A new look at multisets

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Abstract

We consider a new way of looking at multisets, involving linear rather than multiplicative notation and operations which mirror those on natural numbers. The motivation is to create a framework for mathematics which more closely connects with scientific usage in which repetition and relative concepts of equality play an important role. The tropical calculus is introduced as an important tool. and new versions of exclusion/inclusion, the Pigeonhole principle, and Hall's theorem are given.

1 Introduction

Classical set theory adopts some unstated assumptions that colour much of the way we think about mathematics. These include the conventions surrounding the notions of *uniqueness* and *equality*. Traditional mathematics implicitly assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of real numbers, one compact surface of genus three, and so on. Thus the only possible relations between two mathematical objects are 1) they are *equal*, or 2) they are *different*.

In science, and in ordinary life, the situation is not at all like this. In the physical world it is observed that there is enormous repetition. For example, there are many hydrogen atoms, many water molecules, many strands of identical DNA etc. This leads to effectively three possible relations between any two physical objects; they are *different*, they are the *same but separate*, or they are *coinciding and identical*. Scientists do not usually see these distinctions as warranting separate terminologies, but for the sake of definiteness let us say that two physical objects are the **same**, or **equal,** if they are indistinguishable, but possibly separate, and **identical** if they physically coincide. (Thus identity is a refinement of equality.)

Consider for example a water molecule, with two hydrogen atoms, say H^1 and H^2, and one oxygen atom, say O. Clearly H^1 and O are different. However H^1 and H^2 are the same but separate, while H^1 and H^1 are coinciding and identical. Many other examples from daily life can be used to further illustrate these distinctions. Coins of the same denomination and year of minting but which are physically distinct are essentially indistinguishable so we consider them the same (equal) but not identical. Electrons, or amino acids, or grains of sand, recur repeatedly in larger structures where their relative positions may allow us to distinguish them, although individually they may appear to us as the same, despite being obviously separate.

If we consider the situation even more carefully, we realize that the notion of equality is often a relative one. Thus to a shopkeeper two dollar coins are equal even if their year of minting is different, whereas a coin collector would regard them as essentially different.

Is there then a mathematics which more closely mirrors ordinary life and science by distinguishing between the concept of *equality* and the concept of *identity*? I believe that there ought to be, and that then *multisets* rather than sets become the fundamental data structures, and one distinguishes between *concrete* and *generic* operations on these objects.

Loosely speaking, a concrete operation or statement is one that takes into account the particular context in which a (mathematical) object appears, while a generic operation or statement applies independent of context. To return to our example of water molecules, a chemist who refers to 'a hydrogen atom' might be referring to a generic, or non-specific hydrogen atom, or might instead be referring to a specific hydrogen atom in a specific water molecule. Usually the context makes it clear which sense is meant. For many applications, the use of generic operations and concepts suffices. The reader might like to view concrete operations as of secondary, or occasional importance.

Traditional mathematics has no need of such distinctions because of the implicit assumption that all objects exist uniquely in some ideal sense, so there is never, for example, any possible confusion as to which field of real numbers one is talking about.

Classical set theory pays a price for this lack of subtlety, however, in that rather adhoc measures are required to model naturally occurring repetition. A good example is the ubiquitous use of *maps*. In order to effectively talk about four copies of the unit circle S^1, one resorts to indexing schemes, by defining a map from $\{1,2,3,4\}$ to S^1, or a map from S^1 to $\{1,2,3,4\}$, or some equivalent construction. The student beginning a serious study of mathematics is often disconcerted by the profligacy of maps, but of course after a few years one gets used to it, and thereafter considers it a normal state of affairs.

Another weakness in classical mathematics is the reliance on *equivalence relations*. The rational numbers provide a very simple but important example. Most high school students, and all scientists, know that a rational number is an expression of the form

$$\frac{a}{b}$$

where a and b are integers with b non-zero, with the convention that

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

precisely when

$$a_1 b_2 = b_1 a_2.$$

This is a situation where objects that appear different on the page are declared to be equal. However the story is far more complicated than this when viewed by the modern mathematician! To her, the rational number $\frac{3}{4}$ is rather an infinite set–the collection of *all* expressions $\frac{a}{b}$ satisfying $4a = 3b$ with a and $b \neq 0$ integers. But why should we have to consider infinite sets to talk, for example, about three quarters of a pie?

In the same vein, every scientist knows that a real number is a (possibly infinite) decimal expansion, with a special notion of equality that includes for example $2.4699999 \cdots = 2.47$. But to the modern mathematician, a real number is an equivalence class of Cauchy sequences of rational numbers, so an (uncountable–and in general unspecifiable) infinite set of infinite sequences of infinite sets!

The theory we propose here introduces a richer variety of fundamental notions and concepts than classical set theory, which may help us lessen the gap between scientific and mathematical usage of mathematical concepts. It also helps us to count more effectively in a wide variety of situations, and reveals to us, if we care to look, levels of subtlety in theories that are otherwise somewhat invisible. For some specific applications in this direction, see [Wi].

Multisets have of course been studied by combinatorialists and computer scientists, generally as an extension of set theory. Thus for example a multiplicative notation for multisets will be familiar to many readers. The approach here, in contrast, seeks to elevate multisets as the fundamental mathematical objects, with sets rather as (important) special cases. Furthermore, a *linear* notation is used to organize the subject. There are a number of other innovations which the reader will be asked to consider.

Let's summarize the paper quickly. First we introduce a notational system in which finite data structures are clearly identifiable by their form. Operations on multisets are in correspondence with operations on natural numbers, a key point which simplifies much of the subject, and leads also to interesting algebraic structures which we call *realms*. Integral and rational multisets are defined, with applications to more general de Moivre relations. The distinction between *generic* and *concrete* concepts leads to separating the notions of equality $=$ and identity $\equiv$, and to two different concepts of union and intersection. To facilitate computations with multisets, we describe the '*Tropical Calculus*', also called the *minimax* or *max-plus algebra*, which deserves wider recognition as a general and useful tool in multiset theory. Finally we give our versions of the three fundamental theorems in elementary finite set theory: the Pigeonhole Principle, Inclusion/Exclusion, and Hall's theorem. The latter is an 'Inventory Theorem' which encompasses many of the ideas of paper.

Since we are dealing here with an unorthodox approach to mathematics, a few words of explanation are in order as to the nature of the definitions. The reader expecting an axiomatic treatment largely independent of illustrative examples will be disappointed. That is because any such development will, if it is to be immediately understood, be cast already within the pre-existing framework of classical mathematics, with all its implicit, unacknowledged assumptions and conventions. In other words, if you are a classical mathematician and can immediately comprehend my definitions without further expansion and clarification, then I am still working within the usual set-up. Since this is not my intent, it follows that you ought to find some difficulty in adjusting to the concepts.

To smooth the transition to this new way of thinking, you might like to keep the example of our imaginary chemist in mind. When a difficulty involving relations and operations on abstract mathematical objects crops up, try to substitute atoms, molecules, and composite materials, and see if the concepts become clearer. We *could* investigate mathematics in the spirit of a scientist, rather than that of a philosopher. Would that be ... *scientific mathematics*?

2 Natural numbers

The **natural numbers**, or **positive integers**

$$0, 1, 2, 3, \cdots$$

should be familiar. Any particular natural number is an example of a **mathematical object**, or **object**, for short. One of our aims is to provide a consistent framework for creating complex objects from simpler constituents, so many of our examples will feature natural numbers prominently.

The relation between any two natural numbers can be described using one of the **basic relations** $=, \leq$ and $\geq$. The two simplest binary operations are the **union**, or **maximum** of two numbers a and b, here denoted by

$$a \cup b = \max(a, b)$$

and the **intersection**, or **minimum** of a and b, here denoted by

$$a \cap b = \min(a, b).$$

Note the novel use of the symbols $\cup$ and $\cap$ in this context. These operations are commutative, associative, and satisfy the identity laws

$$0 \cup a = a$$
$$0 \cap a = 0$$

the idempotent laws

$$a \cup a = a$$
$$a \cap a = a$$

and the absorption laws

$$a \cup (a \cap b) = a$$
$$a \cap (a \cup b) = a.$$

Natural numbers with the operations of union and intersection form a distributive lattice. Note that union and intersection incorporate the relations of order since $a \leq b$ if and only if $a \cap b = a$.

Arguably the most important of all binary operations is **addition**, denoted by $+$. Addition is commutative, associative, and satisfies

$$0 + a = a$$

$$a + (b \cup c) = (a + b) \cup (a + c)$$
$$a + (b \cap c) = (a + b) \cap (a + c).$$

The operations $\cup, \cap, +$ acting on natural numbers, with the distinguished element 0, and the laws we have listed above, form a **realm**. Unlike the situation in classical mathematics, we do not mean for this definition to involve consideration of the 'collection of all natural numbers' as a completed whole. Although this is not a point to develop here, it is worth noting that there are ways of defining algebraic structures other than to begin with sets and operations on them. In particular, it is possible to consider *operations* as the primary objects of interest, with the only stipulation being that operations act on objects of some particular type(s) to produce objects of some possibly different type(s). This is exactly what we have in mind for a realm.

3 Data Structures

Beginning with the natural numbers, we may create more complicated objects by utilizing the **data structures** of multiset, list, set and ordered set, which we now define.

An unordered collection of mathematical objects with repetitions allowed is a **multiset**. An ordered sequence of objects with repetitions allowed is a **list**. An unordered collection of objects with repetitions not allowed is a **set**. An ordered sequence of objects with repetitions not allowed is an **ordered set**. All of these are themselves objects.

Let's adopt some convenient notation to distinguish between these four fundamental concepts. Multisets will be displayed on the page either in a box, with the elements freely arranged within it, or on the written line between square brackets, with the elements separated by spaces, but *not* by commas, and arranged in an arbitrary linear order between the brackets. When there is any possible confusion arising from spaces being insufficiently clear, we will use the underscore _ to represent a space. Thus

$$X = [5\,5\,8\,7\,8] = [5\_5\_8\_7\_8] = [8\_5\_7\_5\_8]\,.$$

Following generally standard computer usage, lists will be displayed as contained by square brackets, with the elements separated by commas and sequenced from left to right, as in

$$L = [4, 3, 4, 1] \,.$$

A set may be displayed in the same fashion as a multiset, or displayed on the written line in curly brackets { } with the elements again separated by spaces, but not by commas, as in

$$S = \{2\,7\,1\} = [2\_7\_1] = \{2\_7\_1\} = \{1\_2\_7\} \,.$$

An ordered set may be displayed in the same fashion as a list, or displayed in curly brackets { } with the elements separated by commas and sequenced from left to right, as in

$$D = \{4, 6, 3, 2\} \,.$$

Note that the type of the data structure can be deduced from its form. Although it may represent a significant departure from accepted practice, the idea of representing ordered structures with commas and unordered ones with spaces is both natural and pleasant, and square brackets are a very convenient way to remind the reader that we are moving beyond the traditional world of set theory.

The number of times an object x occurs in the data structure A is called the **multiplicity** of x in A and denoted by

$$m_A\,(x) \,.$$

Each individual occurrence of an object x in A is called an **element** of A. The total number of elements of A is called the size of A and denoted by

$$|A| \,.$$

If A is a list, multiset, set, or ordered set with k elements for some natural number k then A will be called a k**-list**, k**-multiset**, k**-set**, or k**-ordered set** respectively. A 2-list will often be called an **ordered pair**. Thus in the example above, the multiset X has five elements so is a 5-multiset, even though only three objects occur in X.

The empty multiset, empty list, empty set, and empty ordered set are all the same object and will be denoted by the symbols ϕ or [ ]. It has size 0.

A data structure composed of mathematical objects is itself regarded as a mathematical object, and so may appear as an element, or an element of an element etc. of larger data structures.

Thus in the examples

$$E = [2\_2\_\,[3\_2]\,\_\,[4, 5, [4, 4]]\,\_1]$$
$$F = [[3, 2]\,, [5]\,, 7, \{3\}]$$
$$G = \{8\_\{2, 3, 4\}\}$$

we see that E is a 5-multiset containing (in an unordered way) three numbers (two of which happen to be the same), a 2-multiset, and a 3-list, the latter of which contains, in order, two numbers and an ordered pair of numbers. F is a 4-list consisting of, in order, an ordered pair of numbers, a 1-multiset, a number, and a 1-set. G is a 2-set containing in an unordered way a number and a 3-ordered set.

Given a list

$$L = [a_1, a_2, \cdots, a_n]$$

we say that a list

$$L' = [a_{i_1}, a_{i_2}, \cdots, a_{i_m}]$$

is a **sublist** of L if $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. (The notion of a submultiset of a multiset is more subtle and will be taken up in a later section.)

If A is a data object and a is an object which appears as an element of A, in the sense that $m_A(a) > 0$, then we write

$$a \in A.$$

Of the four basic types of data structures we have defined, the multiset is probably the most common in everyday life and in science. Our point of view is that the multiset deserves to be the dominant structure in mathematics also. (Computer science with its linear/ordered framework is the exception– there the list is most important.)

4 Relations and operations with multisets

Two multisets A and B are **equal**, or the **same**, if for any object x

$$m_A(x) = m_B(x).$$

A and B are unequal, denoted by $A \neq B$, if they are not equal. A is **less than or equal to** B, denoted by $A \leq B$, if for any object x

$$m_A(x) \leq m_B(x),$$

while A is **less than** B, denoted by $A < B$, if $A \leq B$ and $A \neq B$.

A is **built from** B, or more simply A is **from** B, denoted by $A \subseteq B$, if for any object x

$$x \in A \Rightarrow x \in B.$$

For example if

$$A = [1\_4\_2\_2]$$
$$B = [4\_1\_2\_1\_4]$$

then $B \subseteq A$ and $A \subseteq B$ but $A \neq B$. Note therefore that $\subseteq$ is not a partial order, and that although $A \leq B$ implies $|A| \leq |B|$, and $A < B$ implies $|A| < |B|$, no such relation holds for $\subseteq$. While this definition of $\subseteq$ may seem unfamiliar, it does coincide with the usual one when restricted to sets. It is quite useful in more general contexts.

If A and B are multisets, then the **union**, **intersection**, and **sum** of A and B, denoted by $A \cup B$, $A \cap B$, and $A + B$ respectively, are defined by the rules that for any object x

$$m_{A\cup B}(x) = m_A(x) \cup m_B(x)$$
$$m_{A\cap B}(x) = m_A(x) \cap m_B(x).$$
$$m_{A+B}(x) = m_A(x) + m_B(x)$$

For example if

$$A = [2\_3\_1\_1]$$
$$B = [1\_3\_3\_1\_1]$$

then

$$A \cup B = [2\_3\_3\_1\_1\_1]$$
$$A \cap B = [3\_1\_1]$$
$$A + B = [1\_1\_1\_1\_1\_2\_3\_3\_3].$$

Clearly for any multisets A and B,

$$|A + B| = |A| + |B| = |A \cup B| + |A \cap B|.$$

The main rules satisfied by these operations are the commutative and associative laws

$$\begin{array}{cc} A + B = B + A & (A + B) + C = A + (B + C) \\ A \cup B = B \cup A & (A \cup B) \cup C = A \cup (B \cup C) \\ A \cap B = B \cap A & (A \cap B) \cap C = A \cap (B \cap C) \end{array}$$

the idempotent laws

$$A \cup A = A$$
$$A \cap A = A$$

the identity laws

$$A \cup [\,] = A$$
$$A \cap [\,] = [\,]$$
$$A + [\,] = A$$

and the distributive laws

$$A + (B \cup C) = (A + B) \cup (A + C)$$
$$A + (B \cap C) = (A + B) \cap (A + C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Thus the multiset operations form a realm. Does that mean we are suggesting that there is something called the 'collection' of all multisets? No. It means that the operations $+, \cup$ and $\cap$ obey the realm laws with the empty set acting as the 0 element.

How do we prove all these laws? Each one corresponds exactly to a law for natural numbers. For example, to verify the distributive laws

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A + (B \cup C) &= (A + B) \cup (A + C) \end{aligned}$$

for multisets A, B, C, it suffices to demonstrate the corresponding laws

$$\begin{aligned} a \cup (b \cap c) &= (a \cup b) \cap (a \cup c) \\ a + (b \cup c) &= (a + b) \cup (a + c) \end{aligned}$$

for natural numbers a, b, c, since for any object x we may set $a = m_A(x)$, $b = m_B(x), c = m_C(x)$. Note the difference between this approach and the (usual) idea that to prove two sets equal, one must show that each is contained in the other.

The additive structure on multisets is important for implementing more effective counting procedures in mathematics. To further promote this idea, we introduce the following linear notation. If A is a multiset then let $2A$ denote $A + A$, let $3A$ denote $A + A + A$ and in general for a natural number k let kA denote the sum of k $A's$. Define $0A$ to be ϕ. This then allows us to write multisets as **linear combinations** of other multisets, as in

$$[2\_3\_2\_4\_4\_4] = 2[2] + [3] + 3[4]$$

Thus ideas from linear algebra are incorporated into the theory of multisets, and conversely multisets provide a simple model for linear algebra.

There is also a multiplicative operation for multisets. Define $A \times B$, the **direct product** of the multisets A and B, to be the multiset consisting of all ordered pairs $[a, b]$ with $a \in A$ and $b \in B$. By this we mean that

$$m_{A \times B}([a, b]) = m_A(a)\, m_B(b).$$

For example

$$[1\_2] \times [2\_3\_2] = [[1,2]\_[1,3]\_[1,2]\_[2,2]\_[2,3]\_[2,2]].$$

It follows that

$$A \times [] = [] \times A = [].$$

Clearly

$$|A \times B| = |A|\,|B|.$$

We have distributive laws

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C) \\ A \times (B \cap C) &= (A \times B) \cap (A \times C) \\ A \times (B + C) &= (A \times B) + (A \times C). \end{aligned}$$

However in general the commutative and associative laws do not hold for direct products, that is, in general

$$A \times B \neq B \times A$$
$$(A \times B) \times C \neq A \times (B \times C)\,.$$

A key feature of all the definitions in this section is that they have been **generic**, meaning that they have been independent of the specific position or role played in a larger framework by the multisets involved. For example, the statement

$$[2\_4\_4\_4\_1] \cap [5\_2\_5\_4] = [2\_4]$$

is independent of the relative 'positions' or circumstances of the multisets involved. This will be contrasted in a later section with operations which we call *concrete*, which will depend on a larger context for their interpretation, such as notions relating to inclusions of multisets. It may thus be convenient to use the terms **generic union** and **generic intersection** to describe the operations $\cup$ and $\cap$ introduced in this section.

5 Integral and rational multisets

Let's now widen our definition of multisets to **integral multisets**, which are linear combinations of objects involving coefficients from the integers. If we wish to specify that an object a appears with negative multiplicity in a multiset, we embrace it with round brackets (a), a convention familiar to accountants. Thus

$$A = [2\_4\_(5)\_(5)\_4\_4]$$

is an integral multiset in which the objects 2 and 4 appear with multiplicity 1 and 3 respectively while the object 5 appears with multiplicity -2. For integral multisets the linear notation is especially convenient, so that we also write

$$A = [2] + 3\,[4] - 2\,[5]\,.$$

Note that $|A|$ is still the sum of the multiplicities of all objects appearing in A, thus

$$|A| = 1 + 3 - 2 = 2.$$

Having in this way extended the possible coefficients that can appear to integers, it is but a small step to introduce **rational multisets**, which are linear combinations involving rational coefficients. A particular example is

$$B = \frac{3}{4}\,[5] - \frac{1}{2}\,[2\_7]\,.$$

The multiplicity of an object a in a rational multiset A is the sum of the coefficients of $[a]$ in the linear combination defining A. The size of such a multiset is the sum of the coefficients, and if the size of A is r then we refer to A as an r-set. Thus B above is a $\frac{1}{4}$-set, while

$$C = \frac{1}{2}[7] + \frac{1}{3}[9] + \frac{1}{6}[11]$$

is a 1-set. If we wish to emphasize that all the coefficients of C are in fact strictly positive, we will use the term **positive** multiset.

In practise, many applications will primarily involve positive multisets, with non-positive multisets appearing as intermediaries in calculations. This is much the same as the usual situation with numbers, where the majority of arithmetical calculations are primarily concerned with positive numbers, and negative numbers are often only intermediaries. (A definition of a mathematician: someone who believes that negative numbers are just as important as positive ones.)

By introducing the notation $(-1)\,A = -A$ for the multiset consisting of all the elements of A with multiplicities multiplied by -1, we get some further laws

$$nA + mA = (n+m)A$$
$$n(mA) = (nm)\,A$$

valid for all multisets A and all integers (or rational numbers) n and m.

Now define the **difference** $A - B$ between two multisets A and B as

$$A - B = A + (-B)$$

or equivalently by the rule that for any object x

$$m_{A-B}(x) = m_A(x) - m_B(x).$$

The reader should be warned that this concept does not agree with the standard usage in current set theory, where $A - B$ is used to denote what we would call $A - (A \cap B)$. This standard usage is exceedingly unfortunate, as the benefit obtained by this slight short cut cannot begin to compensate for the accompanying awkwardness, and lack of functoriality. To speak nothing about aesthetics!

Another more subtle warning should be mentioned. Our operation $A - B$ is not on a par with our earlier operations, in the sense that it does not extend unambiguously to *infinite multisets,* which we do not define here, but which play an important role (so we are told) in many branches of mathematics.

We may now derive the 'de Morgan type' laws

$$(-A) \cap (-B) = -(A \cup B)$$
$$(-A) \cup (-B) = -(A \cap B)$$

and their relative versions

$$(A - B) \cap (A - C) = A - (B \cup C)$$
$$(A - B) \cup (A - C) = A - (B \cap C)$$

valid for all (finite) multisets A, B, C. Note that these laws do not require B and C to be less than or equal to A. Also this treatment has the advantage of eliminating reference to a 'universal set U', another dubious concept endemic in undergraduate texts on set theory.

In many situations we want to define sets, or multisets, in terms of certain properties. Thus we might speak about the set S of all the integers between 1 and 100 which are relatively prime to 54 for example. If we denote this property of an integer n by $P(n)$, then we agree to use the convention that

$$S = \{n | P(n)\}.$$

If A is a multiset and f is a function which acts on the elements of A, then we can define a multiset

$$f(A) = [f(a) | a \in A].$$

In such cases our convention is that multiplicities must be firmly respected! That means that each element a of A contributes to the multiset $f(A)$, so that in particular

$$|f(A)| = |A|.$$

For example if $f(x) = x^2$ and $A = [-1\_0\_1\_2]$ then $f(A) = [1\_0\_1\_4]$.

6 Identical multisets and submultisets

We are now going to introduce some concepts concerning *identity* and *submultisets* that are not part of classical set theory or mathematics. These come into view only when we acknowledge the possibility that indistinguishable but separate objects are occurring as components in larger structures. Such a view, as we argued earlier, is warranted by trying to make our mathematics more carefully model real life and science.

Let us remind the reader of the distinction between *objects*, which may occur many times in a given data structure, and the *elements* of the data structure, which are to be considered as separate and distinct. Thus

$$X = [7\_9\_9]$$

has three elements, although only two objects occur as elements. The two $9's$ are separate elements, although they are the same object. This is exactly the situation the chemist faces when considering a water molecule with one oxygen atom and two hydrogen atoms; the latter two hydrogen atoms are distinct and separate although they are the same type of physical object.

Consider the two multisets

$$X = [2\_1\_1\_3\_4\_1\_1\_2]$$
$$Y = [5\_2\_1\_1\_5].$$

Suppose we use the particular order of the elements of X and Y given here to define some multisets. Let A be the multiset consisting of the first three elements of X, let B be the multiset consisting of the last three elements of X, and let C be the multiset consisting of the second, third and fourth elements of Y. Now let us define a fourth multiset D to consist of the third, second and first elements of X. Thus $A = B = C = D = [2\_1\_1]$.

Since each element of A is an element of X, we call A a **submultiset** of X, denoted by $A \sqsubseteq X$. Similarly $B \sqsubseteq X$ and $C \sqsubseteq Y$. Being a submultiset is a stronger concept than being less than or equal to, in the sense that in general $A \sqsubseteq B$ implies $A \leq B$.

If A is a submultiset of B but A is not identical to B, then we write $A \sqsubset B$ and say that A is a **proper submultiset** of B. Thus $A \sqsubset B$ implies $A < B$.

The four multisets are all equal. Nevertheless it is indisputable that the relation between A and D is in some sense different than the relation between B and C. With regard to the particular context in which they were defined, A and D comprised *exactly the same elements* of X. We will thus declare A and D to be **identical**, denoted by

$$A \equiv D.$$

However we do not declare B and C to be identical, since although they both consist of the same *objects*, they do not consist of the same *elements*. Thus identity is a stronger concept than equality, in the sense that in general $A \equiv B$ implies $A = B$.

The **power multiset** $P(A)$ of a multiset A is the multiset of all submultisets of A. As an example, with $D = [2\_1\_1]$,

$$P(D) = [[\ ]\_[2]\_[1]\_[1]\_[2\_1]\_[2\_1]\_[1\_1]\_[2\_1\_1]].$$

With our definition of $P(A)$ it should be clear that

$$|P(A)| = 2^{|A|}$$

since every element of A can either appear or not appear in a submultiset. This illustrates the *general principle* that if we take a concept from set theory, and apply it to a situation where the distinction between some elements of a set becomes blurred, then we should arrive at the corresponding concept from multiset theory.

If A and B are submultisets of a multiset X, then the **concrete intersection** of A and B, denoted by $A \sqcap B$, is the submultiset of X which consists of all those elements of X which are both elements of A and elements of B. Note that elements, not objects, are being discussed here. Similarly the **concrete union** of A and B, denoted by $A \sqcup B$, is the submultiset of X which consists of all elements of X which are either elements of A or elements of B or both.

These operations also satisfy the commutative, associative and distributive laws, and furthermore

$$A \sqcap B \leq A \cap B$$
$$A \sqcup B \geq A \cup B.$$

Having defined both generic and concrete unions and intersections, it is natural to ask about the status of addition $+$. Is it a generic or a concrete operation? The answer is that it is both. This is clarified by precise usage of the relations $=$ and $\equiv$. We have already defined the statement

$$A + B = C$$

for multisets A, B, C to mean that for any object x

$$m_A(x) + m_B(x) = m_C(x).$$

This is a generic definition, in that it is independent of the particular details (position, role) of the multisets A, B, C. However we are now in a position to introduce the statement

$$A + B \equiv C$$

which is a *concrete* definition, affirming that the multiset C consists identically of the elements of A together with the elements of B. Clearly then $A + B \equiv C$ implies $A + B = C$ but not the other way.

7 The max-plus, tropical, or minimax algebra

A useful tool to evaluate complicated expressions involving multisets and the operations $\cup, \cap, +$ is given by the **max-plus algebra**, also known as the **tropical calculus**, or the **minimax algebra**. It seems to have had its origins in control theory, see [**?**],[Cu1],[Cu2],[St]. We will present it here in the context of the realm of natural numbers and then observe that it extends immediately to the realm of multisets. Our notation is intended to simplify the transition to this algebra, but it is nonstandard.

The idea is to introduce two new operations on rational numbers, which we call **tropical addition** and **tropical multiplication**, which allow us to use the more familiar rational manipulations involving $+$ and $\times$ to (cleverly) replace those of $\cup, \cap$ and $+$. For ease of use, an equation flagged with the symbol (T) will be interpreted in the tropical sense, that is, all additions and multiplications that appear will be tropical ones. Indeed, we will agree that all operations that appear in a tropical equation are considered to be tropical. The definitions are as follows.

For rational numbers x, y, z,

$$\begin{aligned} x + y = z \ \ (T) &\Longleftrightarrow x \cup y = z \\ xy = z \ \ (T) &\Longleftrightarrow x + y = z \end{aligned}$$

Both tropical operations are commutative and associative. Tropical multiplication turns the rational numbers into a commutative group with identity 0 and inverse operation $x \to -x$, so that powers correspond to ordinary multiples. That is for any integer n and any rationals x and y,

$$y = x^n \ \ (T) \Longleftrightarrow y = nx.$$

In particular, the multiplicative inverse of x, denoted by x^{-1} (T), is just the rational number $-x$, and so is defined for *all* rational numbers x. This allows us to introduce division, by the rule

$$\frac{x}{y} = xy^{-1} \ \ (T)$$

Thus

$$\frac{x}{y} = z \ \ (T) \Longleftrightarrow x - y = z.$$

Tropical addition enjoys the **idempotent** property

$$x + x = x \ \ (T).$$

Note however that tropical addition does not admit inverses, that is, there is no operation of subtraction. This implies that we are not able to deduce $a = b$ from $a + c = b + c$. Furthermore

$$0 + x = x \ \ (T)$$

if and only if x is positive ($x \geq 0$) while

$$0 + x = 0 \ \ (T)$$

if and only if x is negative ($x \leq 0$).

Addition and multiplication are connected by the distributive law, namely

$$x\,(y + z) = xy + xz \ \ (T).$$

Since

$$x \cap y = x + y - x \cup y$$

we deduce that

$$z = x \cap y \Longleftrightarrow z = \frac{xy}{x + y} \ \ (T)$$

This allows us to convert any expression involving $+, -, \cup, \cap$ into an equivalent tropical rational expression. As an example, here is a new proof of a rather familiar result!

Proposition 1 *The distributive law*

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

holds for all rational numbers a, b, c.

Proof. The corresponding tropical equation is

$$a + \frac{bc}{b + c} = \frac{(a + b)\,(a + c)}{(a + b) + (a + c)} \ \ (T)$$

Both sides of this tropical equation may be simplified by using rational operations, (remember that subtraction is not allowed, however.) The result is

$$\frac{ab+bc+ca}{b+c} = \frac{a^2+ab+ac+bc}{a+b+c} \ (T) \Longleftrightarrow$$
$$(ab+bc+ca)(a+b+c) = (b+c)\left(a^2+ab+ac+bc\right) \ \ (T)$$

But using the idempotent identity, both sides of this tropical equation are

$$a^2b+ab^2+b^2c+bc^2+ca^2+c^2a+abc \ \ (T)$$

so the equation is valid. ■

The following plays a role in the Inventory version of Hall's theorem given in the next section.

Proposition 2 *For any integral or rational multisets R, S, Q,*

$$R \cap S + Q \cap (S - R \cap S) \leq (R+Q) \cap S.$$

Furthermore if Q is a postive multiset, then equality holds.

Proof. Suppose that q, r, s, t are rational numbers with

$$t = (r \cap s) + (q \cap (s - r \cap s)) - ((r+q) \cap s).$$

The corresponding tropical equation is

$$t = \frac{rs}{r+s} \frac{qs\left(\frac{rs}{r+s}\right)^{-1}}{q+s\left(\frac{rs}{r+s}\right)^{-1}} \left(\frac{rqs}{rq+s}\right)^{-1} \ \ (T).$$

Simplifying using rational operations, this becomes

$$t = \frac{rq+s}{rq+r+s} \ \ (T)$$

Then using properties of 0 and the idempotent law for addition,

$$\begin{aligned} t+0 &= \frac{rq+s}{rq+r+s} + \frac{rq+r+s}{rq+r+s} \ \ (T) \\ &= \frac{rq+rq+r+s+s}{rq+r+s} \ \ (T) \\ &= \frac{rq+r+s}{rq+r+s} = 0 \ \ (T) \end{aligned}$$

and so we see that $t \leq 0$. This implies that for any integral or rational multisets R, S, Q,

$$R \cap S + Q \cap (S - R \cap S) \leq (R+Q) \cap S.$$

Now suppose that $q \geq 0$ so that

$$q = q + 0 \ \ (T).$$

In that case

$$\begin{aligned} t &= \frac{r\,(q+0)+s}{rq+r+s} \ \ (T) \\ &= \frac{rq+r+s}{rq+r+s} = 0 \ \ (T) \end{aligned}$$

We deduce that if R, S, Q are multisets with Q positive, then

$$R \cap S + Q \cap (S - R \cap S) = (R + Q) \cap S.$$

■

8 The pigeonhole principle

Theorem 3 *(Pigeonhole principle for multisets) Suppose that $A_1, A_2, \cdots, A_k$ are submultisets of a multiset X satisfying*

$$A_i \sqcap A_j = \phi$$

for all i and j. Then

$$A_1 + A_2 + \cdots + A_k \sqsubseteq X.$$

Proof. Fix an object x occurring in X. All occurrences of x as an element of A_i are by assumption distinct from occurrences of x as an element of A_j, for any i and j. Thus $A_1 + A_2 + \cdots + A_k$ is a submultiset of X. ■

9 Inclusion/Exclusion

Theorem 4 *(Inclusion /Exclusion for multisets) Let $A_1, \ldots A_n$ be multisets. Then*

$$A_1 \cup \cdots \cup A_n = \sum_{1 \leq i \leq n} A_i - \sum_{1 \leq i_1 < i_2 \leq n} A_{i_1} \cap A_{i_2} + \cdots + (-1)^{n+1} A_1 \cap \cdots \cap A_n.$$

Proof. Suppose that x is an object that appears in one of the A_i. If $m_{A_i}(x) = a_i$ then we need to show that

$$a_1 \cup \cdots \cup a_n = \sum_{1 \leq i \leq n} a_i - \sum_{1 \leq i_1 < i_2 \leq n} a_{i_1} \cap a_{i_2} + \cdots + (-1)^{n+1} a_1 \cap \cdots \cap a_n.$$

We may suppose that $a_1 \geq a_2 \geq \cdots \geq a_n$. Then for $i \geq 1$ the number of times a_{i+1} appears on the right hand side is

$$1 - i + \binom{i}{2} - \binom{i}{3} + \cdots \pm \binom{i}{i} = 0$$

while the number of times a_1 appears on the right hand side is 1. ■

Since $|A \pm B| = |A| \pm |B|$ we get, as a Corollary, the usual Principle of Inclusion/Exclusion (extended to multisets), namely

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

10 Hall's theorem

Amazon.com sells books over the internet. Suppose that its current inventory of books is denoted by X, clearly a multiset. A typical internet order $\mathcal{O}$ involves two pieces of data; the total number n of books ordered and the multiset A of books requested, with $n = |A|$. Clearly Amazon can fill this order from its inventory if and only if $A \le X$, in which case a submultiset $F \sqsubseteq X$ must be found with $A = F$, and that multiset F physically removed from the inventory and sent to the customer.

If a **catalogue** $B = [\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_k]$ of orders is received, with $\mathcal{O}_i = [n_i, A_i]$, then a list $L = [F_1, F_2, \cdots, F_k]$ of submultisets of X must be found such that $F_i = A_i$ for $i = 1, \ldots, k$ and such that $F_i \sqcap F_j = \phi$ for all $i \ne j$. Such a **filling** of the catalogue of orders B is possible if and only if

$$A_1 + A_2 + \cdots + A_k \le X.$$

Note that already the discussion of this simple situation involves both generic and concrete concepts.

Even with a sizeable inventory X, large catalogues of orders (say all those received on a given day) may fail to satisfy this condition and Amazon will be forced to ask some customers to modify or delay their orders. To lessen this problem, suppose that Amazon decides to give discounts to customers whose orders are **flexible**, by which we mean that the order $\mathcal{O} = [n, A]$ satisfies $n \le |A|$ rather than equality. In effect this means that the customer wants n books but is willing to have these books chosen, perhaps randomly, from a generally larger multiset A of possibilities.

So let us define a **filling** of a flexible order $\mathcal{O} = [n, A]$ to be a submultiset $F \sqsubseteq X$ with the properties that $F \le A$ and $|F| = n$. If $B = [\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_k]$ is a catalogue of flexible orders, then define its **size** to be

$$s(B) = n_1 + n_2 + \cdots + n_k$$

and its **range** to be the multiset

$$A_B = A_1 + A_2 + \cdots + A_k.$$

A **filling** of such a catalogue B of orders is then a list $L = [F_1, F_2, \cdots, F_k]$ of submultisets of X such that F_i is a filling of order $\mathcal{O}_i$ for i from 1 to k and such that for all $i \ne j$,

$$F_i \sqcap F_j = \phi.$$

The Pigeonhole Principle then implies that

$$F_1 + F_2 + \cdots + F_k \sqsubseteq X.$$

When can a given catalogue B of orders be filled?

Theorem 5 *(Inventory version of Hall's theorem) A catalogue $B = [\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_k]$ of orders $\mathcal{O}_i = [n_i, A_i]$ from a multiset X can be filled if and only if for any sublist $B' = [\mathcal{O}_{i_1}, \mathcal{O}_{i_2}, \cdots, \mathcal{O}_{i_m}]$ of B,*

$$s(B') \leq |A_{B'} \cap X|.$$

Remark 6 *Suppose that X is a set, that each A_i is a subset of X, and that each $n_i = 1$. In that case the theorem states that we can find a system of distinct representatives x_i for the A_i if and only if for any sublist $[A_{i_1}, A_{i_2}, \cdots, A_{i_m}]$ we have $m \leq |(A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_m}) \cap X|$. This is Hall's Theorem ([**?**]).*

Proof. The following proof follows the general form of [Ha]. We have already observed the necessity of the condition. To prove sufficiency, we proceed by induction on the size $n = |B|$. For $n = 1$ it is clear that B can be filled. So suppose that $B = [\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_k]$ is a book of k orders $\mathcal{O}_i = [n_i, A_i]$, i from 1 to k, from a multiset X, that satisfies the condition that for any proper sublist $B' = [\mathcal{O}_{i_1}, \mathcal{O}_{i_2}, \cdots, \mathcal{O}_{i_m}]$ of B,

$$s(B') \leq |A_{B'} \cap X|.$$

Define a sublist $B' = [\mathcal{O}_{i_1}, \mathcal{O}_{i_2}, \cdots, \mathcal{O}_{i_m}]$ of B to be **critical** if

$$s(B') = |A_{B'} \cap X|.$$

We consider two cases. The first case is when there exists a critical proper sublist of B, let us say without any loss of generality

$$B_0 = [\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_m]$$

with

$$A_0 = A_1 + A_2 + \cdots + A_m$$
$$n_0 = n_1 + n_2 + \cdots + n_m$$

for some $m < k$. Since $s(B') = n_0 < n$, by induction there is a filling of B_0, namely a list

$$L = [F_1, F_2, \cdots, F_m]$$

of submultisets of X such that F_i is a filling of order $\mathcal{O}_i$ for i from 1 to m and such that $F_i \sqcap F_j = \phi$ for all i, j from 1 to m. Then define

$$F_0 \equiv F_1 + F_2 + \cdots + F_m \sqsubseteq X.$$

It follows that $F_0 \leq A_0 \cap X$, and since B_0 is a critical sublist $|F_0| = |A_0 \cap X|$, so that

$$F_0 = A_0 \cap X.$$

Let

$$X' \equiv X - F_0 = X - A_0 \cap X$$

and consider the $k - m$ remaining orders

$$B' = [\mathcal{O}_{m+1}, \mathcal{O}_{m+2}, \cdots, \mathcal{O}_k].$$

We need to show that we may fill the book B' from X'. Choose any sublist $B'' = [\mathcal{O}_{j_1}, \mathcal{O}_{j_2}, \cdots, \mathcal{O}_{j_s}]$ of B' and let

$$A'' = A_{j_1} + A_{j_2} + \cdots + A_{j_s}$$
$$n'' = n_{j_1} + n_{j_2} + \cdots + n_{j_s}.$$

Now we know from the previous section that if R, Q, S are multisets with Q positive, then

$$R \cap S + Q \cap (S - R \cap S) = (R + Q) \cap S.$$

Applying this to the positive multisets A_0, A'', X we obtain

$$A_0 \cap X + A'' \cap (X - A_0 \cap X) = (A_0 + A'') \cap X$$

or equivalently

$$A_0 \cap X + A'' \cap X' = (A_0 + A'') \cap X.$$

By assumption the sublist $B_0 + B'' = [\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_m, \mathcal{O}_{j_1}, \mathcal{O}_{j_2}, \cdots, \mathcal{O}_{j_s}]$ satisfies the condition

$$s(B_0 + B'') = n_0 + n'' \leq |(A_0 + A'') \cap X| = |A_0 \cap X| + |A'' \cap X'| = n_0 + |A'' \cap X'|.$$

Thus

$$n'' \leq |A'' \cap X'|.$$

This shows that the book B' from X' satisfies the condition of the theorem and so by our induction hypothesis can be filled, say by a list

$$L' = [F_{m+1}, F_{m+2}, \cdots, F_k].$$

Then the combined list

$$[F_1, F_2, \cdots, F_m, F_{m+1}, F_{m+2}, \cdots, F_k]$$

is a filling of the book B. The second case is where there is no critical proper sublist of B. In this case, suppose that we pick a submultiset $[x_1]$ of X with $x_1 \in A_1$. Let

$$A_1' = A_1 - x_1$$
$$n_1' = n_1 - 1$$

and define $\mathcal{O}_1'$ to be $[n_1', A_1']$. It may be that $n_1' = 0$, in which case $\mathcal{O}_1'$ is an empty order. We claim that the modified book $B' = [\mathcal{O}_1', \mathcal{O}_2, \cdots, \mathcal{O}_k]$ of orders from $X' \equiv X - [x_1]$ satisfies the condition of the theorem. If a sublist B'' from B' does not contain $\mathcal{O}_1'$ then

$$s(B_0 + B'') < |A_{B''} \cap X|$$

since B'' is a sublist of B which contains no critical sublist. Thus since $|X'| = |X| - 1$,

$$s(B_0 + B'') \leq |A_{B''} \cap X'|.$$

If a sublist B'' from B' does contain $\mathcal{O}_1'$ then it is obtained from a sublist B_0'' from B by replacing $\mathcal{O}_1$ by $\mathcal{O}_1'$ so that

$$s(B'') = |B_0''| - 1 < \left|A_{B_0''} \cap X\right| - 1 \leq |A_{B''} \cap X'| + 2 - 1.$$

Thus

$$s(B'') \leq |A_{B''} \cap X'|.$$

This shows the claim, and so by induction there is a filling $L' = [F_1', F_2, \cdots, F_k]$ of the book of orders $B' = [\mathcal{O}_1', \mathcal{O}_2, \cdots, \mathcal{O}_k]$ from X'. To fill the original book $B = [\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_k]$ it suffices to set

$$F_1 \equiv F_1' + [x_1]$$

and note that then $L = [F_1, F_2, \cdots, F_k]$ is a filling of B. ■

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