational trigonometry be properly analyzed and discussed, and the theory implemented if found superior?

If you try your hand at a wide range of problems using rational trigonometry, you will see for yourself the power of this technology. The new algebraic approach is cleaner, faster and more accurate than the old analytic one. Rational trigonometry is far easier to learn and to teach, and so opens up an exciting new vista for mathematics education.

References