Algebraic Structures associated to group actions and sums of Hermitian matrices

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CONTENTS

1 Introduction

This set of lectures is intended to provide an introduction to some of the ideas useful for studying Weyl's problem of describing the possible sets of eigenvalues of the sum $A + B$ of two hermitian matrices both of whose eigenvalues are known. This problem has been studied by Weyl [12], Wielandt [13], Horn [5], V. B. Lidskii [10] and B. V. Lidskii [9]. In 1993 Dooley, Repka and Wildberger [2] published the first solution, utilizing techniques from symplectic geometry and Lie theory. In fact this solution was considerably more general in two important ways. First of all it dealt with sums of orbits in arbitrary simple Lie groups, not just $SU(n)$, and secondly it showed how to describe the *probability* of a given set of eigenvalues occurring, or more precisely determined the underlying probability measure on the set of adjoint orbits which is obtained by convolving invariant measures on given adjoint orbits. As an extra bonus, the convexity of the resulting support was obtained by applying powerful moment map theorems of Guillemin and Sternberg and Atiyah.

The idea of considering the probabilistic aspect of the problem is motivated by the theory of *hypergroups*, which are algebraic objects similar to groups except that the products of two elements are in general probability measures, not singletons, but still retaining the existence of an identity, inverses and associativity. To make up for the increased complexity of probability measures, most hypergroups arising in practise are *commutative* , which makes Fourier analysis possible and indeed extends classical harmonic analysis in many new directions.

Hypergroups were introduced in the 70's by Dunkl [4], Jewett [6] and Spector [11], although in retrospect the idea had been around in various forms much earlier. See [15] for a list of connections.

Our aim is to introduce the reader gently into this perhaps unfamiliar world of multiplying elements and getting probability measures. Once accustomed to this point of view, the reader may well start seeing hypergroups all around. In fact they are at least as ubiquitous as groups and furthermore much of group theory (both finite and continuous) is clarified by a consideration of these objects. Another benefit is that one sees that even in the simplest kind of situation where a group acts on a space, there is likely to be an attendant algebraic structure of a hypergroup-like nature.

We will begin by focusing on the simple case of finite commutative hypergroups and give examples from classical geometry, finite group theory, including the class and character hypergroup of a finite group, and classical number theory. We then develop some abstract results which show

how the familiar Pontryagin duality theory for finite abelian groups generalizes to a slightly larger class of *signed hypergroups* (for which negative probabilities are forced upon us!)

We focus on two important theorems which illustrate the duality between a hypergroup and its dual, and then introduce continuous versions of these results by starting off with convolutions of spheres in n dimensional space, moving on to general symmetric spaces, and then finally we attempt to at least describe the form of the answer to Weyl's problem given in [2].

It is important to note that the approach given here to the sums of eigenvalues problem is only one of many possible. Horn's important conjecture in this subject, which gives recursive descriptions of the faces of the polytope of possible sets of eigenvalues has been recently established by Klyachko [7]. In related work Knutson and Tao [8] have an approach which utilizes honeycombs.

2 Particle descriptions of finite commutative hypergroups.

We consider a particular model for abstract particle collisions, with the following assumptions. Let $\{c_0, c_1, \dots, c_n\}$ be a collection of particle types, with c_0 being a **photon**. If we collide c_i with c_j, the result is a single particle, and the probability of obtaining c_k is n_{ij}^k and is fixed. We write

$$c_i c_j = \sum_k n_{ij}^k c_k.$$

• The photon is absorbed in any particle collision.

• Every particle c_i has an **anti-particle** c_i^* (which may coincide with c_i) with the defining property that c_0 occurs with strictly positive probability in $c_i c_j$ iff $c_j = c_i^*$.

• All collision probabilities n_{ij}^k are invariant under interchange of particles with anti-particles. Furthermore $(c_i^*)^* = c_i$ for all i.

• Finally, the results of collisions are independent of their order in time and space. Thus when we consider collision of 3 particles c_i, c_j, c_k, the final outcome is independent of the order of occurrence of collisions.

If these conditions are satisfied then we say $K = \{c_0, c_1, \dots, c_n\}$ forms a **finite commutative hypergroup** (of particles).

Given such a hypergroup we introduce an important concept which plays the role of the size (or mass) of the particles and allows us to assign a total size to the entire collection of particles. If $c_i c_i^* = \dfrac{1}{w} c_0 + \dots$ then define $w(c_i) = w(c_i^*) = w$ the **weight** of c_i. Define $w(K) = \sum_i w(c_i)$ as the weight of the hypergroup K.

Example. [The Golden hypergroup] $K = \{c_0, c_1, c_2\}$

$$\begin{aligned} c_1^2 &= \tfrac{1}{2}c_0 + \tfrac{1}{2}c_2 \\ c_1c_2 &= \tfrac{1}{2}c_1 + \tfrac{1}{2}c_2 \\ c_2^2 &= \tfrac{1}{2}c_0 + \tfrac{1}{2}c_1. \end{aligned}$$

Note that $c_i^* = c_i$ for all i. A hypergroup satisfying this property is called **Hermitian**. Note also that $w(K) = 5$.

Example. Hypergroups of order two.

If a hypergroup has only two elements then we may write it as $K = \{c_0, c_1\}$ with structure equation

$$c_1^2 = \alpha c_0 + (1-\alpha)c_1$$

for some $\alpha \in (0, 1]$. Alternatively by introducing $w = \alpha^{-1}$ we may write this as

$$c_1^2 = \frac{1}{w}c_0 + \frac{w-1}{w}c_1,$$

where $w \in [1, \infty)$. Thus $w(K) = 1 + w$. In the special case of $w = 1$ we recover the commutative group $\mathbb{Z}_2$.

Example. Hypergroups of order three.

If a hypergroup has three elements the situation is much richer and already connects with interesting combinatorics of strongly regular graphs. Suppose that $K = \{c_0, c_1, c_2\}$. There are then two separate cases depending on whether K is Hermitian or not.

Case 1. Suppose that K is not Hermitian so that $c_1^* = c_2$.

$\boxed{\textbf{Exercise 1}}$ Show that the structure equations must be

$$\begin{aligned} c_1^2 &= \frac{w-1}{2w}c_1 + \frac{w+1}{2w}c_2 \\ c_1c_2 &= \frac{1}{w}c_0 + \frac{w-1}{2w}c_1 + \frac{w-1}{2w}c_2 \\ c_2^2 &= \frac{w+1}{2w}c_1 + \frac{w-1}{2w}c_2 \end{aligned}$$

for some $w \in [1, \infty)$. Here $w(K) = 1 + 2w$. If $w = 1$ we get $\mathbb{Z}_3$.

Case 2. Suppose that K is Hermitian so that $c_i^* = c_i$.

$\boxed{\textbf{Exercise 2}}$ Show that there is a 3-parameter family of Hermitian hypergroups. Show that the minimal weight of such a hypergroup is 4.

3 Precise Definitions

Hypergroups were introduced in the mid-seventies by Dunkl, Jewett and Spector, primarily in the locally compact situation, which we discuss later. In retrospect, we see the idea has been around in different guises for a long time. Here are some occurrences of related objects. For precise references, see Wildberger [14].

C-algebras	(Kawada)
generalized translation operators	(Levitan)
hypercomplex systems	(Berezansky & Kalyushny, Vainermann)
pseudo groups	(Brauer)
paragroups	(Ocneanu)
superselection sectors	(Doplicher, Haag & Roberts, Longo)
Bose-Mesner algebras	
centralizer algebras	
Racah-Wigner algebras	(Sharpe)
table algebras	(Arad & Blau)
association schemes	(Bannai & Ito)
fusion rule algebras	(Verlinde, Witten).
Hecke algebras	

We now give a proper mathematical definition of the main concept. A **finite commutative hypergroup** is a pair (K, A), where A is an associative commutative *-algebra with unit c_0 over $\mathbb{C}$ and $K = \{c_0, c_1, \ldots, c_n\}$ is a basis of A satisfying

1) $K^* = K$

2) the structure constants $n_{ij}^k \in \mathbb{C}$ defined by

$$c_i c_j = \sum_k n_{ij}^k c_k$$

satisfy $n_{ij}^k \in \mathbb{R}$, $\sum\limits_k n_{ij}^k = 1$ and

$$\begin{array}{rcl} c_i^* = c_j & \iff & n_{ij}^0 > 0 \\ c_i^* \neq c_j & \iff & n_{ij}^0 = 0. \end{array}$$

3) $n_{ij}^k \geq 0$,

If the condition $n_{ij}^k \geq 0$ is removed, call K a **commutative signed hypergroup**. Here is probably the main fact about such objects.

Proposition 3.1 *Let (K, A) be a commutative signed hypergroup. Then the algebra A is semisimple.*

Proof. If $a \in A$ is non-zero then the coefficient of c_0 in aa^* is also non-zero, so that $aa^* \neq 0$. Then $(aa^*)(aa^*)^* = a^2 (a^2)^* \neq 0$ so that a^2 is non-zero. Therefore A has no non-zero nilpotent elements and so is semisimple. $\square$

A commutative semisimple *-algebra of dimension $n+1$ is necessarily isomorphic to $\mathbb{C}^{n+1}$. So we see that the 'algebra' side of a commutative hypergroup is rather trivial. What is interesting here is the particular choice of a basis of this algebra which acts as much like a group as possible while preserving the probabilistic nature of products.

4 Convolving spheres in a Platonic solid

Our first example gives a new slant to the familiar symmetry inherent in objects called distance regular graphs, which includes the graphs of vertices of Platonic solids.

For simplicity let X be the graph consisting of the vertices and edges of a cube. Fix a vertex x_0. For $i = 0, 1, 2, 3$ let S_i denote the sphere of radius i from x_0. Thus S_1 contains the three neighbouring vertices of x_0.

For $i, j, k \in \{0, 1, 2, 3\}$ define n_{ij}^k as follows. Choose a random point y in S_i. Choose a random point z of distance j from y. Let n_{ij}^k be the probability that z is in S_k.

Thus, for example,

$$n_{12}^1 = \frac{2}{3} \qquad n_{12}^3 = \frac{1}{3} \qquad n_{12}^0 = n_{12}^2 = 0,$$

since the probability of taking successive steps of sizes one and two and ending up at distance one from the starting vertex is $\frac{2}{3}$, while the probability of ending up at distance two from the starting vertex is $\frac{1}{3}$.

Introduce a hypergroup $K = \{c_0, c_1, c_2, c_3\}$ with structure equations $c_i c_j = \sum_k n_{ij}^k c_k$. Leaving out the equations involving c_0, we have

$$\begin{aligned} c_1^2 &= \frac{1}{3}c_0 + \frac{2}{3}c_2 & \qquad c_1 c_3 &= c_2 \\ c_1 c_2 &= \frac{2}{3}c_1 + \frac{1}{3}c_3 & \qquad c_3^2 &= c_0 \\ c_2^2 &= \frac{1}{3}c_0 + \frac{2}{3}c_2 & \qquad c_2 c_3 &= c_1 \end{aligned}$$

This table allows us in principle to calculate any probabilities involving more complicated series of steps on our cube. For example suppose we want to know the likely distance from the origin after a nearest neighbour random walk of length 3. This corresponds to computing c_1^3, which is

$$\begin{aligned} c_1^3 = c_1^2 c_1 &= (\frac{1}{3}c_0 + \frac{2}{3}c_2)c_1 = \frac{1}{3}c_1 + \frac{2}{3}(\frac{2}{3}c_1 + \frac{1}{3}c_3) \\ &= \frac{7}{9}c_1 + \frac{2}{9}c_3. \end{aligned}$$

It means that we have a $\frac{7}{9}$ probability of being in S_1, and a $\frac{2}{9}$ probability of being in S_3 after 3 steps.

$\boxed{\textbf{Exercise 3}}$ Find the probability of returning to x_0 after 6 steps of a random walk. Repeat for n steps.

We refer to the product above as 'convolving spheres' in X. It is well defined in this case because X is a **distance transitive** graph: if $d(x, y) = d(x', y')$ then there is an automorphism σ of X with $\sigma(x) = x'$ and $\sigma(y) = y'$. Indeed for any distance transitive graph X, the hypergroup $K(X)$ of spheres in X is well defined. More generally, a graph X is **distance regular** if the same procedure of convolving spheres is well defined and produces a hypergroup $K(Y)$ independent of base point x_0. Not all distance regular graphs are distance transitive.

5 Multi-sets and the class and character hypergroups of a finite group.

Given a non-commutative group G, there are two canonical hypergroups associated to G, the *class hypergroup* and the *character hypergroup* of G, which end up being dual in a precise sense to be described later. An efficient way of thinking of the former is to first develop some ideas about multi-sets to promote effective **counting** on G.

To begin with let G be a finite set. A **multi-set** of G, or an m-**set** of G, is an unordered collection X of elements of G, each occurring with finite multiplicity. If, for example, $G = \{e, g, h, \ldots, k\}$ then $A = [g, h, g, g, k, h]$ and $B = [g, k, k, \ell, k, \ell, k]$ are multi-sets with $|A| = 6$ and $|B| = 7$. We also write $A = 3g + 2h + k$ and $B = g + 4k + 2\ell$. We may form the **sum** $A + B = 4g + 2h + 5k + 2\ell$, the **union** $A \cup B = 3g + 4k + 2h + 2\ell$, the **intersection** $A \cap B = g + k$, and even the **difference** $A - B = 2g + 2h - 3k - 2\ell$, the latter more properly regarded as a **virtual** multi-set.

Let $m_A(x)$ denote the multiplicity of the element x in A and let

$$|A| = \sum_{x \in A} m_A(x)|x|$$

be the *mass* of the multi-set A. If $|A| = \lambda$ then we will refer to A as a λ-set.

In the special case when all elements x of A are 'atoms' in our theory then the mass of A is just the sum of all multiplicities of elements of A, a concept which coincides with the size of a set. However, we allow the possibility of A having elements which are themselves multi-sets with masses larger than one. In this case the notion of mass and size *do not coincide.*

Of particular interest for us will be 1-sets, which can be regarded as generalizations of elements of a set or alternatively as probability distributions, or averages on a given set. Here is a basic result which is a convenient warm-up for those unfamiliar with manipulations of multi-sets.

Theorem 5.1 *(Inclusion/Exclusion for multi-sets)*

Let $A_1, \ldots, A_n$ be multi-sets of a finite set G. Then

$$\begin{aligned} A_1 \cup \ldots \cup A_n &= A_1 + \ldots + A_n - (A_1 \cap A_2 + \ldots + A_{n-1} \cap A_n) \\ &\quad +(A_1 \cap A_2 \cap A_3 + \ldots) - \ldots + (-1)^{n+1}(A_1 \cap \ldots \cap A_n). \end{aligned}$$

$\boxed{\textbf{Exercise 4}}$ Give a proof.

The usual Inclusion/Exclusion theorem is an immediate consequence of the above, obtained simply by taking absolute values of all entries.

Now suppose G is a finite group and let us see how naturally to apply multi-set terminology to allow us to simplify the group multiplication. For $x, y \in G$ let $x^y = y^{-1}xy$. Let A, B be multi-sets of G. Define

$$\begin{aligned} AB &= [ab \,|\, a \in A,\ b \in B], \\ A^B &= [a^b \,|\, a \in A,\ b \in B], \\ A^{-1} &= [a^{-1} \,|\, a \in A] = A^*. \end{aligned}$$

Note that

$$|AB| = |A|\,|B| = |A^B| \quad \text{and} \quad |A^{-1}| = |A|.$$

A multi-set X of G is G-**invariant** if $X^g = X,\ \forall\, g \in G$. Let $C_0, C_1, \ldots, C_n$ be the conjugacy class of G with $C_0 = \{e\}$. These are G-invariant sets and any G-invariant multi-set is a sum of C_i's. Thus

$$C_iC_j = \sum_{k=0}^{n} N_{ij}^k C_k \tag{5.1}$$

for some $N_{ij}^k \in \mathbb{N}$ called the **class structure constants** of G.

Fundamental Problem. Given a group G, determine explicitly the class structure constants N_{ij}^k.

The function $d(C_i) = |C_i|$ is a **dimension function**; it satisfies

$$\begin{aligned} d(C_i)d(C_j) &= \sum_k N_{ij}^k d(C_k) \quad (5.2) \\ \text{and} \qquad d(C_i^*) &= d(C_i). \end{aligned}$$

Now, defining

$$c_i = \frac{C_i}{d(C_i)},$$

we see that

$$c_i c_j = \frac{1}{d(C_i)d(C_j)} \sum_k N_{ij}^k \; d(C_k) \; c_k = \sum_k n_{ij}^k \; c_k$$

where

$$n_{ij}^k = \frac{N_{ij}^k d(C_k)}{d(C_i)d(C_j)}.$$

Note that $\sum_k n_{ij}^k = 1$ because of (5.2). This makes $K(G) = \{c_0, c_1, \ldots, c_n\}$ into a hypergroup, called the **class hypergroup** of G. It is easy to see that $w(K(G)) = |G|$. One may think of the c_i as particular types of multi-sets, in fact 1-sets.

Example. Let $G = A_4$ with conjugacy classes

$C_0 = \{e\}$ $\quad C_1 = \{(12)(34), (13)(24), (14)(23)\}$ $\quad C_2 = \{(123), (142), (243), (134)\}$ and $C_3 = \{(132), (124), (234), (143)\}$. Then

$$\begin{aligned} C_1^2 &= 3C_0 + 2C_1 & C_2^2 &= 4C_3 \\ C_1C_2 &= 3C_2 & C_2C_3 &= 4C_0 + 4C_1 \\ C_1C_3 &= 3C_3 & C_3^2 &= 4C_2. \end{aligned}$$

The corresponding hypergroup equations for $K(A_4) = \{c_0, c_1, c_2, c_3\}$ are

$$\begin{aligned} c_1^2 &= \tfrac{1}{3}c_0 + \tfrac{2}{3}c_1 & c_2^2 &= c_3 \\ c_1c_2 &= c_2 & c_2c_3 &= \tfrac{1}{4}c_0 + \tfrac{3}{4}c_1 \\ c_1c_3 &= c_3 & c_3^2 &= c_2. \end{aligned}$$

Note that $w(K(G)) = 1 + 3 + 4 + 4 = 12 = |G|$.

Let G be a finite group and $G^\wedge = \{\pi_0, \pi_1, \dots, \pi_n\}$ its irreducible representations. For each π_i let π_i^* be the dual representation. Then we know there exist $M_{ij}^k \in \mathbb{N}$ such that

$$\pi_i \otimes \pi_j = \sum_k M_{ij}^k \, \pi_k.$$

Furthermore $\pi_i \otimes \pi_i^* = \pi_0 + \dots$. This makes $G^\wedge$ into a commutative **fusion rule algebra**. Let χ_i be the associated **character** of π_i, the function on G defined by

$$\chi_i(g) = \operatorname{tr} \pi_i(g).$$

Note that $\chi_i(e) = d_i = \dim \pi_i$. In fact

$$d_i \otimes d_j = \sum_k M_{ij}^k \, d_k,$$

so d_i is a 'dimension function' of the fusion rule algebra and allows us to renormalize to get a hypergroup. Set $X_i = \frac{\chi_i}{d_i}$ (note that $X_i(e) = 1$). Then

$$X_i X_j = \sum_k m_{ij}^k \, X_k, \tag{5.3}$$

where

$$m_{ij}^k = \frac{d_k M_{ij}^k}{d_i d_j}$$

and where $\sum_k m_{ij}^k = 1, \quad \forall\, i,j$. Let $K(G^\wedge) = \{X_0, X_1, \dots, X_n\}$ with structure constants (5.3). The weight of an element X_i is

$$w(X_i) = \left(m_{ii^*}^0\right)^{-1} = \left(\frac{d_0 M_{ii^*}^0}{d_i d_{i^*}}\right)^{-1} = d_i^2,$$

so that $w(K(G^\wedge)) = \sum_i d_i^2 = |G| = w(K(G))$. We call $K(G^\wedge)$ the **character hypergroup** of G.

Example. Let $G = A_4$ with conjugacy classes C_0, C_1, C_2, C_3 as before. Here is the familiar **group character table**, giving the values of the 4 irreducible characters on the classes:

	C_0	C_1	C_2	C_3
χ_0	1	1	1	1
χ_1	3	-1	0	0
χ_2	1	1	σ	σ^2
χ_3	1	1	σ^2	σ

where $\sigma = e^{\frac{2\pi i}{3}}$

Here are the values of the **normalized** characters:

	C_0	C_1	C_2	C_3
X_0	1	1	1	1
X_1	1	$-\frac{1}{3}$	0	0
X_2	1	1	σ	σ^2
X_3	1	1	σ^2	σ

As functions on G, the multiplication of the X_i's is pointwise. Without any knowledge of the representations of G, we may thus compute the structure constants of $K(G^\wedge) = \{X_0, X_1, X_2, X_3\}$ to be

$$
\begin{aligned}
X_1^2 &= \frac{1}{9}X_0 + \frac{2}{3}X_1 + \frac{1}{9}X_2 + \frac{1}{9}X_3 & X_2^2 &= X_3 \\
X_1X_2 &= X_1 & X_2X_3 &= X_0 \\
X_1X_3 &= X_1 & X_3^2 &= X_2
\end{aligned}
$$

6 Hypergroups of Gaussian periods

Let p be a prime, $\mathbb{Z}_p = \{1, e^{\frac{2\pi i}{p}}, e^{\frac{2\pi i(2)}{p}}, \ldots, e^{\frac{2\pi i(p-1)}{p}}\}$. Let $C_0 = \{1\}$, $C_1 = \{x \in \mathbb{Z}_p \mid x \neq 1,\, x = y^2$ for some $y \in \mathbb{Z}_p\}$; $C_2 = \{x \in \mathbb{Z}_p \mid x \neq y^2$ for any y in $\mathbb{Z}_p\}$. Then

$$\mathbb{Z}_p = C_0 \cup C_1 \cup C_2$$

is a disjoint union and there exist $N_{ij}^k \in \mathbb{N}$ such that $C_iC_j = \sum_k N_{ij}^k C_k$, where we use the multi-set conventions defined above in the group $\mathbb{Z}_p$.

Define $C_i^* = \overline{C}_i = C_i^{-1}$ as before.

Example. $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$ $\quad C_0 = \{0\}$ $\quad C_1 = \{1,2,4\}$ $\quad C_2 = \{3,5,6\}$

$$
\begin{aligned}
C_1^2 &= C_1 + 2C_2 \\
C_1C_2 &= 3C_0 + C_1 + C_2 \\
C_2^2 &= 2C_1 + C_2.
\end{aligned}
$$

A dimension function is $d(C_0) = 1, d(C_1) = d(C_2) = 3$. Normalizing as in the previous case we get

$$
\begin{aligned}
c_1^2 &= \frac{1}{3}c_1 + \frac{2}{3}c_2 \\
c_1c_2 &= \frac{1}{3}c_0 + \frac{1}{3}c_1 + \frac{1}{3}c_2 \\
c_2^2 &= \frac{1}{3}c_1 + \frac{2}{3}c_2.
\end{aligned}
$$

Note: This is a non-Hermitian hypergroup of order three. It can be obtained from the result of Exercise 1 by setting $w = 3$. If -1 is not a quadratic residue, we get a Hermitian hypergroup.

For general p the structure constants N_{ij}^k, called **cyclotomic constants**, were determined by Gauss. There is a natural generalization of this problem to the case where we consider the subgroup of all kth powers in $\mathbb{Z}_p$ for some k dividing $p-1$. The images of this subgroup under the automorphism group of $\mathbb{Z}_p$ form a hypergroup with $(p-1)/k$ elements. Finding the structure constants in this situation is much more difficult than in quadratic case, and despite much work the general problem still seems to be unsolved. For more in this direction, see [16].

$\boxed{\textbf{Exercise 5}}$. Find the number of solutions $(x, y, z, w) \in \mathbb{Z}_p^4$ to $x^2 + 3y^2 + 4z^2 \equiv 2w^2 \pmod 7$, where we think of $\mathbb{Z}_p$ as $\{0, \ldots, p-1\}$.

7 Algebraic structures on homogeneous spaces

There is an interesting and not widely studied algebraic structure on a homogeneous space G/H. This structure is close to being a hypergroup but is not a hypergroup. Nevertheless it deserves our attention because it is so universal and so simple. Closely related is the hypergroup of H orbits on G/H, which we study later and which is indeed a bona-fide hypergroup, at least when H is compact, though not necessarily commutative.

For simplicity let us suppose that G is a finite group with a subgroup H. A multi-set $A \sqsubset G$ will be called H *left-invariant* if $yA = A$ for all $y \in H$, H *right-invariant* if $Ay = A$ for all $y \in H$, and H *bi-invariant* if it is both H left-invariant and H right-invariant. The subgroup H is itself H bi-invariant. For any $x \in G$, the multi-sets xH and Hx are right and left-invariant respectively and are called left and right cosets respectively, while the multi-set HxH is bi-invariant.

Note carefully that at this point our notation unfortunately but necessarily diverges from the standard usage of HxH as a set. For us HxH is a multi-set of G, which happens to be *uniform*, in that all elements have the same multiplicity.

We say a group G *left-acts* on a set A if there is a function from $G \times X$ to X that sends (x, a) to xa subject to the condition there exists $x \in G$ with $xa = a'$. A *right-action* of G on A requires $(ax)x' = a(xx')$ for all $x', x \in G$ and $a \in A$. A left-action is *transitive* if for all $a, a' \in A$ there exists $x \in G$ with $xa = a'$, and similarly for a right action.

We pay more attention than usual to the roles of left and right here to avoid ambiguity in the subject.

Suppose that H is a subgroup of G, and that h is the associated 1-set. Note that $h^2 = h$. More generally let $H = H_0, H_1, \cdots, H_l$ be a list of the distinct left cosets with $h = h_0, h_1, \cdots, h_l$ the associated $i1$-*cosets*

$$h_j = \frac{1}{|H_j|} H_j.$$

Define the *homogeneous space* G/H to be the set $\{h_0, h_1, \cdots, h_l\}$ of left 1-cosets of H. Then G left-acts transitively on G/H by multiplication in G of 1-sets. Conversely, given a transitive left-action of G on A, pick $a_0 \in A$ and let $H = \{x \in G |\ xa_0 = a_0\} = G_{a_0}$ be the stabilizer of a_0 and h the associated 1-set. Then the map $xh \leftrightarrow xa_0$ sets up a bijection between G/H and A which respects the left-action of G on both.

In an entirely analogous fashion G right-acts on the homogeneous space $H\backslash G$ consisting of right 1-cosets of H.

Since the product of H right-invariant 1-sets is an H right-invariant 1-set there exist constants p_{ij}^k such that

$$h_i h_j = \sum_{k=0}^{l} p_{ij}^k h_k.$$

This is an associative 'product' on G/H which shares some of the properties of a hypergroup, for example the p_{ij}^k are probabilities. However in general h_0 is only a right identity, that is $h_i h_0 = h_i$ for all i. There are inverses but they are not unique, and the product is generally not commutative. But no matter, the structure constants p_{ij}^k are computable invariants of the quotient G/H which in the special case when H is a normal subgroup reduces to a group multiplication product which we call the *quotient* group G/H.

This product structure deserves to be more widely known and studied. It is in some sense a 'collapse' or 'amalgamation' of the group structure on G to a homogeneous space G/H or $H\backslash G$.

As a consequence, we see that any pointed homogeneous space (A, a_0) *carries an algebraic structure.*

Example The group A_4 right-acts transitively on the vertices $\{v_1, v_2, v_3, v_4\}$ of a regular tetrahedron X. The stabilizer subgroup of v_1 is $H = H_0 = \{e, (234), (243)\}$, and its other right cosets are $H_1 = \{(12)(34), (124), (123)\}$, $H_2 = \{(13)(24), (132), (134)\}$ and $H_3 = \{(14)(23), (143), (142)\}$.

Thus $X \simeq H\backslash G$, with the following multiplication law.

	h_0	h_1	h_2	h_3
h_0	h_0	$\frac{1}{3}(h_1+h_2+h_3)$	$\frac{1}{3}(h_1+h_2+h_3)$	$\frac{1}{3}(h_1+h_2+h_3)$
h_1	h_1	$\frac{1}{3}(h_0+h_2+h_3)$	$\frac{1}{3}(h_0+h_2+h_3)$	$\frac{1}{3}(h_0+h_2+h_3)$
h_2	h_2	$\frac{1}{3}(h_0+h_1+h_3)$	$\frac{1}{3}(h_0+h_1+h_3)$	$\frac{1}{3}(h_0+h_1+h_3)$
h_3	h_3	$\frac{1}{3}(h_0+h_1+h_2)$	$\frac{1}{3}(h_0+h_1+h_2)$	$\frac{1}{3}(h_0+h_1+h_2)$

8 Characters of commutative groups and signed hypergroups

The theory of finite abelian groups and more generally locally compact abelian groups is particularly useful and beautiful because of the existence of a perfect duality and accompanying Fourier transform. Let us recall, in a form amenable to generalization, some basic facts about harmonic analysis on a finite abelian group $G = \{g_0 = e, g_1, \dots, g_n\}$ written multiplicatively.

A **character** of G is a function $\chi : G \to \mathbb{C}$ satisfying

$$\begin{aligned} \chi(g_i)\chi(g_j) &= \chi(g_i g_j) \\ \chi(g_i^{-1}) &= \overline{\chi(g_i)}. \end{aligned}$$

The function identically one is a character, call it χ_0. The set of all characters is denoted by $G^\wedge$.

Theorem 8.1 *1) The characters form a basis for the space $F(G)$ of all complex valued functions on G, so that $|G| = |G^\wedge|$ and we may write $G^\wedge = \{\chi_0, \chi_1, \dots, \chi_n\}$.*

2) $G^\wedge$ is itself an abelian group under pointwise multiplication.

3) Characters are orthonormal with respect to the inner product

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_k \chi_i(g_k)\, \overline{\chi_j(g_k)}.$$

4) For $g \in G$, $X_g(\chi) = \chi(g)$ is a character of $G^\wedge$ and $g \to X_g$ establishes the isomorphism $G \simeq (G^\wedge)^\wedge$.

Now suppose $K = \{c_0, c_1, \dots, c_n\}$ is a commutative signed hypergroup. Recall this means that the structure constants n_{ij}^k, while real, are allowed to be negative, subject still to $\sum_k n_{ij}^k = 1, \quad \forall\, i, j$ and subject to the crucial conditions

$$c_i^* = c_j \iff n_{ij}^0 > 0$$

$$c_i^* \neq c_j \iff n_{ij}^0 = 0.$$

A **character** of K is a function $\chi : K \to \mathbb{C}$ satisfying

$$\begin{aligned}\chi(c_i)\chi(c_j) &= \sum_k n_{ij}^k \chi(c_k)\\ \chi(c_i^*) &= \overline{\chi(c_i)}.\end{aligned}$$

The function identically one is a character, call it χ_0. The set of all characters is denoted by $K^\wedge$.

Here is the corresponding theorem for finite commutative hypergroups. See [17] for more details.

Theorem 8.2 *1) The characters form a basis for the space $F(K)$ of all functions on K, so $|K| = |K^\wedge|$. We write $K^\wedge = \{\chi_0, \chi_1, \ldots, \chi_n\}$.*

2) $K^\wedge$ is itself a signed hypergroup under pointwise multiplication and conjugation.

3) Characters are **orthogonal** *with respect to the inner product*

$$\langle \chi_i, \chi_j \rangle = \frac{1}{w(K)} \sum_k \chi_i(c_k)\overline{\chi_j(c_k)}\, w(c_k).$$

4) For $c \in K$, $X_c(\chi) = \chi(c)$ is a character of $K^\wedge$ and $c \to X_c$ establishes the isomorphism $K \simeq (K^\wedge)^\wedge$.

$\boxed{\textbf{Exercise 6}}$.

(1) Determine the characters of all the hypergroups discussed so far, i.e., complete the **hypergroup** character table in each case.

	c_0	c_1	$\ldots$	c_n
χ_0	1	1	$\ldots$	1
χ_1	1			
$\vdots$	$\vdots$			
χ_n	1			

(2) Determine the **structure constants** of all the hypergroup duals from the above tables.

Example. Convolution of spheres in a cube gave $K = \{c_0, c_1, c_2, c_3\}$ with structure equations

$$\begin{aligned}c_1^2 &= \tfrac{1}{3}c_0 + \tfrac{2}{3}c_2 & c_1c_3 &= c_2\\ c_1c_2 &= \tfrac{2}{3}c_1 + \tfrac{1}{3}c_3 & c_3^2 &= c_0\\ c_2^2 &= \tfrac{1}{3}c_0 + \tfrac{2}{3}c_2 & c_2c_3 &= c_1\end{aligned}$$

Here is the hypergroup character table, with the weight of all elements $c_i \in K$ and $\chi_i \in K^\wedge$ given, as well as the equations of the dual signed hypergroup (a hypergroup in this case). The circled numbers are the weights of the elements of K and $K^\wedge$.

		1	3	3	1
		c_0	c_1	c_2	c_3
1	χ_0	1	1	1	1
3	χ_1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	1
3	χ_2	1	$\frac{1}{3}$	$\frac{1}{3}$	-1
1	χ_3	1	-1	-1	-1

$$\begin{aligned} \chi_1^2 &= \frac{1}{3}\chi_0 + \frac{2}{3}\chi_1 & \qquad \chi_1\chi_3 &= \chi_2 \\ \chi_1\chi_2 &= \frac{1}{3}\chi_3 + \frac{2}{3}\chi_2 & \chi_3^2 &= \chi_0 \\ \chi_2^2 &= \frac{1}{3}\chi_0 + \frac{2}{3}\chi_1 & \chi_2\chi_3 &= \chi_1 \end{aligned}$$

Application: (Solution to Exercise 3) The duality allows us to realize K as a set of functions on $K^\wedge$. This realization turns convolution into pointwise multiplication (one of the main points of Fourier analysis!).

Thus c_1^n can be realized as the function with values 1, $\left(-\frac{1}{3}\right)^n$, $\left(\frac{1}{3}\right)^n$, and $(-1)^n$ on χ_0, χ_1, χ_2, and χ_3 respectively. Since each c_i is orthogonal to c_j, $c_i \neq c_j$, if the coefficient of c_0 in c_1^n is m then

$$\langle c_i^n, c_0\rangle = \langle m\, c_0 + \dots, c_0\rangle = m\langle c_0, c_0\rangle = m.$$

Thus

$$m = \frac{1}{8}\left(1 + 3\left(-\frac{1}{3}\right)^n + 3\left(\frac{1}{3}\right)^n + (-1)^n\right)$$

is the probability of being at S_0 after n steps of a nearest neighbour random walk. When $n = 6$ this is

$$\frac{1}{8}\left(2 + 6\left(\frac{1}{3}\right)^6\right) = \frac{1}{4}\left(1 + \left(\frac{1}{3}\right)^5\right).$$

9 Two important theorems

Let G be a finite group. Let $K(G) = \{c_0, c_1, \dots, c_n\}$ denote its class hypergroup and let $K(G^\wedge) = \{X_0, X_1, \dots, X_n\}$ denote its character hypergroup.

Theorem 9.1 $K(G)^\wedge \simeq K(G^\wedge)$.

This duality, which goes back to Frobenius' original paper on the subject, is easy to describe. Recall that each normalized character X_i on G is a function on G (with value one at e) and so defines a character of the hypergroup $K(G)$ by the pairing

$$(X_i, c_j) = X_i(g) \qquad \text{for any} \quad g \in C_j.$$

As a consequence the character theory of G and the multiplicities of tensor products of representations of G are completely determined by the class hypergroup $K(G)$.

Note: The map $G \to K(G)$ is $1:1$ on simple groups, a fact known only as a result of the classification of simple groups.

Suppose a group H acts on a finite abelian group G by automorphisms. Let the set of H orbits on G be $\{\mathcal{O}_0 = \{e\},\ \mathcal{O}_1, \ldots, \mathcal{O}_n\}$. Then using multi-set notation, there exist $N_{ij}^k \in \mathbb{Z}^+$ such that

$$\mathcal{O}_i\mathcal{O}_j = \sum_k N_{ij}^k \mathcal{O}_k$$

and $d_i = |\mathcal{O}_i|$ is a dimension function. Setting $c_i = \dfrac{\mathcal{O}_i}{d_i}$ we get

$$c_i c_j = \sum_k n_{ij}^k\, c_k = \sum_k \frac{d_k N_{ij}^k}{d_i d_j} c_k$$

which makes $\{c_0, c_1, \ldots c_n\}$ into a commutative hypergroup, of weight $|G|$, called the hypergroup of H orbits on G, and denoted by $K(G;H)$. This is a very rich source of examples.

Now if H acts on G, it also acts on $G^\wedge$ by the rule

$$(h\chi)(g) = \chi(h^{-1}g).$$

Thus we may also consider the hypergroup of H orbits on $G^\wedge$, say $K(G^\wedge, H) = \{X_0, X_1, \ldots, X_n\}$.

Theorem 9.2 $K(G^\wedge; H) \simeq K(G;H)^\wedge$.

As before we may describe the pairing explicitly. Let $\mathcal{O}_i$ be an H-orbit in G and $\mathcal{U}_j$ an H-orbit in $G^\wedge$. Let c_i and X_j be the elements in $K(G;H)$ and $K(G^\wedge; H)$ respectively determined by $\mathcal{O}_i$ and $\mathcal{U}_j$. Then

$$(c_i, X_j) = \frac{1}{|\mathcal{O}_i|\ |\mathcal{U}_j|} \sum_{g \in \mathcal{O}_i} \sum_{\chi \in \mathcal{U}_j} \chi(g).$$

Exercise 7 Prove this gives the isomorphism of the above Theorem.

By H-invariance, it is possible to rewrite this pairing also as

$$\begin{aligned}(c_i, X_j) &= \frac{1}{|\mathcal{U}_j|}\sum_{\chi\in\mathcal{U}_j}\chi(g) \text{ for some (any) } g\in\mathcal{O}_i \\ &= \frac{1}{|\mathcal{O}_i|}\sum_{g\in\mathcal{O}_i}\chi(g) \text{ for some (any) } \chi\in\mathcal{U}_j.\end{aligned}$$

In other words X_j is just the Fourier transform of the probability measure on the orbit $\mathcal{U}_j \subseteq G^\wedge$.

Note. Any representation π of a group H in a vector space V over a finite field F_q gives rise to a hypergroup $K(V;H)$ of orbits. We refer to this as 'convolution' of orbits, or more precisely as convolution of uniform probability measures on these orbits, since we always normalize by $|\mathcal{O}_i|$ to get a hypergroup. In each of these cases, the dual hypergroup is then obtained from the orbits of H in the dual vector space V^* by taking the Fourier transforms of the invariant measures on the dual orbits.

Note. The general set-up here has applications far beyond finite group theory. Implicit in this duality of hypergroups is a partial explanation of why the Kirillov orbit method is such a powerful tool in the representation theory of Lie groups. In a nutshell the characters of (say) a compact Lie group are dual to the hypergroup of conjugacy classes, which in turn is closely related to the hypergroup of adjoint orbits via the wrapping map. Since the dual of the hypergroup of adjoint orbits is the hypergroup of co-adjoint orbits, the fundamental connection between characters and coadjoint orbits is quite natural. The second formula above is essentially the Kirillov character formula in the context.

For more information in this direction, see Wildberger [15].

10 Convolution of spheres in $\mathbb{R}^n$

Suppose we start at the origin in $\mathbb{R}^2$ and take a random step of length r_1, and then a random step of length r_2. Here random means we choose a direction using a uniform probability measure on $[0, 2\pi]$. What is the probability that we end up at distance r from the origin? More precisely we seek the probability density function, $f_{r_1,r_2}(r)$ of the quantity r. If θ is the angle between the first and second step then the cosine law states that $r^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$, so that $2rdr = 2r_1r_2\sin\theta d\theta$. Because of the rotational symmetry, $f_{r_1,r_2}(r)dr$ will be the measure of that part of the circle of radius r_2 for which the angle is in $[\theta, \theta + d\theta]$, that is $\frac{d\theta}{\pi}$. So

$$f_{r_1,r_2}(r) = \frac{1}{\pi}\frac{2r}{2r_1r_2\sin\theta} = \frac{1}{\pi h} = \frac{2r}{4\pi\,\text{Area}\,(\Delta OPQ)}.$$

Using Heron's Theorem, we get alternatively

$$f_{r_1,r_2}(r) \quad = \quad \frac{2r}{\pi[(r^2-(r_1-r_2)^2)((r_1+r_2)^2-r^2)]^{\frac{1}{2}}}$$

$$\text{for} \qquad |r_1-r_2| \le r \le r_1+r_2.$$

To calculate the same probability in n space, we need to know the ratio of the surface area of a strip of the sphere at an angle θ to $\theta + d\theta$ from a fixed radius to the total surface area. This ratio is

$$\frac{\sin^{n-2}\theta}{c_n}\,d\theta,$$

where

$$c_n = \int_0^\pi \sin^{n-2}\theta\,d\theta = \frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}}{\Gamma(\frac{n}{2})}.$$

This gives the probability density

$$f^{(n)}_{r_1,r_2}(r) \quad = \quad \frac{2r}{c_n}\,\frac{[(r^2-(r_1-r_2)^2)((r_1+r_2)^2-r^2)]^{\frac{n-3}{2}}}{(2r_1r_2)^{n-2}} \tag{10.1}$$

$$\text{for} \qquad |r_1-r_2| \le r \le r_1+r_2.$$

Note that in the special case of $n = 3$ the formula reduces to

$$f^{(3)}_{r_1,r_2}(r) = \frac{r}{2r_1r_2}$$

which is rather remarkably a linear function of r. If in three dimensional space we renormalize so that the uniform surface measure ν_r on the sphere of radius r has mass

$$\int_{S_r} d\nu_r = \frac{r}{2},$$

then

$$\nu_{r_1} * \nu_{r_2} = \int_{|r_1-r_2|}^{r_1+r_2} \nu_r\,d\mu(r).$$

Remark. Recall that orthogonal projection of surface measure of a two-dimensional sphere onto a diameter is uniform. This useful fact, well known to those teaching advanced calculus, is thus curiously related to the uniform measure appearing in the formula for $\nu_{r_1} * \nu_{r_2}$.

The convolution structure on spheres in n space described above may be transferred to give a locally compact commutative hypergroup on $\mathbb{R}^+$. The convolution

$$\delta_{r_1} *_n \delta_{r_2} = f^{(n)}_{r_1,r_2}(r)\ \chi_{[|r_1-r_2|,r_1+r_2]}\,dr \tag{10.2}$$

extends by linearity and continuity to an algebra structure on the space $M_c(\mathbb{R}^+)$ of compactly supported Borel measures on $\mathbb{R}^+$. This algebra structure has in a generalized sense a basis consisting of the delta functions, and with respect to this basis the axioms of a hypergroup are satisfied. The resulting structure is called a *locally compact commutative hypergroup*, and obeys the following axioms, where we write K instead of $\mathbb{R}^+$, in general a locally compact topological space, and $M_c(K)$ instead of $M_c(\mathbb{R}^+)$:

(1) (Closure) The product of delta functions $\delta_x * \delta_y$ for $x, y \in K$ is a compactly supported probability measure which varies continuously with x and y.

(2) (Associativity) $M_c(K)$ is associative.

(3) (Identity) $\exists\, e \in K$ so that δ_e is the identity.

(4) (Inverses) $\forall\, x \in K$ $\exists$ unique element $x^* \in K$ with the property that e is contained in the support of the measure $\delta_x * \delta_{x^*}$. Furthermore $(\delta_x)^* = \delta_{x^*}$ and $*$ extends to an algebra involution.

For more details, see Jewett [6] or Bloom and Heyer [1].

(5) (Commutativity) $M(K)$ is commutative.

If G is a locally compact group, $M(G)$ satisfies the above Axioms (1)-(4). The point is that there are situations when K has no underlying group structure yet $M(K)$ still 'acts like a group algebra' in the sense of satisfying the above Axioms.

Note that for each $n \geq 0$ (not just $n \in \mathbb{Z}^+$) the formula for $\delta_{r_1} *_n \delta_{r_2}$ gives a hypergroup structure on $\mathbb{R}^+$. This allows us to interpolate results from radial harmonic analysis on $\mathbb{R}^n$ to non-integral values of n.

11 Convolutions on symmetric spaces

Let $X = G/K$ be a symmetric space, K the compact subgroup which fixes a point $O \in X$. Let $\mathcal{O}_1$, $\mathcal{O}_2$ be K orbits about O. Choose a point P on $\mathcal{O}_1$ randomly using the unique K-invariant probability measure on it. The isotropy subgroup of G at P is conjugate to K so it makes sense to talk about the translate of the orbit $\mathcal{O}_2$ around O to P, say $\mathcal{O}_2'$. In fact we may choose any isometry ρ in G which sends O to P and let $\mathcal{O}_2' = \rho(\mathcal{O}_1)$. Choose a point Q randomly on $\mathcal{O}_2'$ and consider the probability density that Q lies on the general K orbit $\mathcal{O}$ about O. We interpret this as a probability measure on the space $K(X)$ of K orbits on X, which is the product of the Dirac masses at $\mathcal{O}_1$ and $\mathcal{O}_2$.

For the sphere S^n, $K(X) \simeq [0,\pi]$ indexes the set of 'hypercircles' centered at a fixed point O. The spherical cosine law

$$\cos r = \cos r_1 \cos r_2 + \sin r_1 \sin r_2 \cos\theta$$

and a similar analysis to the one in the previous section gives a probability density

$$g^{(n)}_{r_1,r_2}(r) = \frac{\sin r}{c_n}\,\frac{[(\cos(r_1-r_2)-\cos r)(\cos r - \cos(r_1+r_2))]^{\frac{n-3}{2}}}{[(\sin r_1)(\sin r_2)]^{n-2}}.$$

This is a nonlinear analog of the linear convolution of spheres in $\mathbb{R}^n$.

A *character* of a hypergroup K is a bounded function $X : K \to \mathbb{C}$, such that

$$\begin{aligned} X(x)X(y) &= \int_K X(z)\delta_x * \delta_y(z) \qquad \forall\, x,y \in K \\ X(x^*) &= \overline{X(x)} \qquad \forall\, x \in K\,. \end{aligned}$$

We let $K^\wedge$ denote the set of characters. If G is a compact Lie group acting on a vector space V,

$$K(V;G)^\wedge \cong K(V^\wedge;G).$$

The Fourier transform of the invariant probability measure μ_M on any G orbit $M \subset V^\wedge$ is X_M.

This is a continuous version of Theorem 4.2. It means that the way to get characters of the hypergroup of orbits of G on V is to take Fourier transforms of orbits of G in the dual vector space $V^\wedge$.

In the case of spheres in n dimensional space, corresponding to the action of the orthogonal group $O(n)$, the Fourier transforms of uniform probability measure on the sphere of radius λ is

$$\phi^{(n)}_\lambda(r) = \mu^\wedge_\lambda(r) = \frac{J_{\frac{n-2}{2}}(\lambda_r)}{(\lambda_r/2)^{\frac{n-2}{2}}}\Gamma(\frac{n}{2}).$$

where $J_k(x)$ denotes the Bessel function of order K

$$J_k(x) = (\frac{x}{2})^k \sum_{m=0}^{\infty} \frac{(-1)^m X^{2m}}{2^{2m} m!\Gamma(m+k+1)}.$$

Of special note is the case when $n=3$ for which we get

$$\phi^{(3)}_\lambda(r) = \frac{\sin\lambda_r}{\lambda_r}.$$

For the case of the orthogonal group $O(n-1)$ acting on the sphere S^n we have seen that the hypergroup of orbits is an interval

$$K(X) \simeq [0,\pi]$$

and that $K^\wedge \simeq \{0, 1, 2, \dots\}$.

In this case the characters are given explicitly by

$$\psi_k^n(r) = \frac{k!\Gamma(n-1)}{\Gamma(n-1+k)}\, P_k^{\frac{n-1}{2}}(\cos r),$$

where P_k^ℓ is the Gegenbauer polynomial defined by

$$(1 - 2tz + z^2)^{-\lambda} = \sum_{k=0}^{\infty} P_k^\lambda(f) z^k.$$

For $\lambda = \frac{1}{2}$ this is the Legendre polynomial and for $\lambda = 1$ this is the Tchebycheff polynomial

$$U_k(\cos r) = \frac{\sin(k+1)r}{\sin r}$$

so that

$$\psi_k^3(r) = \frac{\sin(t+1)r}{(k+1)\sin r}.$$

The trigonometric relation

$$U_\ell U_m = \sum_{\substack{k=|\ell-m| \\ k\equiv \ell+m (\mathrm{mod}\, 2)}}^{l+m} U_k$$

yields the following hypergroup structure on $K(S^3)^\wedge = \{\psi_0^3, \psi_1^3, \dots\}$

$$\psi_l^3 * \psi_m^3 = \sum_{k=|\ell-m|}^{|\ell+m|} \frac{(k+1)}{(\ell+1)(m+1)} \psi_k^3.$$

Note that $\phi_{k+1}^3(r) = \psi_k^3 \dfrac{\sin r}{r}$.

This means that the nonlinear hypergroup of $O(3)$ orbits on the 3-sphere is related to the hypergroup of $O(3)$ orbits on $K(\mathbb{R}^3)$ by the factor $j(X) = \dfrac{\sin r}{r}$, which is in fact an analytic square root of the determinant of the exponential map from the tangent space of S^3 to S^3 itself.

12 The Main Problems.

We now specialize to the situation when a compact Lie group G acts on a Euclidean vector space V via a linear action $\pi : G \to GL(V)$. Given $X \in V$, the set $\pi(G)X = \mathcal{O}$ is the *orbit* through X, and carries a unique G-invariant probability measure μ_O. There are then some natural problems that arise.

Problem 1 Describe the orbits of G on V, both individually and as a collection $K = K(V; G)$.

Problem 2 Describe the hypergroup structure of K; that is, find the measures $\varphi(O, O', O'')$ such that

$$\mu_{\mathcal{O}} * \mu_{\mathcal{O}'} = \int_K \varphi(\mathcal{O}, \mathcal{O}', \mathcal{O}'') \mu_{\mathcal{O}''} \, d\mathcal{O}''$$

with respect to some measure $d\mathcal{O}$ on K.

Problem 3 Describe the Fourier transform of the measures $\mu_{\mathcal{O}}$ explicitly (these will be the characters of the dual hypergroup $K(V^*; G)$.

Note that from abstract considerations each orbit O is a homogeneous space of G, that is of the form G/H for some subgroup H of G. What we are interested in here is the geometry of the imbedding of O in the Euclidean space V. For example, we might be interested in the projection of μ_O onto subspaces W of V.

Main Example. If $\mathfrak{g}$ is the Lie algebra of G, then G acts on $\mathfrak{g}$ via the adjoint action, the derivative of the conjugation of action of G on itself. The orbits of this action are called *adjoint orbits*. If $\mathfrak{g}^*$ is the linear dual of $\mathfrak{g}$, then G acts on $\mathfrak{g}^*$ by the rule

$$(g \cdot f)(x) = f(g^{-1} \cdot X) \qquad f \in \mathfrak{g}^*, \, X \in \mathfrak{g}, \, g \in G$$

and the orbits of this action are the co-adjoint orbits. When G is compact, we can always find a G-invariant positive definite inner product $(\, , \,)$ which allows us to identify $\mathfrak{g}$ and $\mathfrak{g}^*$, so that the adjoint and co-adjoint orbit pictures coincide. It is important to realize that this is not true generally.

Nevertheless for any Lie group G, the co-adjoint orbits are symplectic manifolds, that is they carry closed non-degenerate 2-forms, and the study of these orbits is intimately tied up with symplectic geometry and associated moment maps. We will see that this feature has important consequences for our program of understanding orbit convolutions.

Example

Let $G = SU(n)$, the group of all determinant-one $n \times n$ unitary matrices, so that $\mathfrak{g} = su(n)$ is the Lie algebra of all traceless skew Hermitian matrices. The adjoint action is then conjugation, so that the orbit $\mathcal{O}_X$ consists of all matrices in $\mathfrak{g}$ with the same eigenvalues as X. Note that multiplication by i transforms a skew Hermitian matrix to a Hermitian one, and that by adding a suitable multiple of the identity we may obtain any Hermitian matrix. Thus the orbits here are essentially the same as the orbits of G on the space of Hermitian matrices.

Since any $X \in \mathfrak{g}$ can be diagonalized, each orbit $\mathcal{O}$ intersects the abelian subalgebra

$$\mathfrak{t} = \{X \in su(n) |\ X = \begin{vmatrix} i\lambda_1 & & 0 \\ & \ddots & \\ 0 & & i\lambda_n \end{vmatrix}, \lambda_i \in \mathbb{R},\ \sum_{j=1}^{n} \lambda_j = 0\}$$

in a finite number of places and intersects the cone $\mathfrak{t}^+ = \{X \in \mathfrak{t} \mid \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\}$ in exactly one place. Thus $\mathfrak{t}^+$ forms a transversal to the set of G orbits in $\mathfrak{g}$. If $X = \mathrm{diag}(i\lambda_1, \ldots, i\lambda_n)$ then $\mathcal{O}_X \cap \mathfrak{t} = S_n(i\lambda_1, \ldots, i\lambda_n)$ where S_n, the symmetric group, acts in the obvious fashion. Note that $|\mathcal{O}_X \cap \mathfrak{t}\ | = n!$ if and only if all eigenvalues of X are distinct, that is, if X is *regular*.

13 Sums of Orbits and Weyl's Eigenvalue Problem

In the general case, orbits of G in $\mathfrak{g}$ are parametrized by a cone $\mathfrak{t}^+$ in a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$.

This may be described as follows. If we denote the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ by ad, the elements $ad(H)$, $H \in \mathfrak{t}$ are semisimple and simultaneously commute; it follows that

$$\mathfrak{g} = \mathfrak{t} \oplus \alpha\mathfrak{g}_\alpha,$$

where each $\alpha \in \mathfrak{t}^*$, and where $\mathfrak{g}_\alpha$ is a 2-dimensional subspace of $\mathfrak{g}$ on which $ad(H)$ has the form

$$\alpha(H) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for each $H \in \mathfrak{t}$. The set R of all such α and their negatives are called the *roots* of $\mathfrak{g}$ and form an abstract root system in $\mathfrak{t}^*$. Each $\alpha \in R$ determines a hyperplane $\mathcal{H}_\alpha$ in $\mathfrak{t}$. If we let s_α denote the reflection in this hyperplane then W, the Weyl group of G, is the finite group generated by the reflections s_α, $\alpha \in R$. Any connected component of $\mathfrak{t}\ - \cup_{\alpha \in R}\mathcal{H}_\alpha$ is called a Weyl chamber; these open polyhedral cones are in $1 : 1$ correspondence with the elements of W since W acts simply transitively on the set of all Weyl chambers.

Let us choose a distinguished Weyl chamber C. The set of roots $\alpha \in R$ which are positive on C is denoted R^+ and called the **positive** roots. The closure $\overline{C}$ is a closed polyhedral cone and is a fundamental domain for the action of W on $\mathfrak{t}$, that is, every W orbit on $\mathfrak{t}$ meets $\overline{C}$ in a unique point.

It turns out that $\overline{C}$ also parametrizes the G orbits on $\mathfrak{g}$. In fact for any adjoint orbit $\mathcal{O}$, $\mathcal{O} \oplus \overline{C}$ consists of exactly one point.

Example. $SU(n)$. The roots are the functions α_{jk} for $j \neq k$, where

$$\alpha_{jk}(H) = \alpha_{jk}(\begin{vmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_n \end{vmatrix}) = \lambda_j - \lambda_k.$$

If we let C be the chamber containing, say diag $i(n, n-1, \cdots, 1)$ then $\alpha_{jk} \in R^+ \Leftrightarrow j < k$. The reflection $s_{\alpha_{jk}}$ just interchanges the j^{th} and k^{th} entries of H, so that $W = \langle s_{\alpha_{jk}} \rangle = S_n$. The hyperplane $\mathcal{H}_{\alpha_{jk}}$ consists of all $H \in \mathfrak{t}$ with $\lambda_j = \lambda_k$.

Let $p : \mathfrak{g} \to \mathfrak{t}$ be the orthogonal projection with respect to $(\ , \ )$. For any $\lambda \in \overline{C}$, let μ_λ by the G-invariant probability measure on the orbit $\mathcal{O}_\lambda$ through λ. It turns out that to describe convolution of orbital measures it is convenient to work with renormalized measures, not probability measures. For $\lambda \in C$, let v_λ be the G-invariant probability measure on $\mathcal{O}_\lambda$ with total mass

$$\prod_{\alpha \in R^+} (\alpha, \lambda) \Big/ \prod_{\alpha \in R^+} (\alpha, \rho).$$

Here ρ is one-half the sum of the positive roots, and plays something of the role of Planck's constant in this theory. Let T_λ be the push-down of v_λ to $\mathfrak{t}$ via p. The particular measure T_ρ plays an important role in the theory.

Theorem 13.1 *Let $\lambda \in \overline{C}$ and $\mathcal{O}_\lambda$ be the associated orbit through λ.*

1. *$p(\mathcal{O}_\lambda) = \mathrm{conv}(W \cdot \lambda) = D_\lambda$, the convex hull of the W orbit of λ in T.*

2. *T_λ is a piece-wise polynomial measure on D_λ.*

Part 1 of the theorem is a classical result of Schur and Horn for the case $G = SU(n)$ which was extended by Kostant to the general case. Part 2 is a special case of a result of Duistermat and Heckman coming from symplectic geometry.

We will need specific information about the measures T_λ. Let F_α denote Lebesgue measure on the ray through $\alpha \in \mathfrak{t}$ and let

$$P = \prod_{\alpha \in R^+} * F_{-\alpha}$$

denote the convolution product of $F_{-\alpha}$, $\alpha \in R^+$. This is well defined since $\{\alpha \mid \alpha \in R^+\}$ lie in a convex cone. More canonical perhaps is the W anti-invariant measure on $\mathfrak{t}$

$$Q = \sum_{w \in W} \mathrm{sgn}(w) w(P),$$

where $\mathrm{sgn}(w)$ is $(-1)^r$ if r is the number of reflections in an expression for w. Let δ_x be the delta function at $x \in \mathfrak{t}$.

Theorem 13.2 [1] *For $\lambda \in C$ (interior of $\overline{C}$),*

$$T_\lambda = \sum_{w \in W} \operatorname{sgn}(w)\delta_{w(\lambda)} * P = 1/|W| \sum_{w \in W} \operatorname{sgn}(w)\delta_{w(\lambda)} * Q.$$

There is remarkable cancellation in these formulae.

Exercise Work out these formulas for the special cases $su(2)$ and $su(3)$.

We are now in a position to at least state a solution to the old problem of Weyl which asked us to describe the possible sets of eigenvalues for the sum $A+B$ of two Hermitian matrices A and B both of whose eigenvalues are known. This solution works out the *probability measure* for the possible sets of eigenvalues, whose support is the set in question. In fact the solution given here, which was the first complete solution to Weyl's problem, tackled the general case of convolving any two adjoint orbits of any compact simple Lie group.

Theorem 13.3 [1] *For $\lambda, \lambda' \in C$,*

$$v_\lambda * v_{\lambda'} = \int_C \phi(\lambda, \lambda'; \lambda'') v_{\lambda''} d\lambda'',$$

where

$$\phi(\lambda, \lambda'; \cdot\,) = \sum_{w \in W} T_\lambda * \delta_{w(\lambda')}(\lambda'').$$

Of course there are still many questions to be settled about the specific nature of this measure. In [2] an inductive approach to determining the vertices of the support is given. The question about determining the faces, at least in the case of SU(n), involves Horn's Conjecture, now proved by Klyachko [7], and Knutson and Tao [8]. To determine corresponding information about the faces for the general case is certainly an interesting and important problem.

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