HYPERGROUPS, SYMMETRIC SPACES, AND WRAPPING MAPS

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Abstract. We discuss the connection between the classical theory of symmetric spaces and the more recent theory of hypergroups, with particular emphasis on the special case of a compact Lie group and a remarkable map called the wrapping map.

§0. Introduction.

This paper attempts an exposition of the connection between symmetric spaces and hypergroup theory. This connection is not new to those involved in hypergroups, as the classical symmetric spaces have always provided a very rich source of examples for the theory. However for those working in symmetric spaces, the usefulness of the hypergroup point of view may not be well known. It provides a new way of looking at some of the fundamental aspects of harmonic analysis on these spaces. In addition the language of hypergroups forces us to come to grips with concrete formulae for convolutions (often involving special functions) which the general theory often allows us to finesse. On the other hand, the rich and explicit structures underlying Lie theory allow us to write down explicit convolution structures for certain hypergroups that appear which suggest new directions for hypergroup theory as well.

A symmetric space X is a Riemannian manifold with a lot of symmetry; more specifically we require that for every point $x \in X$ there is a global isometry of X that fixes x and reverses every geodesic through x. Such spaces and harmonic analysis on them have been the subject of much study since the work of Cartan (see for example Cartan [Cn], Gelfand [Ge], Helgason [He1], [He2]).

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Let G denote the group of isometries of X and let K be the compact subgroup of G which fixes a specified point $O \in X$. Then $X \simeq G/K$ and there is a natural connection between harmonic analysis on the space X and right K–invariant harmonic analysis on G. In many applications one is especially interested in 'spherical' harmonic analysis on X— the study of functions on X which are further invariant under the action of K on X, or equivalently the study of K bi-invariant functions (or measures, or operators...) on G.

A hypergroup is a very useful generalization of a group. This notion was introduced into harmonic analysis in the 70's by Dunkl [Du], Jewett [Je] and Spector [Sp]— see also the surveys of Ross [Ros], Heyer [Hey] and Litvinov [Li] and the recent book of Bloom and Heyer [BH].

A hypergroup can be defined to be a locally compact space $\mathcal{K}$ for which the Borel measures $M(\mathcal{K})$ form a $*$-algebra satisfying (essentially) the following axioms

(1) (Closure) The product of Dirac delta functions $\delta_x * \delta_y$, for $x, y \in \mathcal{K}$, is always a compactly supported probability measure which varies continuously with x and y.
(2) (Associativity) The algebra $M(\mathcal{K})$ is associative.
(3) (Existence of an Identity) There exists an element $e \in \mathcal{K}$ such that δ_e is the identity.
(4) (Existence of Inverses) For every $x \in \mathcal{K}$ there exists a unique element x^* such that e is contained in the support of the measure $\delta_x * \delta_{x^*}$. Furthermore $(\delta_x)^* = \delta_{x^*}$.

These axioms are neither exactly precise nor standard; for a careful discussion see Jewett [Je].

So what is the relationship between these two concepts? Well, every symmetric space X has associated to it a canonical hypergroup $\mathcal{K}(X)$, which we call the *spherical* hypergroup of X. With the notation above, the underlying set for $\mathcal{K}(X)$ is the set of K orbits on X. The algebra structure on the measures on $\mathcal{K}(X)$ is induced from the convolution structure of K–invariant measures on X, which in turn may be defined by identifying K–invariant measures on X with K-bi-invariant measures on G and then using group convolution. The resulting hypergroup is independent of the choice of base point O.

This convolution is admittedly somewhat abstract; for this reason, and also for purposes of generalization, we will describe an alternate, more geometric approach to this convolution using elementary probability.

Actually there is a second hypergroup associated to X which is in some sense a linearization of $\mathcal{K}(X)$. The tangent space to X at the point O is a vector space V on which the group K acts linearly and we may convolve K–invariant measures in V. The resulting hypergroup has underlying set the collection of K orbits on V; we denote it by $\mathcal{K}(V)$.

With this setup, we may relate aspects of harmonic analysis on the symmetric space X with harmonic analysis on the hypergroup $\mathcal{K}(X)$. The point is that $\mathcal{K}(X)$ is a *commutative* hypergroup. Commutative hypergroups have a character theory much like that for locally compact abelian groups; that is, there is a notion of dual object, of Fourier transform, of Haar measure and Plancherel measure. There are however new features as well; the dual object is not always a hypergroup (although for many interesting examples it is), the Plancherel measure may be supported on a proper subset of the dual, etc. In any case, we find notions of *commutative* harmonic analysis relating to the geometry of the space X.

An interesting special case occurs when X is a compact Lie group G. The approach above then implies that harmonic analysis on the Lie group G and (commutative) harmonic analysis on the spherical hypergroup $\mathcal{K}(G)$ are closely connected (see Wildberger [Wi2]). What makes this point of view fruitful are two crucial facts. The first is that the linearization of $\mathcal{K}(G)$, namely $\mathcal{K}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, can be explicitly described using the geometry of root systems. The second is that there is a remarkable relation between the hypergroups $\mathcal{K}(G)$ and $\mathcal{K}(\mathfrak{g})$ given by a result which we call the wrapping theorem.

Of course there is a well known infinitesimal relation between harmonic analysis on a symmetric space and on its tangent space. Thus as we inflate a sphere, the sphere approaches its tangent plane, Legendre polynomials tend to Bessel functions, the Laplace Beltrami operator tends to the Euclidean Laplacian etc. The theory of group contractions (see Clerc [Cl], Dooley [Do], Dooley and Rice [DR]) provides a framework in which to generalize these phenomena. However what we are alluding to here is something considerably stronger— a *global* connection between harmonic analysis on X and on V for the special case $X = G$. Evidence for such a connection was hinted at in Wildberger [Wi1].

There is evidence that this relation can be extended (perhaps in a modified form) to more general symmetric spaces. Such a generalization will probably be related to Rouvière's e function (see Rouvière [Rou1], [Rou2], [Rou3]).

The existence of the wrapping map has numerous implications, including a simple explanation for why the unitary dual of G is in bijection with the set of integral coadjoint orbits and why the characters of the irreducible representations are given by the Kirillov character formula, an explanation for the naturalness of the Duflo isomorphism, and an argument for why the Laplace Beltrami operator is *not* the correct Laplacian for harmonic analysis on G.

We now describe the contents of this paper.

In §1 we describe the convolution structures associated to the simplest symmetric spaces— the spheres S^n and their tangent spaces $\mathbb{R}^n$. We use a down to earth, probabilistic approach to convolution.

In §2 we introduce the notion of a character of a hypergroup and relate a character of the spherical hypergroup $\mathcal{K}(X)$ to the more traditional notion of a spherical function on X. The spherical transform on the symmetric space (in the sense of Helgason [He2]), is essentially the Fourier transform on the commutative hypergroup $\mathcal{K}(X)$. We describe the dual hypergroups for the case of S^n and note a curious global relation between the characters of the spherical hypergroups $\mathcal{K}(S^n)$ and the corresponding linearizations $\mathcal{K}(\mathbb{R}^n)$ exactly when $n = 1, 3$. These happen to be the values of n for which S^n is a Lie group.

In §3 we depart from our main theme to show how the ideas discussed have applications to other areas— in particular to the harmonic analysis on a homogeneous tree. We give a somewhat nonstandard derivation of the characters of the corresponding spherical hypergroup (essentially a Hecke algebra) by using the hypergroup $\mathcal{K}(S^3)^\wedge$ discussed in the previous section.

In the remainder of the paper we try to clarify the phenomenon mentioned above for the spheres S^1 and S^3 by studying the hypergroup associated to a general compact semisimple Lie group G when regarded as a symmetric space $G \simeq (G \times G)/G$.

In §4 we describe the explicit structure of the linearization $\mathcal{K}(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G. This is joint work with Dooley and Repka. Here the underlying set $\mathcal{K}$ is a closed cone $\overline{C}$ in the Lie algebra $\mathfrak{t}$ of a maximal torus of G and the convolution of delta functions is given explicitly as a signed sum over the Weyl group of translations of the push down of the invariant measure on one of the orbits to $\mathfrak{t}$, at least when both orbits are generic. One of the interesting aspects of this hypergroup is that the support of such a convolution is always a convex polytope in $\overline{C}$.

In §5 we show how to relate the 'linear' hypergroup $\mathcal{K}(\mathfrak{g})$ to the 'nonlinear' hypergroup $\mathcal{K}(G)$ by the consideration of a map Φ from distributions on $\mathfrak{g}$ of compact support to distributions on G given by

$$\langle \Phi(\mu), f\rangle = \langle \mu, j\tilde{f}\rangle$$

where μ is a distribution of compact support on $\mathfrak{g}$, f is a smooth function on G, $\tilde{f} = f \circ \exp$ is its lift to the Lie algebra via the exponential map, and j is the square root of the Jacobian of the exponential map. The consideration of Φ, which we call the *wrapping map*, is motivated by the work of Harish Chandra [Ha], Kashiwara–Vergne [KV], Duflo [Du], and Helgason [He2].

Theorem 5.1. *([DW]) Let μ and ν be two* $\mathrm{Ad}(G)$*–invariant distributions of compact support on $\mathfrak{g}$. Then*

$$\Phi(\mu) * \Phi(\nu) = \Phi(\mu * \nu)$$

where the convolution on the left is group convolution and the convolution on the right is Euclidean convolution in $\mathfrak{g}$.

We call this result the *wrapping theorem*. Harish Chandra knew this result when one of the distributions was supported at $0 \in \mathfrak{g}$, and Frenkel [Fr] had proven this result for functions in unpublished work.

We show how to use it to provide a new way of establishing the unitary dual of G as the set of integral coadjoint orbits and the validity of the Kirillov character formula essentially without any knowledge of the detailed structure of G. The proof of the wrapping theorem itself, however, relies on the traditional machinery of representation theory; finding a structure free proof would be very interesting.

In §6 we show how the wrapping theorem relates invariant differential operators on G to constant coefficient differential operators on $\mathfrak{g}$. This shows why for many purposes of harmonic analysis (i.e. the heat or wave equations), the 'shifted' Laplacian is more natural than the usual Laplace Beltrami operator.

Some problems are distributed throughout the paper.

§1. Spherical convolution on a symmetric space.

The simplest examples of symmetric spaces X are the plane $\mathbb{R}^2$, the sphere S^2, and the hyperbolic plane H^2. For each of these the group K of isometries fixing a fixed point O is a circle. The point O is always a K orbit, and in the case of S^2, the antipodal point is also an orbit; all other orbits are circles centered at O. The set $\mathcal{K}$ of all K orbits on X is homeomorphic to $\mathcal{K} = \mathbb{R}^+$, $[0, \pi]$, and $\mathbb{R}^+$ for $\mathbb{R}^2$, S^2, and H^2 respectively. In each case $\mathcal{K}$ carries a natural convolution structure which makes it into an algebraic object called a hypergroup. Let us describe how this structure comes about.

We begin with a discussion of the simple case of $\mathbb{R}^2$. Two circles of radii r_1 and r_2, may be convolved by identifying each with the unique rotationally invariant probability measure supported on it and using the structure of $\mathbb{R}^2$ as abelian group to convolve these measures. The result is a rotationally invariant measure which can be decomposed as a suitable integral of the basic invariant measures on circles. However it is more convenient for purposes of generalization to think probabilistically.

Thus we randomly pick a point P of distance r_1 from O (using, as we do henceforth, the invariant probability measure just mentioned), and then randomly pick a point Q of distance r_2 from P, and ask what is the probability density $f_{r_1,r_2}(r)$ for Q to be of distance r from O?

If θ is the angle OPQ, then the cosine law states that

$$r^2 = \ r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta$$

from which we see that $2r\ dr = \ 2r_1 r_2 \sin\theta\ d\theta$. Because of the rotational symmetry, $f_{r_1,r_2}(r)dr$ will be the measure of that portion of the circle around P for which the angle OPQ is in $[\theta, \theta + d\theta]$, that is $d\theta/\pi$. Thus for $|r_1 - r_2| \le r \le r_1 + r_2$,

$$f_{r_1,r_2}(r) = \frac{1}{\pi}\frac{2r}{2r_1r_2\sin\theta} = \frac{2r}{4\pi Area(\triangle(OPQ))}.$$

Elementary trigonometry (or Heron's Theorem) lets us rewrite this as

$$f_{r_1,r_2}(r) = \frac{2r}{\pi[(r^2-(r_1-r_2)^2)((r_1+r_2)^2-r^2)]^{1/2}}\,\chi_{[|r_1-r_2|,r_1+r_2]}$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$.

The function $f_{r_1,r_2}(r)$ determines the convolution structure of rotationally invariant measures in the plane. By identifying the circle of radius r with the point $r \in \mathbb{R}^+$, we may transfer this convolution structure to a convolution structure on the set $M(\mathbb{R}^+)$ of Borel measures on $\mathbb{R}^+$; specifically we write

$$\delta_{r_1} * \delta_{r_2} = f_{r_1,r_2}(r)dr$$

where δ_{r_i} is the Dirac measure at r_i. This convolution extends by linearity and continuity to an algebra structure on $M(\mathbb{R}^+)$; see Jewett [Je] for a wider discussion of this fundamental example. We further make $M(\mathbb{R}^+)$ into a $*$-algebra by introducing the involution $x^* = x$. Setting $\mathcal{K} = \mathbb{R}^+$, we see that the $*$-algebra $M(\mathcal{K})$ satisfies the axioms 1)– 4) of the introduction and so $\mathcal{K}$ is a *hypergroup.*

Recall that a hypergroup $\mathcal{K}$ is called *commutative* if $M(\mathcal{K})$ is commutative, *Hermitian* if the involution $x \to x^*$ is the identity, and *discrete* or *compact* if $\mathcal{K}$ is discrete or compact respectively. Hermitian hypergroups are always commutative. We will say further that $\mathcal{K}$ is *conical* if $\mathcal{K}$ is homeomorphic to a cone in some $\mathbb{R}^n$ in such a way that $e = 0$.

The set of circles centered at the origin in $\mathbb{R}^2$ thus forms a Hermitian conical hypergroup $\mathcal{K} \simeq \mathbb{R}^+$.

Let us quickly generalize the above reasoning to $\mathbb{R}^n$ viewed as a symmetric space, so that K is $O(n, \mathbb{R})$ and the K orbits are spheres centered at O. The only essential change in the above calculation is that now the portion of the sphere for which the angle OPQ lies in $[\theta, \theta + d\theta]$ has measure $\sin^{n-2}\theta d\theta/c_n$ where

$$c_n = \int_0^\pi \sin^{n-2}\theta d\theta = \frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}}{\Gamma(\frac{n}{2})}.$$

We get the new probability density

$$f^n_{r_1,r_2}(r) = \frac{2r}{c_n}\frac{[(r^2-(r_1-r_2)^2)((r_1+r_2)^2-r^2)]^{\frac{n-3}{2}}}{(2r_1r_2)^{n-2}}\chi_{[|r_1-r_2|,r_1+r_2]}.$$

Thus for each $n \in \mathbb{Z}, n > 1$, we get a new hypergroup structure on $\mathbb{R}^+$, namely

$$\delta_{r_1} *_n \delta_{r_2} = \ f^n_{r_1,r_2}(r)dr.$$

Note for future reference the somewhat special case when $n = 3$; in this case if we renormalize the invariant measure on the sphere of radius r to have total mass r, then the convolution of two such measures is written as a uniform integral of similiar measures.

The probabilistic approach to convolution outlined above generalizes to define the spherical hypergroup $\mathcal{K}(X)$ for any symmetric space X. For any two K orbits $\mathcal{O}_1, \mathcal{O}_2$ about O, we choose a point P on $\mathcal{O}_1$ randomly using the unique K–invariant probability measure which exists on it. Now the isotropy subgroup of G at P, that is those isometries of X which fix P, is conjugate in G to K, so that it makes sense to talk about the translate of the orbit $\mathcal{O}_2$ around O to P, say $\mathcal{O}_2'$. In fact we may choose any isometry ρ in G which sends O to P, and consider the image of $\mathcal{O}_2$ under ρ. Choose a point Q randomly on this orbit $\mathcal{O}_2'$, and consider the probability density that Q lies on the general K orbit $\mathcal{O}$ about O. We interpret this as a probability measure on the space $\mathcal{K}(X)$ of K orbits on X which is the product of the Dirac masses at $\mathcal{O}_1$ and $\mathcal{O}_2$.

For the sphere S^n, the spherical hypergroup $\mathcal{K}(X) \simeq [0, \pi]$ indexes the set of small hypercircles centered at a fixed point O. To find the hypergroup structure, we use the spherical cosine law

$$\cos r = \ \cos r_1 \cos r_2 + \sin r_1 \sin r_2 \cos\theta$$

and the previous analysis for $\mathbb{R}^n$ to obtain the probability density

$$g^n_{r_1,r_2}(r) = \frac{\sin r}{c_n} \frac{[(\cos(r_1 - r_2) - \cos r)(\cos r - \cos(r_1 + r_2))]^{\frac{n-3}{2}}}{[(\sin r_1)(\sin r_2)]^{n-2}} \chi_{[|r_1 - r_2|, r_1 + r_2]}(r)$$

where all constants are taken modπ. One can check that for small r, r_1, r_2 this density is approximated by $f^n_{r_1,r_2}(r)$.

At this point it may be useful to discuss a point which may be puzzling the reader. Why do we introduce the notion of a hypergroup and not simply work with the algebras of K–invariant measures or functions (or even distributions) on X? The answer is that the hypergroup notion carries not only the algebra structure but also information regarding a privileged (generalized) basis of the algebra. This is a subtle but important distinction. We are expressing interest not so much in an abstract algebra but in the actual structure constants of that algebra with respect to a particular (generalized) basis.

This is perhaps more clearly illustrated in the finite case: consider a finite group G as a homogenous space $G \simeq (G \times G)/G$. Then the orbits of the isotropy group of the identity are just the conjugacy classes. The algebra of central functions is as an abstract algebra rather trivial (it is isomorphic to $\mathbb{C}^n$ where n is the number of conjugacy classes), but as a hypergroup it is decidedly nontrivial.

As a result, when we discuss hypergroup structures, we can not easily avoid the actual determination of explicit structure constants and equations. This is why the theory of special functions becomes very important for us. On the other hand, the theory of hypergroups sheds new light on the significance and meaning of certain aspects of special function theory.

Let us conclude this section with some problems.

Problem 1. *Determine for each symmetric space X the explicit structure of the associated spherical hypergroup $\mathcal{K}(X)$.*

Problem 2. *Does the hypergroup $\mathcal{K}(X)$ determine X? If so, can we recover X from $\mathcal{K}(X)$?*

Problem 3. *For what more general Riemannian manifolds X does the spherical hypergroup $\mathcal{K}(X)$ make sense?*

As a partial answer to the last question, we note that if (G, K) is a Gelfand pair and $X = G/K$ then $\mathcal{K}(X)$ is well defined and that this set up includes both the symmetric space case and the action of a compact group on a vector space.

§2. Characters and spherical functions

All of the hypergroups that we deal with in this paper are commutative, so we restrict our attention from now on to commutative hypergroups $\mathcal{K}$.

A *character* of $\mathcal{K}$ is a bounded function $\psi : \mathcal{K} \to \mathbb{C}$ such that

(1) $\psi(x)\psi(y) = \int_{\mathcal{K}} \psi(z)\delta_x * \delta_y(z)$ for all $x, y \in \mathcal{K}$
(2) $\psi(x^*) = \overline{\psi(x)}$ for all $x \in \mathcal{K}$

The set of all characters of $\mathcal{K}$ is denoted $\mathcal{K}^\wedge$.

Although in general there does not appear to be a complete duality between $\mathcal{K}$ and $\mathcal{K}^\wedge$ as for commutative locally compact groups, there are many important cases where $\mathcal{K}^\wedge$ is indeed a hypergroup under pointwise multiplication and complex congugation, and where the corresponding dual object of $\mathcal{K}^\wedge$ is isomorphic to the original $\mathcal{K}$.

In the case when $\mathcal{K}$ is the spherical hypergroup of a symmetric space X as above, a character $\psi \in \mathcal{K}^\wedge$ may be regarded as a function on X constant on K orbits. This turns out to be a bounded spherical function on X in the sense of Helgason [He2]. The Fourier transform on $\mathcal{K}(X)$ becomes what is known in symmetric space theory as the spherical transform on X. Thus the hypergroup point of view provides a new language to discuss spherical harmonic analysis.

Since we are emphasizing the *algebraic* content of the hypergroups $\mathcal{K}(X)$, it is natural to ask the following question.

Problem 4. *Does the set $\mathcal{K}(X)^\wedge$ carry a hypergroup structure and if so can one describe this explicitly?*

To give the reader a sense for these characters and their algebraic structure, we will describe the situation in the basic example of the sphere S^n and its tangent space $\mathbb{R}^n$. The situation for the symmetric space $X = \mathbb{R}^n$ is simplified by the fact that $\mathbb{R}^n$ is a vector space.

Suppose more generally that we have a linear action of a compact Lie group K on a finite dimensional vector space V. The orbits of K on V form a hypergroup $\mathcal{K}(V;K)$. However K also acts on the dual vector space V^* by the rule $(kf)(X) = f(k^{-1}X)$ for $k \in K$, $f \in V^*$ and $X \in V$. We thus may also form the hypergroup $\mathcal{K}(V^*;K)$. The main point is now that

$$\mathcal{K}(V;K)^\wedge \simeq \mathcal{K}(V^*;K).$$

This means that every orbit $\mathcal{O} \subset V^*$ determines a character $\chi_\mathcal{O}$ of $\mathcal{K}(V;K)$ and conversely. We can furthermore exhibit this character explicitly. If $\mu_\mathcal{O}$ is the invariant probability measure on $\mathcal{O}$ and $\mu_\mathcal{M}$ is the invariant probability measure on an orbit $\mathcal{M} \subset V$, then

$$\chi_\mathcal{O}(\mathcal{M}) = \int_\mathcal{O}\int_\mathcal{M} e^{if(X)} d\mu_\mathcal{M}(X) d\mu_\mathcal{O}(f).$$

By choosing any $X_0 \in \mathcal{M}$ and any $f_0 \in \mathcal{O}$, we may use the K–invariance in the above integral to rewrite it as

$$\chi_\mathcal{O}(\mathcal{M}) = \int_\mathcal{O} e^{if(X_0)} d\mu_\mathcal{O}(f) = \int_\mathcal{M} e^{if_0(X)} d\mu_\mathcal{M}(X).$$

In other words, characters of $\mathcal{K}(V;K)$ are given by Fourier transforms of the invariant measures on orbits in V^*. Multiplication of characters corresponds on the Fourier transform side to convolution of the corresponding orbits in V^*. Since the actions of K on V and V^* are isomorphic, $\mathcal{K}(V;K)^\wedge$ is thus isomorphic to $\mathcal{K}(V;K)$.

Returning now to the case of $V = \mathbb{R}^n$, we see that any character of $\mathcal{K}(V)$ is given by the Fourier transform of a sphere $S_\lambda \subset V^*$ of some radius $\lambda \geq 0$. To determine the Fourier transform of the invariant probability measure μ_λ on this sphere, we pick a specified direction in V^* - say the z axis, and consider the (orthogonal) push down of μ_λ onto this axis. This is the measure on $[-\lambda, \lambda]$ given by

$$\frac{(\lambda^2 - z^2)^{\frac{n-3}{2}}}{c_n \lambda^{n-2}} dz.$$

The Fourier transform of μ_λ evaluated at a point with coordinate r on the dual axis is thus

$$\begin{aligned} \mu_\lambda^\wedge(r) &= \frac{1}{c_n \lambda^{n-2}} \int_{-\lambda}^{\lambda} e^{irz} (\lambda^2 - z^2)^{\frac{n-3}{2}} dz \\ &= \frac{1}{c_n} \int_{-1}^{1} e^{i\lambda r w} (1 - w^2)^{\frac{n-3}{2}} dw \\ &= \frac{J_{\frac{n-2}{2}}(\lambda r)}{(lr/2)^{\frac{n-2}{2}}} \Gamma(\frac{n}{2}). \end{aligned}$$

Here J_k denotes the Bessel function of order k (see for example Erdélyi [Er]),

$$J_k(x) = (\frac{x}{2})^k \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(m+k+1)}.$$

Summarizing, we have $\mathcal{K}(\mathbb{R}^n) \simeq \mathbb{R}^+ = \{r \geq 0\}$, $\mathcal{K}(\mathbb{R}^n)^\wedge \simeq \mathbb{R}^+ = \{\lambda \geq 0\}$, and for any $\lambda \in \mathbb{R}^+$ the corresponding character is

$$\phi_\lambda^n(r) = \frac{J_{\frac{n-2}{2}}(\lambda r)}{(\lambda r/2)^{\frac{n-2}{2}}} \Gamma(\frac{n}{2}).$$

For the special case $n = 3$, we get

$$\phi_\lambda^3(r) = \frac{\sin \lambda r}{\lambda r}.$$

Since the structures of $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{K}(\mathbb{R}^n)^\wedge$ are isomorphic, the characters satisfy the multiplication law

$$\phi_{\lambda_1}^n \phi_{\lambda_2}^n = \int_{|\lambda_1 - \lambda_2|}^{\lambda_1 + \lambda_2} \frac{2l}{c_n} \frac{[(\lambda^2 - (\lambda_1 - \lambda_2)^2)((\lambda_1 + \lambda_2)^2 - \lambda^2)]^{\frac{n-3}{2}}}{(2\lambda_1 \lambda_2)^{n-2}} \phi_\lambda^n \, d\lambda.$$

Note that this implies a corresponding formula for products of Bessel functions.

For the sphere $X = S^n$, the characters of $\mathcal{K}(X) \simeq [0,\pi]$ are indexed by $k = 0, 1, 2, \cdots$ and are the zonal spherical functions

$$\psi_k^n(r) = \frac{k!\Gamma(n-1)}{\Gamma(n-1+k)} P_k^{\frac{n-1}{2}}(\cos r)$$

where P_k^l is the Gegenbauer polynomial defined by

$$(1-2tz+z^2)^{-\lambda} = \sum_{k=0}^{\infty} P_k^{\lambda}(t) z^k$$

for $|t| \le 1$ and $\lambda > 0$, and where we have normalized so that $\psi_k^n(0) = 1$.

The Gegenbauer polynomial $P_k^{\lambda}(t)$ reduces for $\lambda = 1/2$ to the Legendre polynomial $P_k^{1/2}(t) = P_k(t)$ and for $\lambda = 1$ to the Tchebycheff polynomial $P_k^1(t) = U_k(t)$. For the latter we have the explicit formula

$$U_k(\cos r) = \frac{\sin(k+1)r}{\sin r}$$

so that

$$\psi_k^3(r) = \frac{\sin(k+1)r}{(k+1)\sin r}.$$

The trigonometric relation

$$U_l U_m = \sum_{\substack{k=|l-m| \\ k \equiv l+m (mod 2)}}^{l+m} U_k$$

then yields the following hypergroup structure for $\mathcal{K}(S^3)^\wedge = \{\psi_0^3, \psi_1^3, \cdots\}$:

$$\psi_l^3 * \psi_m^3 = \sum_{\substack{k=|l-m| \\ k \equiv l+m (mod 2)}}^{l+m} \frac{k+1}{(l+1)(m+1)} \psi_k^3.$$

In this case double duality holds; that is every character of $\mathcal{K}(S^3)^\wedge$ is evaluation at an element of $\mathcal{K}(S^3)$.

The reader familiar with Lie theory will recognize that if we consider S^3 as a Lie group (either as the unit quaternions in $\mathbb{H} \simeq \mathbb{R}^4$, say, or as $SU(2)$) then the spherical functions ψ_k^3 are exactly the normalized irreducible characters of the group (this means they have the value 1 at e) and the above hypergroup equation just expresses the Clebsch Gordan relation for products of these characters. Note that this has the seemingly curious consequence that the characters of $SU(2)$ as a Lie group are determined entirely by the structure of $SU(2)$ as a Riemannian manifold. This is equally true of any compact Lie group.

There is a general formula for the product of spherical functions on the sphere S^n. It is

$$\begin{aligned}\psi_l^n * \psi_m^n = \sum_{\substack{k=|l-m| \\ k \equiv l+m (mod 2)}}^{l+m} & \frac{(k+\frac{n-1}{2})\Gamma(g+n-1)}{[\Gamma(\frac{n-1}{2})]^2\Gamma(g+\frac{n+1}{2})} \\ & \times \frac{\Gamma(g-l+\frac{n-1}{2})\Gamma(g-m+\frac{n-1}{2})\Gamma(g-k+\frac{n-1}{2})}{\Gamma(g-l+1)\Gamma(g-m+1)\Gamma(g-k+1)} \\ & \times \frac{l!m!\Gamma(n-1)}{\Gamma(n+l-1)\Gamma(n+m-1)}\ \psi_k^n\end{aligned}$$

where $l+m+k=2g$. This result is a consequence of equation IX §4 (7) of Vilenkin [Vi] (slightly corrected). Despite its complexity, it is really a rather fundamental formula— it describes the hypergroup structure of $\mathcal{K}(S^n)^\wedge$.

Note that for any $k = 0, 1, \cdots$, we have

$$\phi_{k+1}^3(r) = \psi_k^3(r)\frac{\sin r}{r}.$$

This is a global relationship between the characters of $\mathcal{K}(\mathbb{R}^3)$ and the characters of $\mathcal{K}(S^3)$ which involves the exponential map $\exp : \mathbb{R}^3 \to S^3$ and the factor $\sin r/r$ which is the square root of the Jacobian of the exponential map. A similiar relation also holds (rather trivially) for the sphere S^1 and its tangent space $\mathbb{R}$. These are exactly the spheres which are compact Lie groups. We will generalize this phenomenon to this wider setting with the help of the wrapping map in §5.

§3. Hypergroups associated to homogeneous trees.

We may use the approach of the previous sections to associate hypergroups to more general geometric objects. Consider for example a homogeneous tree T_q of degree $q+1$. If we fix an origin O then the sphere S_n of radius n about O has $(q+1)q^{n-1}$ elements for $n \geq 1$ and of course 1 element for $n = 0$. The subgroup of isometries of T_q fixing O acts transitively on each S_n, so the probabilistic convolution of these spheres is well defined and results in a spherical hypergroup $\mathcal{K}(T_q) = \{s_0, s_1, \cdots\}$ with structure equations

$$\begin{aligned}s_m * s_n = \frac{q}{q+1}s_{n+m} + \frac{q-1}{q}\sum_{j=1}^{min(m,n)-1} \frac{1}{(q+1)q^{j-1}}s_{n+m-2j} \\ + \frac{1}{(q+1)q^{min(m,n)-1}}s_{|m-n|}.\end{aligned}$$

This is a discrete hypergroup on $\mathbb{N}$ with identity s_0. In particular,

$$s_1 * s_n = \frac{q}{q+1} s_{n+1} + \frac{1}{q+1} s_{n-1} \tag{3.1}$$

for $n \geq 1$ and the multiplication is completely determined by this latter formula. If we renormalize and set $\Theta_n = |S_n| s_n$ for all n then we get the relations

$$\Theta_1 * \Theta_n = \Theta_{n+1} + q\Theta_{n-1}$$

for $n \geq 1$ and

$$\Theta_1 * \Theta_1 = \Theta_2 + (q+1)\Theta_0.$$

These relations identify the so called *Hecke* algebra. Note that the hypergroup structure can be described by a single equation while the Hecke algebra structure requires two.

We may now ask, what are the spherical functions on a homogeneous tree? Or in other words, what are the characters of the hypergroup $\mathcal{K}(T_q)$? This may be done directly by analysing the spectrum of the Laplacian on the tree (see for example Cartier [Ca], Letac [Le]) but we may also proceed by noticing that there is some connection between the above hypergroup and the hypergroup $\mathcal{K}(S^3)^\wedge \simeq \mathcal{K}(SU(2))^\wedge = \{\psi_0^3, \psi_1^3, \cdots\}$ introduced in §2. In fact suppose we set

$$r_m = q^{\frac{2-m}{2}} \frac{m+1}{q+1}\ \psi_m^3 - q^{\frac{-m}{2}} \frac{m-1}{q+1}\ \psi_{m-2}^3$$

for $m \geq 1$ (with the convention that $\psi_{-1}^3 = 0$) and set $r_0 = \psi_0^3$ in the algebra spanned by the ψ_m^3's. Then by checking the relation Eq. (3.1) one may establish that $\{r_0, r_1, \cdots\}$ forms a hypergroup isomorphic to $\mathcal{K}(T_q)$.

From this it follows that characters of $\mathcal{K}(T_q)$ are related to characters of $\mathcal{K}(S^3)^\wedge$, that is, elements of $\mathcal{K}(S^3)$ as given in the previous section. Thus for any $\theta \in \mathbb{C}$, we may define a function A_θ on $\mathcal{K}(T_q)$ such that

$$A_\theta(s_m) = \frac{q \sin(m+1)\theta - \sin(m-1)\theta}{q^{m/2}(q+1)\sin\theta}$$

If we set $\lambda = e^{i\theta}$ then

$$A_\theta(s_m) = A_\lambda(s_m) = \frac{q(\lambda^{m+1} - \lambda^{-m-1}) - (\lambda^{m-1} - \lambda^{-m+1})}{q^{m/2}(q+1)(\lambda + \lambda^{-1})}.$$

A_λ is bounded, and so a character of $\mathcal{K}(T_q)$, if and only if $q^{-1/2} \leq |\lambda| \leq q^{1/2}$.

It seems that $\mathcal{K}(T_q)$ is a deformation of the hypergroup $\mathcal{K}(SU(2))^\wedge$ in some sense (see Ross and Xu [RX] for a discussion of deformations of hypergroups). Furthermore if one looks at the idea of a random walk on a building, one is naturally led to a hypergroup which seems to be, in the homogeneous case, a deformation of the hypergroup $\mathcal{K}(SU(3))^\wedge$ (see Cartwright and Młotkowski [CM]).

Problem 5. *Are there geometric objects whose associated hypergroups are modifications of the hypergroups $\mathcal{K}(SU(n))^\wedge$? What about for more general Lie groups?*

For additional information on the connection between hypergroups and more general graphs (such as strongly-regular graphs), see Wildberger [Wi3], [Wi4], [Wi5].

§4. The hypergroup structure of $\mathcal{K}(\mathfrak{g})$.

Let $\mathfrak{g}$ be the Lie algebra of a compact semisimple Lie group G. We now wish to describe somewhat explicitly the hypergroup $\mathcal{K}(\mathfrak{g})$ of adjoint orbits in $\mathfrak{g}$ under Euclidean convolution. This turns out to be a conical hypergroup which exhibits a rather remarkable convexity property.

Let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of a maximal torus $T \subset G$. Denoting the adjoint representation of $\mathfrak{g}$ (respectively G) on $\mathfrak{g}$ by ad (respectively Ad), the elements $ad(H), H \in \mathfrak{t}$ are semisimple and simultaneously commute; it follows that we may decompose $\mathfrak{g}$ as

$$\mathfrak{g} \;=\; \mathfrak{t} \oplus_\alpha \mathfrak{g}_\alpha$$

where each $\alpha \in \mathfrak{t}^*$, and where $\mathfrak{g}_\alpha$ is a 2 dimensional subspace of $\mathfrak{g}$ in which $ad(H)$ has the form

$$\alpha(H)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for each $H \in t$. Let Σ be the set consisting of all such α and their negatives; the *roots* of $\mathfrak{g}$. Each $\alpha \in \Sigma$ determines a hyperplane $\mathcal{H}_\alpha$ in $\mathfrak{t}$.

By introducing a positive definite, non degenerate, Ad(G)–invariant inner product $(\;,\;)$ on $\mathfrak{g}$ (for example, the negative of the Killing form), we may conveniently identify $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\mathfrak{t} \simeq \mathfrak{t}^*$. The set Σ is then invariant under the reflection s_α through the hyperplane $\mathcal{H}_\alpha$ so that Σ is a classical root system. The finite group W generated by all the s_α is the Weyl group; it is a group of isometries of $\mathfrak{t}$.

Any connected component of $\mathfrak{t} \setminus \cup_{\alpha\in\Sigma} \mathcal{H}_\alpha$ is called a Weyl chamber; these open polyhedral cones are in $1:1$ correspondence with the elements of W since W acts simply transitively on the set of all Weyl chambers. Let us arbitrarily choose a distinguished Weyl chamber C. The set of roots $\alpha \in \Sigma$ which are positive on C is denoted Σ^+; these are the *positive* roots. The closure $\overline{C}$ is a closed poyhedral cone and is a fundamental domain for the action of W on $\mathfrak{t}$. This means that every W orbit on $\mathfrak{t}$ meets $\overline{C}$ in a unique point.

One of the fundamental principles of the subject is that there is an intimate connection between Ad(G)–invariant objects on $\mathfrak{g}$ and W–invariant (or anti-invariant) objects on $\mathfrak{t}$. In particular, for any adjoint orbit $\mathcal{O} \subset \mathfrak{g}$, $\mathcal{O} \cap\ \mathfrak{t}$ is exactly a single W orbit, so that $\mathcal{O} \cap \overline{C}$ always consists of a single point. Thus $\overline{C}$ exactly parametrizes the set of all adjoint orbits; in other words, $\mathcal{K}(\mathfrak{g})$ as a set is the closed polyhedral cone $\overline{C} \subset \mathfrak{t}$. Thus $\mathcal{K}(\mathfrak{g})$ is always a conical hypergroup. For a rank one Lie algebra this cone is isomorphic to $\mathbb{R}^+$.

If $p : \mathfrak{g} \to \mathfrak{t}$ denotes the orthogonal projection and $\mathcal{O}$ is any adjoint orbit with say $\mathcal{O} \cap \overline{C} = X$, then $p(\mathcal{O}) = D_X = \text{conv } W(X)$, the convex hull of the W orbit of X. This is a classical result of matrix theory due to Schur and Horn for the case of $\mathfrak{g} = su(n)$, generalized by Kostant [Ko].

Suppose that $X \in C$. Since the adjoint orbit $\mathcal{O}$ is a homogeneous space for the compact Lie group G, it carries a unique $\text{Ad}(G)$–invariant measure $\nu = \nu_X$ such that the total mass is $\prod_{\alpha\in\Sigma^+} \alpha(X)/\prod_{\alpha\in\Sigma^+}(\alpha,\rho)$ where ρ is one-half the sum of the positive roots. From Kostant's theorem we know that the pushdown $T_X = \ p^*(\nu)$ of this measure onto $\mathfrak{t}$ will have support D_X. It will be important to have more detailed information about this pushdown.

For $\alpha \in \Sigma$, let F_α be the measure on $\mathfrak{t}^* \simeq \mathfrak{t}$ which is Lebesgue measure on the half line through α. Since $\{\alpha|\alpha \in \Sigma^+\}$ lies in a convex cone, the convolution product

$$P = \prod_{\alpha\in\Sigma^+} * F_\alpha$$

is a well defined measure on $\mathfrak{t}$; in fact it is a continuous function of polynomial growth. More canonical is the W anti-invariant measure

$$Q = \sum_{w\in W} \text{sgn}(w) w(P)$$

where for any $w \in W$, $\text{sgn}(w)$ is the determinant of w and where the action of w on measures is induced from its action on $\mathfrak{t}$. Now the formula we require is

$$-T_X = \sum_{w\in W} \text{sgn}(w)\delta_{w(X)} * P = \frac{1}{|W|}\sum_{w\in W} \text{sgn}(w)\delta_{w(X)} * Q$$

where the convolutions are in $\mathfrak{t}$ (and just amount to translations).

For the case when $\mathfrak{g} = su(2)$ the orbit $\mathcal{O}_X$ is just a sphere in $\mathbb{R}^3$, and the above formula tells us that the orthogonal projection of the surface area of a sphere onto a line through the center is the uniform measure on the image. In probabilistic language, the z coordinate of a point chosen at random on a unit sphere is equally distributed in $[-1,1]$— a well known result from advanced calculus.

We now come to our description of the hypergroup structure of $K(\mathfrak{g})$, at least for generic orbits.

Theorem 4.1. *([DRW]) The convolution $\nu_X * \nu_Y$ for $X, Y \in C$ is given by*

$$\nu_X * \nu_Y = \int_C \phi(X, Y, Z)\ \nu_Z\ dZ$$

where $\phi(X, Y, Z) = \sum_{w \in W} \operatorname{sgn}(w) T_X * \delta_{w(Y)}(Z)$.

For example in the case of $\mathfrak{g} = su(2)$, the normalized measure assigns to the sphere of radius r the mass r, and the statement is then that the convolution of these radial measures decomposes into an integral of similiar measures with the weight determined by the projection measure discussed previously ie a constant measure on an interval.

Note in general that there is some remarkable cancellation involved in the nature of the measures T_X and their translates to insure that the result is symmetric in X and Y. To determine the convolution of orbits which are not generic, ie orbits of the form $\mathcal{O}_X$ where X is in $\overline{C}$ but not in C, one needs to modify the above result by taking an appropriate L'Hopital's rule limit. We refer the reader to [DRW] for the details.

In the special case that the translate $D_X * \delta_Y$ lies entirely in $\overline{C}$, the sum in Theorem 4.1 reduces to one term and the function ϕ has support exactly $D_X * \delta_Y$. In general the support of ϕ is always contained in $D_X * \delta_Y$. Furthermore this support is always a convex polytope. These facts are closely related to convexity results of symplectic geometry due to Atiyah [At], Guillemin and Sternberg [GS1], [GS2] and Kirwan [Ki].

Note that there is some sense in which the measure $\delta_X * \delta_Y$ is a linear function of the variables X and Y. It seems of some interest to clarify this statement and so to define a *linear* conical hypergroup. The above remarks then motivate us to define a *convex* linear conical hypergroup as a linear conical hypergroup $\mathcal{K}$ such that the supports of all the measures $\delta_x * \delta_y$ are convex in $\mathcal{K}$.

Problem 6. *Define and classify linear convex conical hypergroups $\mathcal{K} \subset \mathbb{R}^n$.*

§5. The wrapping map

We now investigate the relation between the hypergroups $\mathcal{K}(\mathfrak{g})$ and $\mathcal{K}(G)$ for a compact connected semisimple Lie group G. The first is essentially a linear approximation to the second. One therefore expects only an infinitesimal relation between them as predicted by the theory of group contractions. However in fact a much more remarkable relation exists between the two which has consequences for the study of harmonic analysis, differential equations and random walks on the group G.

It will be convenient to describe this relation in the more general framework of distribution theory— this will allow us to apply the results to differential operators and equations.

The exponential map $\exp : \mathfrak{g} \to G$ is surjective for a compact Lie group. Let $j(X)$ denote the square root of the Jacobian of exp (with respect to Lebesgue measure dX on $\mathfrak{g}$ and Haar measure dg on G) which satisfies $j(0) = 1$. Then j is an analytic $\mathrm{Ad}(G)$-invariant function on $\mathfrak{g}$. For $G = S^3$ one has $j(X) = \sin r/r$.

For any C^∞ function ψ on G let $\tilde{\psi}(X) = \psi(\exp X)$ be its lift to $\mathfrak{g}$. Define a map $\Phi : \mathcal{E}_c(\mathfrak{g}) \to \mathcal{E}(G)$ from the space $\mathcal{E}_c(\mathfrak{g})$ of distributions of compact support on $\mathfrak{g}$ to the space $\mathcal{E}(G)$ of distributions on G by

$$\langle \Phi(\mu), \psi \rangle = \langle \mu, j\tilde{\psi} \rangle$$

for any $\mu \in \mathcal{E}_c(\mathfrak{g})$ and $\psi \in C^\infty(G)$. The same formula allows us to define $\Phi(f)$ for any L^1 function f on $\mathfrak{g}$.

The following result we will call the *wrapping theorem.*

Theorem 5.1. *([DW]) Let μ and ν be two $\mathrm{Ad}(G)$–invariant distributions of compact support on $\mathfrak{g}$ or two $\mathrm{Ad}(G)$–invariant functions in $L^1(\mathfrak{g})$. Then*

$$\Phi(\mu) * \Phi(\nu) = \Phi(\mu * \nu)$$

where the convolution on the left is group convolution on G and the convolution on the right is Euclidean convolution in $\mathfrak{g}$.

Special cases of the wrapping theorem were known to Harish Chandra, Duflo, and Frenkel. Our proof of the theorem is reasonably standard using classical results of harmonic analysis on the group G: namely the Weyl and Kirillov character formulae. It would be nice to have a structure free proof.

The wrapping theorem allows us to deduce the structure of the hypergroup $\mathcal{K}(G)$ from that of the hypergroup $\mathcal{K}(\mathfrak{g})$. In other words, to convolve two generic conjugacy classes C_1, C_2 of G, we wrap the convolution of any two orbits $\mathcal{O}_1, \mathcal{O}_2$ for which $\exp(\mathcal{O}_i) = C_i$ and include the appropriate scaling factor coming from j. Thus the results of the previous section may be applied to answer specific questions about convolution of central measures on a compact Lie group.

Another use of the theorem is to provide a simple conceptual approach to the fundamentals of harmonic analysis on the group G. In particular, the classification of the unitary dual $G^\wedge$ by highest weights, or equivalently by integral coadjoint orbits, is a consequence of the wrapping theorem and some generalities. Since this is an interesting point, we will now explain this remark in more detail.

Recall from the discussion of §2 that the dual object for the hypergroup of adjoint orbits $\mathcal{K}(\mathfrak{g})$ is nothing but the hypergroup of coadjoint orbits $\mathcal{K}(\mathfrak{g}^*)$. More specifically, any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ gives a character $\chi_{\mathcal{O}}$ of $\mathcal{K}(\mathfrak{g})$ by the rule

$$\chi_{\mathcal{O}}(\mathcal{M}) = \int_{\mathcal{M}} e^{if_0(X)} d\mu_{\mathcal{M}}(X) = \int_{\mathcal{O}} e^{if(X_0)} d\mu_O(f)$$

for any $f_0 \in \mathcal{O}$ and any X_0 in the adjoint orbit $\mathcal{M}$ and all characters arise this way.

A classical result of Weyl implies that the dual object of the hypergroup $\mathcal{K}(G)$ is just the hypergroup $\mathcal{K}(G^\wedge)$ of normalized characters of G. The wrapping theorem implies that if we take any normalized character χ of the group G, then the map that sends $\mathcal{M}$ to $\langle \Phi(\mu_{\mathcal{M}}), \chi \rangle$ is a character of the hypergroup $\mathcal{K}(\mathfrak{g})$. But by the remarks above, any such character is necessarily of the form $\chi_{\mathcal{O}}$ for some coadjoint orbit $\mathcal{O} \subset \mathfrak{g}$.

Not every coadjoint orbit $\mathcal{O}$ may arise this way due to the obvious constraint that the Fourier transform of μ_O must factor through the wrapping map, which turns out to be just the condition that the orbit $\mathcal{O}$ is *integral.*

Thus characters of the group G are in correspondence with integral coadjoint orbits and given such an orbit, the corresponding character is given by a formula involving the push down of the Fourier transform of the invariant measure on that orbit (this is known as the Kirillov character formula).

§6. Differential equations on a compact Lie group

The wrapping map has some other consequences concerning G–invariant differential equations on G and the relation with G–invariant constant coefficient differential equations on $\mathfrak{g}$. To see the connection, recall that any constant coefficient differential operator on $\mathfrak{g}$ can be realized by Euclidean convolution with a distribution supported at $0 \in \mathfrak{g}$, that this space of distributions forms an algebra under convolution isomorphic to the full polynomial algebra $S(\mathfrak{g})$ under Fourier transform, and that the $\mathrm{Ad}(G)$–invariant operators correspond to the subalgebra of $\mathrm{Ad}(G)$–invariant distributions supported at 0, which under Fourier transform can be identified with the $\mathrm{Ad}(G)$–invariant polynomials $I(\mathfrak{g}) \subset S(\mathfrak{g})$.

On the other hand, left invariant (say) differential operators on G can be realized by group convolution with distributions on G supported at $e \in G$. The space of all distributions on G whose support is e forms an algebra under group convolution which is nothing other than the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ (which is noncommutative). The subset of G–invariant distributions (with respect to the conjugation action of G on itself) forms a subalgebra $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ which turns out to be commutative.

The wrapping map Φ restricts to a map $\Phi : S(\mathfrak{g}) \to U(\mathfrak{g})$ which is in fact an isomorphism of linear spaces and maps $I(\mathfrak{g})$ onto $D(\mathfrak{g})$. What the wrapping theorem asserts in this context is that $\Phi : I(\mathfrak{g}) \to Z(\mathfrak{g})$ is an algebra isomorphism. This is a result of Harish Chandra, and due to its generalization to arbitrary Lie algebras by Duflo, this restriction is known as the Duflo isomorphism.

If the definition of the wrapping map had not included the factor $j(X)$, we would still have a map from $S(\mathfrak{g})$ to $U(\mathfrak{g})$ - this map is useful in the theory of enveloping algebras and is known as the symmetrization map. However this map does *not* have the property that it is an algebra isomorphism from $I(\mathfrak{g})$ to $Z(\mathfrak{g})$. This difference between the symmetrization map and the wrapping map has consequences.

One of the most standard concerns the Laplace–Beltrami operator $\triangle_G$ on G. Since this is defined by a sum of squares of vector fields on G coming from an orthonormal basis say of the Lie algebra $\mathfrak{g}$, it is nothing but the image of the usual Euclidean Laplacian (identified with a distribution supported at 0) under the symmetrization map.

This is however, *not* always the correct Laplacian on G for many aspects of harmonic analysis. This is remedied by also considering the image of the Euclidean Laplacian under the wrapping map. In other words, suppose that we consider the distribution $\delta_\triangle$ on $\mathfrak{g}$ defined by $\langle \delta_\triangle, f\rangle = \triangle_{\mathfrak{g}} f(0)$ for $f \in C_0^\infty(\mathfrak{g})$ and then define $L_G(\psi) = \psi * \Phi(\delta_\triangle)$ for $\psi \in C^\infty(G)$. This operator is sometimes called the shifted Laplacian.

In general the difference between the two Laplacians L_G and $\triangle_G$ consists of what physicists call a 'curvature' term which in this case is the constant $\dim \mathfrak{g}/12$ (see Helgason [He2]). However this small difference manifests itself— for example if G is an odd dimensional Lie group then the usual wave equation does not satisfy a Huygens' principle while the shifted wave equation does (see Fegan [Fe1]). Furthermore Fegan [Fe2] has also shown that the asymptotics of the fundamental solution to the shifted heat equation on G are simpler than for the nonshifted heat equation.

Similiar remarks hold for classical PDE's on spheres— see the discussion in Taylor [Ta].

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