Hypergroups associated to random walks on the Platonic solids

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Abstract

We explicitly construct the convolution hypergroups of spheres in the vertex graphs of the classical Platonic solids, their character tables, and the structure of their dual signed hypergroups. This allows us to easily determine certain aspects of the random walks on these graphs. The character tables share many of the pleasant properties of character tables of finite groups.
§0. Introduction

The notion of hypergroup was introduced by Dunkl [1], Jewett [1] and Spector [1] and the theory of finite commutative hypergroups has now made contact with many other branches of mathematics including group theory, conformal field theory and algebraic combinatorics (see the survey Wildberger [3]). In this paper we demonstrate the close connection with the theory of distance-regular graphs by explicitly constructing the hypergroups associated to the classical Platonic solids, and using the harmonic analysis of these hypergroups to study the random walks on these graphs.

Random walks on polyhedra have of course been studied before; we mention the work of Göbel and Jagers [1], Letac and Takacs [1] and van Slijpe [1], [2]. See the works by Biggs [1] and by Brouwer, Cohen and Neumaier [1] for information on distance regular and other graphs.

Any distance transitive graph $X$ has associated to it a finite Hermitian hypergroup $\mathcal{K}(X) = \{c_0, c_1, \ldots, c_n\}$ where $n$ is the diameter of $X$. This hypergroup is the convolution structure on the spheres of various radii about some fixed point. The element $c_1$ is closely related to the Laplacian of the graph. The point of this paper is that questions about random walks on $X$ can often be easily converted to questions about the algebraic structure of $\mathcal{K}(X)$. These questions can be then be answered if we have established the basic ingredient of harmonic analysis on $\mathcal{K}(X)$—namely the character table. This character table shares many of the pleasant properties of those of finite groups such as orthogonality relations and algebraic integrality.

In this paper we explicitly write down the hypergroup structures $\mathcal{K}(X)$ for the vertex-edge graphs $X$ of the classical Platonic solids. We determine the character tables by some easy algebraic manipulations and then proceed to write down the structure equations of the dual objects (which also turn out to be hypergroups.) Finally we show how this information allows us to obtain explicit answers to questions about random walks using only commutative harmonic analysis.

As a by-product, we establish the eigenvalues of these graphs (see for example Schwenk and Wilson). It seems that in general the use of the entire hypergroup structure $\mathcal{K}(X)$ poses some advantages over consideration of the single operator $c_1$, the Laplacian.
In order to discuss duality, we develop the basic facts on hypergroups in the more
general context of signed hypergroups.

§1. Harmonic analysis on signed hypergroups

We briefly review here the main facts about harmonic analysis on commutative signed
hypergroups; proofs may be found in Wildberger [1], [2].

A finite signed hypergroup is a subset $K = \{c_0, c_1, \ldots, c_n\}$ of a $*$-algebra $A$ with unit
$c_0$ satisfying

A1) $K$ is a basis of $A$

A2) $K^* = K$

A3) The structure constants $n^k_{ij} \in \mathbb{C}$ defined by

$$c_i c_j = \sum_k n^k_{ij} c_k$$

satisfy the conditions

$$c_i^* = c_j \iff n^0_{ij} > 0$$

$$c_i^* \neq c_j \iff n^0_{ij} = 0.$$  

A4) $n^k_{ij} \in \mathbb{R}.$

All signed hypergroups considered here will be commutative (that means $A$ is commu-
tative). If in addition we have $c_i^* = c_i$ for all $i$ then $K$ is called Hermitian. If $n^k_{ij} \geq 0 \forall i, j, k$
then $K$ is called a hypergroup. If $c_i^* = c_j$, then the positive quantity $\omega(c_i) = (n^0_{ij})^{-1}$ is
called the weight of $c_i$.

A character of $K$ is a function $\chi : K \to \mathbb{C}$ satisfying

$$\chi(c_i) \chi(c_j) = \sum_k n^k_{ij} \chi(c_k).$$  \hfill (1.3)

If $K$ is Hermitian, such a character $\chi$ is necessarily real-valued. There are exactly $n + 1$
characters which we label $K^\wedge = \{\chi_0, \chi_1, \ldots, \chi_n\}$ with $\chi_0$ being the function identically
one. These functions are linearly independent; in fact they are orthogonal with respect to
the inner product

$$\langle f, g \rangle = \frac{1}{\omega(K)} \sum_k \omega(c_k) f(c_k) \overline{g(c_k)} \quad f, g \in C(K)$$  \hfill (1.4)
where \( \omega(\mathcal{K}) = \sum_k \omega(c_k) \), and \( C(\mathcal{K}) \) denotes the space of all complex functions on \( \mathcal{K} \). Under pointwise multiplication we may thus write

\[
\chi_i \chi_j = \sum_k m_{ij}^k \chi_k. \tag{1.5}
\]

Then it turns out that \( \mathcal{K}^\wedge \subseteq C(\mathcal{K}) \) is also a signed hypergroup, and that furthermore the signed hypergroups \( \mathcal{K} \) and \( (\mathcal{K}^\wedge)^\wedge \) are naturally isomorphic (in the obvious sense) under the map \( c_i \rightarrow \hat{c}_i \) where \( \hat{c}_i(\chi_j) = \chi_j(c_i) \ \forall \chi_j \in \mathcal{K}^\wedge \). This means that we can obtain an explicit realization of any signed hypergroup as a signed hypergroup of functions on the dual. The functions \( \hat{c}_i \in C(\mathcal{K}^\wedge) \) are then orthogonal with respect to the inner product

\[
\langle \phi, \psi \rangle = \frac{1}{\omega(\mathcal{K}^\wedge)} \sum_k \omega(\chi_k)\phi(\chi_k)\overline{\psi(\chi_k)} \quad \phi, \psi \in C(\mathcal{K}^\wedge). \tag{1.6}
\]

We also note that \( \omega(\mathcal{K}^\wedge) = \omega(\mathcal{K}) \).

The main problem of harmonic analysis on an signed hypergroup \( \mathcal{K} \) is thus the determination of the ‘character table’ – the matrix of values \( \chi_i(c_j) \).

Finally we note the useful relations

\[
n_{ij}^k \omega(c_i) = n_{jk}^i \omega(c_k) \tag{1.7}
\]

and

\[
\langle \chi_i, \chi_i \rangle = \frac{1}{\omega(\chi_i)}. \tag{1.8}
\]

§2. Hypergroups associated to distance transitive graphs

Now suppose a graph \( X \) is distance transitive. This means that the automorphism group \( G \) has the following property. If \((x_1, x_2)\) and \((y_1, y_2)\) are two pairs of vertices with \( d(x_1, x_2) = d(y_1, y_2) \) then there exists a \( g \in G \) with \( g(x_i) = y_i \) for \( i = 1, 2 \). Here \( d \) denotes the usual distance between vertices.

Let \( x_0 \) be a fixed vertex. Let \( S_i \) be the set of all vertices of \( X \) of distance \( i \) from \( x_0 \) for \( i = 0, \ldots, n = \text{diameter}(X) \). There is then a natural convolution of these spheres which defines a Hermitian hypergroup as follows. Let a vertex \( y \) be chosen at random on \( S_i \) and let \( z \) be a vertex chosen at random at distance \( j \) from \( y \). Let \( n_{ij}^k \) denote the probability
that $z$ is in $S_k$. Introducing basis elements $c_0, c_1, \ldots, c_n$ we get a Hermitian hypergroup structure

$$c_i c_j = \sum_k n_{ij}^k c_k.$$  

Here $c_i$ can be considered as the uniform probability measure on $S_i$, and we denote the hypergroup by $\mathcal{K}(X)$.

We now determine the hypergroup structures $K(X)$, their dual signed hypergroups and character tables for each of the classical Platonic solids.

i) **Tetrahedron**  Here $\mathcal{K}(X) = \{c_0, c_1\}$ with the single non-trivial equation

$$c_1^2 = \frac{1}{3}c_0 + \frac{2}{3}c_1$$

and character table

<table>
<thead>
<tr>
<th></th>
<th>$c_0$</th>
<th>$c_1$</th>
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</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>$-\frac{1}{3}$</td>
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</table>

It is then immediately clear that $\mathcal{K}(X)^\wedge \simeq \mathcal{K}(X)$.

ii) **Octahedron**  $\mathcal{K}(X) = \{c_0, c_1, c_2\}$ with equations

$$c_1^2 = \frac{1}{4}c_0 + \frac{1}{2}c_1 + \frac{1}{4}c_2$$

$$c_1c_2 = c_1$$

$$c_2^2 = c_0$$

The character table is computed without difficulty to be

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<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

and the structure equations of $\mathcal{K}(X)^\wedge$ are

$$\chi_1^2 = \frac{1}{2}\chi_0 + \frac{1}{2}\chi_1$$

$$\chi_1\chi_2 = \chi_2$$

$$\chi_2^2 = \frac{1}{3}\chi_0 + \frac{2}{3}\chi_2.$$
The hypergroup $K(X)$ is in this case isomorphic to the representation hypergroup of the symmetric group $S_3$.

**iii) Cube** $K(X) = \{c_0, c_1, c_2, c_3\}$ with equations

\[
\begin{align*}
    c_1 &= \frac{1}{3}c_0 + \frac{2}{3}c_2 \\
    c_2 &= \frac{1}{3}c_0 + \frac{2}{3}c_2 \\
    c_1c_2 &= \frac{2}{3}c_1 + \frac{1}{3}c_3 \\
    c_2c_3 &= c_1 \\
    c_1c_3 &= c_2 \\
    c_2 &= c_0 \\
    c_3 &= c_0
\end{align*}
\]

The last equation forces $\chi_i(c_3) = \pm 1$ for all $i$ from which it is easy to construct the character table

<table>
<thead>
<tr>
<th></th>
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<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

From the symmetry of the table we see that $K(X)^\wedge \simeq K(X)$.

**iv) Icosahedron** $K(X) = \{c_0, c_1, c_2, c_3\}$ with equations

\[
\begin{align*}
    c_1 &= \frac{1}{5}c_0 + \frac{2}{5}c_1 + \frac{2}{5}c_2 \\
    c_2 &= \frac{1}{5}c_0 + \frac{2}{5}c_1 + \frac{2}{5}c_1 \\
    c_1c_2 &= \frac{2}{5}c_1 + \frac{2}{5}c_2 + \frac{1}{5}c_3 \\
    c_2c_3 &= c_1 \\
    c_1c_3 &= c_2 \\
    c_2 &= c_0 \\
    c_3 &= c_0
\end{align*}
\]

Again the last equations simplify the situation. We obtain

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<tr>
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<th>$c_0$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>$-\frac{2}{5}$</td>
<td>$-\frac{2}{5}$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>$-\frac{1}{\sqrt{5}}$</td>
<td>$\frac{1}{\sqrt{5}}$</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{5}}$</td>
<td>$-\frac{1}{\sqrt{5}}$</td>
<td>-1</td>
</tr>
</tbody>
</table>

Now it is more work to construct the dual equations – we use $\langle \chi_i\chi_j, \chi_k \rangle = \frac{m_{ij}^k}{\omega(\chi_k)}$ where (1.8) allows us to compute $\omega(\chi_1) = 5, \omega(\chi_2) = 3$ and $\omega(\chi_3) = 3$. 

6
\[
\begin{align*}
\chi_1^2 &= \frac{1}{5} \chi_0 + \frac{4}{5} \chi_1 \\
\chi_2^2 &= \frac{1}{3} \chi_0 + \frac{2}{3} \chi_1 \\
\chi_1 \chi_2 &= \frac{2}{5} \chi_2 + \frac{3}{5} \chi_3 \\
\chi_1 \chi_3 &= \frac{3}{5} \chi_2 + \frac{2}{5} \chi_3 \\
\chi_2 \chi_3 &= \chi_1 \\
\chi_3 &= \frac{1}{3} \chi_0 + \frac{2}{3} \chi_1
\end{align*}
\]

v) Dodecahedron Here \( K(X) = \{c_0, c_1, c_2, c_3, c_4, c_5\} \) and with the help of a model or diagram one may verify the structural equations

\[
\begin{align*}
c_1^2 &= \frac{1}{3} c_0 + \frac{2}{3} c_2 \\
c_1 c_2 &= \frac{1}{3} c_1 + \frac{1}{3} c_2 + \frac{1}{3} c_3 \\
c_1 c_3 &= \frac{1}{3} c_2 + \frac{1}{3} c_3 + \frac{1}{3} c_4 \\
c_1 c_4 &= \frac{2}{3} c_3 + \frac{1}{3} c_5 \\
c_1 c_5 &= c_4 \\
c_2^2 &= \frac{1}{6} c_0 + \frac{1}{6} c_1 + \frac{1}{6} c_2 + \frac{1}{3} c_3 + \frac{1}{6} c_4 \\
c_2 c_3 &= \frac{1}{6} c_1 + \frac{1}{3} c_2 + \frac{1}{3} c_3 + \frac{1}{6} c_4 + \frac{1}{6} c_5 \\
c_2 c_4 &= \frac{1}{3} c_2 + \frac{1}{3} c_3 + \frac{1}{3} c_4 \\
c_2 c_5 &= c_3 \\
c_3^2 &= \frac{1}{6} c_0 + \frac{1}{6} c_1 + \frac{1}{6} c_2 + \frac{1}{3} c_3 + \frac{1}{6} c_4 \\
c_3 c_4 &= \frac{1}{3} c_1 + \frac{1}{3} c_2 + \frac{1}{3} c_3 \\
c_3 c_5 &= c_2 \\
c_4^2 &= \frac{1}{3} c_0 + \frac{2}{3} c_2 \\
c_4 c_5 &= c_1 \\
c_5 &= c_0
\end{align*}
\]

We find the characters by considering the possibilities for \( c_5 \). If \( c_5 = 1 \), then \( c_1 = c_4 \) and \( c_2 = c_3 \) so the quadratic equations become \( 3c_1^2 = 1 + 2c_2 \) and \( 3c_1 c_2 = c_1 + 2c_2 \) which can be solved to get \( c_1 = -\frac{1}{3} \) or \( -\frac{2}{3} \) and \( c_2 = -\frac{1}{3} \) or \( \frac{1}{6} \) respectively. The case \( c_5 = -1 \) is similar.

The character table is thus

| \(|\chi_0\)| | \(|\chi_1\)| | \(|\chi_2\)| | \(|\chi_3\)| | \(|\chi_4\)| | \(|\chi_5\)| |
|---|---|---|---|---|---|
| \(c_0\) | 1 | 1 | 1 | 1 | 1 |
| \(c_1\) | 1 | 0 | -\frac{1}{2} | \frac{1}{2} | 0 | -1 |
| \(c_2\) | 1 | \frac{1}{3} | -\frac{1}{3} | -\frac{1}{3} | \frac{1}{3} | 1 |
| \(c_3\) | 1 | \frac{\sqrt{5}}{3} | \frac{1}{3} | -\frac{1}{3} | -\frac{\sqrt{5}}{3} | -1 |
| \(c_4\) | 1 | -\frac{2}{3} | \frac{1}{6} | \frac{1}{6} | -\frac{2}{3} | 1 |
| \(c_5\) | 1 | -\frac{\sqrt{5}}{3} | \frac{1}{3} | -\frac{1}{3} | \frac{\sqrt{5}}{3} | -1 |

The weights of \( c_1, \ldots, c_5 \) are respectively 3, 6, 6, 3 and 1 while the weights of \( \chi_1, \ldots, \chi_5 \)
are respectively 4,5,3,4 and 3. The structure of $K(X)^\wedge$ can then be computed to be

\[
\begin{align*}
\chi_1^2 &= \frac{1}{4} \chi_2 + \frac{1}{4} \chi_2 + \frac{1}{2} \chi_4 \\
\chi_2^2 &= \frac{1}{5} \chi_0 + \frac{4}{9} \chi_2 + \frac{16}{45} \chi_4 \\
\chi_1 \chi_2 &= \frac{1}{5} \chi_1 + \frac{2}{5} \chi_3 + \frac{2}{5} \chi_5 \\
\chi_2 \chi_3 &= \frac{8}{15} \chi_1 + \frac{2}{5} \chi_3 + \frac{1}{15} \chi_5 \\
\chi_1 \chi_3 &= \frac{2}{3} \chi_2 + \frac{1}{3} \chi_4 = \chi_1 \chi_5 \\
\chi_2 \chi_4 &= \frac{4}{9} \chi_2 + \frac{8}{9} \chi_4 \\
\chi_1 \chi_4 &= \frac{1}{2} \chi_1 + \frac{1}{4} \chi_3 + \frac{1}{4} \chi_5 \\
\chi_2 \chi_5 &= \frac{8}{15} \chi_1 + \frac{1}{15} \chi_3 + \frac{2}{5} \chi_5 \\
\chi_3^2 &= \frac{1}{3} \chi_0 + \frac{2}{3} \chi_2 \\
\chi_4^2 &= \frac{1}{4} \chi_0 + \frac{25}{36} \chi_2 + \frac{1}{18} \chi_4 \\
\chi_3 \chi_4 &= \frac{1}{3} \chi_1 + \frac{2}{3} \chi_5 \\
\chi_4 \chi_5 &= \frac{1}{3} \chi_1 + \frac{2}{3} \chi_3 \\
\chi_3 \chi_5 &= \frac{1}{9} \chi_2 + \frac{8}{9} \chi_4 \\
\chi_5^2 &= \frac{1}{3} \chi_0 + \frac{2}{3} \chi_2
\end{align*}
\]

It should be noted that in all the cases we have considered, the dual signed hypergroup turned out to be a hypergroup. This is known as the Krein condition in the theory of strongly regular graphs (see Scott [1]).

§3. Random walks

Armed with the character tables of the previous section, we may now easily provide explicit answers to questions concerning random walks on these graphs.

**Theorem 3.1.** Let $X$ be the vertex-edge graph of one of the classical Platonic solids. Then the probability of a random walk on $X$ returning to the starting point after $n$ steps is

1) Tetrahedron: \( p = \frac{1}{4} \left[ 1 + 3\left(-\frac{1}{3}\right)^n \right] \)
2) Octahedron: \( p = \frac{1}{6} \left[ 1 + 2\left(-\frac{1}{2}\right)^n \right] \)
3) Cube: \( p = \frac{1}{8} \left[ 1 + 3\left(\frac{1}{3}\right)^n + 3\left(-\frac{1}{3}\right)^n + (-1)^n \right] \)
4) Icosahedron: \( p = \frac{1}{12} \left[ 1 + 5\left(-\frac{1}{5}\right)^n + 3\left(-\frac{1}{\sqrt{5}}\right)^n + 3\left(\frac{1}{\sqrt{5}}\right)^n \right] \)
5) Dodecahedron: \( p = \frac{1}{20} \left[ 1 + 5\left(\frac{1}{3}\right)^n + 3\left(\frac{\sqrt{5}}{3}\right)^n + 4\left(-\frac{2}{3}\right)^n + 3\left(-\frac{\sqrt{5}}{3}\right)^n \right] \).

**Proof.** The probability we require is the coefficient of $c_0$ in $c^n_1$, or $\langle c_0, c^n_1 \rangle$, where we identify $c_i$ with the function $\hat{c}_i$ on $K(X)$. But from the tables computed above, we may read out these inner products to be as given. \(\square\)

While these probabilities have surely been evaluated before, we would like to emphasize how easily they follow from the character tables of the previous section. We may in a
similiar vein analyse more complicated random walks. As an example, suppose a random walk on the dodecahedron takes steps of size 0, 1, 2 or 3 with respective probabilities \( \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \) and 0. Then the probability \( p \) that the distance from the starting point is exactly 2 after \( n \) steps will be the coefficient of \( c_2 \) in \( \left( \frac{1}{4}c_0 + \frac{1}{4}c_1 + \frac{1}{2}c_2 \right)^n \), that is,

\[
p = \omega(c_2)\langle c_2, \left( \frac{1}{4}c_0 + \frac{1}{4}c_1 + \frac{1}{2}c_2 \right)^n \rangle.
\]

This is then

\[
p = \frac{5}{20} \left[ 1 - \frac{1}{5} \left( \frac{1}{4} - \frac{1}{4} \times \frac{1}{5} - \frac{1}{2} \times \frac{1}{5} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1}{4} + \frac{1}{4\sqrt{5}} - \frac{1}{2\sqrt{5}} \right)^n + \frac{1}{\sqrt{5}} \left( \frac{1}{4} - \frac{1}{4\sqrt{5}} + \frac{1}{2\sqrt{5}} \right)^n \right]
\]

\[
= \frac{1}{4} \left[ 1 - \frac{1}{5} \left( \frac{1}{10} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1}{4} - \frac{1}{4\sqrt{5}} \right)^n + \frac{1}{\sqrt{5}} \left( \frac{1}{4} + \frac{1}{4\sqrt{5}} \right)^n \right].
\]

Bibliography


