FINITE COMMUTATIVE HYPERGROUPS AND APPLICATIONS FROM GROUP THEORY TO CONFORMAL FIELD THEORY

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§0. Introduction.

Finite abelian groups are among the most simple and useful of all algebraic objects, possessing a pleasant but non-trivial classification, a perfect theory of duality, and a penchant for cropping up in interesting places.

When one thinks of generalizations of this theory within the finite setting, the theory of non-abelian finite groups immediately comes to mind. This theory has now achieved a very high state of development and sophistication. It comes as a surprise perhaps, that there is *another* generalization whose simplicity and range of application rivals that of the non-abelian groups. This is the theory of *finite commutative hypergroups.*

This paper hopes to introduce these fascinating and important objects to the wider mathematical and physical community. Special cases, generalizations or variants of these objects have been studied under a plethora of different names : let us mention the work of Kawada [Kaw] on C-algebras, Levitan's work [Le] on generalized translation operators, Brauer's work [Br] on pseudo-groups, Hecke algebras, hypercomplex systems (see Berezansky and Kalyushnyi [BK], Vainermann [Va]), paragroups (Ocneanu [O]), superselection sectors (Doplicher, Haag and Roberts [DHR1], [DHR2], Longo [Lo]), Bose Mesner algebras (Bose and Mesner [BM]), Racah- Wigner algebras (see Sharp [Sh]), centralizer algebras, table algebras (Arad and Blau [AB]), association schemes (see Bannai and Ito [BI]) and the fusion rules of conformal field theories (Verlinde [Ve], Moore and Seiberg [MS]) .

The general theory of hypergroups was introduced by Dunkl [Du1], Jewett [Je], and Spector [Sp] (see also the surveys of Ross [R] and Litvinov [Li]) and treats finite commutative hypergroups as a special case. There has thus been an element of re-discovery and occasional duplication in the history of the subject. This also means there is a unique opportunity for workers in different areas to benefit from the results and insights of others working in seemingly unrelated fields. We hope this paper will facilitate such a cross-disciplinary exchange.

From this author's point of view the first work in the subject is Frobenius' landmark paper of 1896: 'Über Gruppencharaktere' [Fr], where the class hypergroup $\mathcal{K}(G)$ of a finite group G first makes its (implicit) appearance and is used as the

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corner-stone for the development of harmonic analysis on G. One of the somewhat heretical points we would like to emphasize is that what is usually known as 'non-abelian harmonic analysis on a group G' is largely equivalent to 'abelian harmonic analysis on the commutative hypergroup $\mathcal{K}(G)$'. In fact this seems to have been Frobenius' original orientation- representation theory came later.

This insight opens up the possibility of a re-evaluation of harmonic analysis on general groups - for a discussion of the implications for Lie theory, see [Wil2].

So what is a finite commutative hypergroup? We will give a precise mathematical definition in the next section; here is a precise *physical* definition. A finite commutative hypergroup is a finite collection of particles, (strictly speaking, particle *types*) say $\{c_0, c_1, \cdots, c_n\}$, which are allowed to interact by colliding. When two particles collide, they coalesce to form a third particle. The results of collisions are however not definite; if we collide c_i with c_j the probability of emerging with the single particle c_k is $n_{ij}^k$and is fixed. The particle c_0 is absorbed in any collision- we call it a photon. Each particle has an anti-particle which is uniquely specified by the following rule: collision of two particles has a non-zero probability of resulting in a photon if and only if the two particles are each others anti-particles. Particle interactions are independent of their order in time or their position in space. The structure of the entire system is thus determined completely by the probabilities n_{ij}^k which are invariant under interchange of all particles with their antiparticles.

At this point the reader unfamiliar with hypergroups is encouraged to write down *mathematical* axioms that correspond to the above.

This simple physical description already makes it clear (at least to this author) that hypergroups are important objects. Although no claim is made that real particles interact exactly as above what is true is that physicists have long had need of mathematical models which behave this way - this is one reason at least for the ever popular role of group representations in physics. The general theory provides physicists with a whole variety of other models for such interactions - models arising from other aspects of group theory, from combinatorics, number theory, Von Neumann algebras, graph theory and quantum groups.

When we come to discuss the harmonic analysis of such objects, we will need to introduce more general objects- so-called *signed* hypergroups (these were originally called *ensembles* in [Wil1]). From a physical standpoint, this means nothing more than letting the probabilities n_{ij}^k become negative (or greater than 1). While such a suggestion may seem drastic, the mathematics more or less forces it upon one. This may provide some additional motivation for considering the potential role of negative probabilities in physics- an issue already raised by Feynmann [Fe].

The outline of this paper is as follows.

In §1 we introduce the basic objects of interest and some of the main motivating examples. Although we are primarily interested in finite commutative hypergroups, both for the harmonic analysis and applications it is necessary to consider somewhat more general objects. We thus proceed in a very open ended fashion by first introducing 'generalized hypergroups' and then proceeding to the more specialized case of 'signed hypergroups' and 'hypergroups'.

§2 introduces the basic setup for harmonic analysis on a signed hypergroup. Restricting one's attention to hypergroups does not exactly lead to a workable theory, as already recognized by Dunkl, Jewett, and Spector, since the dual of such an object is not in general also a hypergroup. When we go to signed hypergroups, however, these problems disappear and one has a theory of duality and Fourier transform almost as pleasant as that for finite abelian groups.

In §3 we discuss the important relation with finite group theory, illustrate the situation explicitly with the case of S_3, and describe some interesting results of Arad and Blau, and of Arad, Fisman and Blau on products of group characters which were proven in the hypergroup framework and thus hold in much greater generality.

In §4 we introduce the connection with distance regular graphs, random walks on graphs, and the theory of strongly regular graphs. Again we try to illustrate the main ideas through examples. We mention the connection with systems of orthogonal polynomials and finite measures on the real line.

In §5 we show that the classical theory of cyclotomy contains implicitly the study of certain 'integral' hypergroups with a cyclic symmetry in the structure constants. These structure constants are closely related to the numbers of solutions of Diophantine equations of 'Fermat' type. A rigidity result of the author is presented.

In §6 we discuss association schemes. These are very general combinatorial objects which include many of the previous examples as special cases. Roughly speaking these are hypergroups having renormalizations which can be realized by $0, 1$ matrices. This subject has been highly developed by algebraic combinatorialists and is a rich source of examples and results.

In §7 we present a connection with Information Theory, define the entropy of a general element in the algebra of a hypergroup, and state a version of the Second Law of Thermodynamics due to the author that asserts that entropy is nondecreasing in particle collisions. An application to multiplicities of group representations occurring in a tensor product is given.

In §8 we briefly indicate how hypergroups arise from the study of inclusions of Π_1 factors and the Jones index theory. The types of hypergroup that occur in this context have been studied by McMullen [M], McMullen and Price [MP] and Sunder [Su] and are very similiar to objects called fusion rule algebras by physicists.

Fusion rule algebras arise naturally in 2D conformal field theories, and in particular finite ones appear in the 'rational' field theories. §9 attempts to give a brief introduction to these objects and show how they may be viewed mathematically as arising from the representation theory of affine Lie algebras, from the truncated tensor products of representations of a quantum deformation of a classical enveloping algebra when q is a root of unity, or from the superselection sector theory of the C^*-algebra of observables.

Some open problems are distributed throughout the article.

Obviously because of the range of topics treated we can not hope to be more than sketchy in many places, and unfortunately can not mention all the people who have contributed to various aspects of the subject. It is hoped that the reader will use the references given to access the wider literature and that workers in different areas will perhaps be encouraged to compare their various diverse approaches and results.

§1. Basic definitions and examples

A (finite) *generalized hypergroup* is a pair $(\mathcal{K}, \mathcal{A})$ where $\mathcal{A}$ is a *-algebra with unit c_0 over $\mathbb{C}$ and $\mathcal{K} = \{c_0, c_1, \ldots, c_n\}$ is a subset of $\mathcal{A}$ satisfying

(A1) $\mathcal{K}$ is a basis of $\mathcal{A}$.
(A2) $\mathcal{K}^* = \mathcal{K}$.
(A3) The structure constants $n_{ij}^k \in \mathbb{C}$ defined by

$$c_i c_j = \sum_k n_{ij}^k c_k$$

satisfy the conditions

$$\begin{array}{rcl} c_i^* = c_j & \Leftrightarrow & n_{ij}^0 > 0 \\ c_i^* \neq c_j & \Leftrightarrow & n_{ij}^0 = 0. \end{array}$$

$\mathcal{K}$ is called *hermitian* if $c_i^* = c_i$ for all i, *commutative* if $c_i c_j = c_j c_i$ for all i, j, *real* if $n_{ij}^k \in \mathbb{R}$ for all i, j, k, *positive* if $n_{ij}^k \geq 0$ for all i, j, k and *normalized* if $\sum_k n_{ij}^k = 1$ for all i, j. A generalized hypergroup which is both positive and normalized will be called a *hypergroup*. A generalized hypergroup which is both real and normalized will be called a *signed hypergroup*.

In this paper, all generalized hypergroups will be commutative.

The *weight* of an element $c_i \in \mathcal{K}$ is $\omega(c_i) = (n_{ij}^0)^{-1}$ where $c_j = c_i^*$. The *total weight* of $\mathcal{K}$ is $\omega(\mathcal{K}) = \sum_i \omega(c_i)$. If $\mathcal{K}$ is a hypergroup then the set of elements of weight 1 forms a group, and $\mathcal{K}$ is an abelian group if and only if all its elements have weight 1.

We now introduce some broad classes of examples. Let G be any finite group with conjugacy classes $C_0 = \{e\},\ C_1, \ldots, C_n$. Identifying C_i with the element $\sum_{g \in C_i} g$ in the group algebra $\mathbb{C}G$, we obtain structure equations

$$C_i C_j = \sum_k N_{ij}^k C_k \tag{1.1}$$

where N_{ij}^k are non-negative integers. Let $c_i = C_i/|C_i|$ so that (1.1) becomes

$$c_i c_j = \sum_k n_{ij}^k c_k$$

where

$$n_{ij}^k = \frac{N_{ij}^k |C_k|}{|C_i|\ |C_j|}.$$

Then $\{c_0, c_1, \ldots, c_n\}$ becomes a hypergroup (where $c_i^* = 1/|C_i| \sum_{g \in C_i} g^{-1}$) which we call the *class hypergroup* of G and denote by $\mathcal{K}(G)$. Note that $\omega(c_i) = |C_i|$ and $\omega(\mathcal{K}(G)) = |G|$.

Now let $\{\rho_0, \rho_1, \ldots, \rho_n\}$ be the irreducible representations of G (up to equivalence) and suppose that

$$\rho_i \otimes \rho_j = \sum_k M_{ij}^k \rho_k.$$

The M_{ij}^k are non-negative integers and if $d_i = \dim \rho_i$ then

$$d_i d_j = \sum M_{ij}^k d_k.$$

This means that if χ_i is the *normalized* character of ρ_i, that is $\chi_i(g) = tr\rho_i(g)/d_i$, then we obtain a hypergroup $\{\chi_0, \chi_1, \ldots, \chi_n\}$ with

$$\chi_i \chi_j = \sum_k m_{ij}^k \chi_k$$

where

$$m_{ij}^k = \frac{M_{ij}^k d_k}{d_i d_j}$$

and where $\chi_i^* = \overline{\chi}_i$. This hypergroup we call the *representation* or *character hypergroup* of G and denote by $\mathcal{K}(G^\wedge)$. Note that $\omega(\chi_i) = d_i^2$ and $\omega(\mathcal{K}(G^\wedge)) = |G|$.

Finally, let H be any subgroup of Aut$G$ for any finite abelian group G. Then the probability measures on the H orbits on G form a hypergroup under convolution in an analogous way as did the conjugacy classes of G. This hypergroup we denote by $\mathcal{K}(G; H)$. This is a very rich source of examples. Note that in particular any finite dimensional representation of a group H in a vector space over a finite field yields such a hypergroup.

§2. Harmonic analysis on a commutative signed hypergroup

Let $\mathcal{K} = \{c_0, c_1, \ldots, c_n\} \subset \mathcal{A}$ be a signed hypergroup (which, as throughout, is commutative) with structure constants n_{ij}^k. A *character* of $\mathcal{K}$ is a function $\chi : \mathcal{K} \to \mathbb{C}$ satisfying

$$\chi(c_i)\chi(c_j) = \sum_k n_{ij}^k \, \chi(c_k).$$

The function identically one is a character; we call it χ_0. The set of all characters is denoted by $\mathcal{K}^\wedge$. Here are some basic facts of harmonic analysis on $\mathcal{K}$, following [Wil1],[Wil7].

(1) There exists a basis $\{e_0, e_1, \ldots, e_n\}$ of $\mathcal{A}$ consisting of mutually orthogonal idempotents satisfying

$$c_i e_j = \chi_j(c_i) e_j \tag{2.1}$$

for some functions χ_j on $\mathcal{K}$.

(2) $\mathcal{K}^\wedge = \{\chi_0, \chi_1, \ldots, \chi_n\}$ and each χ_j satisfies $\chi_j(c_i^*) = \overline{\chi_j(c_i)}$.

(3) The $\{\chi_j\}$ are orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\omega(\mathcal{K})} \sum_i f(c_i)\overline{g(c_i)}\omega(c_i). \tag{2.2}$$

(4) $\mathcal{K}^\wedge$ is itself a signed hypergroup under pointwise multiplication and complex conjugation. Thus $\omega(\chi_j)$ is well-defined (and positive.)
(5) $\omega(\mathcal{K}^\wedge) = \omega(\mathcal{K})$.
(6) $(\mathcal{K}^\wedge)^\wedge \simeq \mathcal{K}$ under the obvious and canonical map.
(7) The relationships between the two bases $\{c_0, \ldots, c_n\}$ and $\{e_0, \ldots, e_n\}$ of $\mathcal{A}$ are given by

$$e_j = \frac{\omega(\chi_j)}{\omega(\mathcal{K})} \sum_i \omega(c_i)\overline{\chi_j(c_i)} c_i$$
$$c_i = \sum_j \chi_j(c_i) e_j$$

In particular we have the following formulae for the Haar measure e_0 and unit c_0:

$$e_0 = \frac{1}{\omega(\mathcal{K})} \sum_i \omega(c_i) c_i$$
$$c_0 = \sum_j e_j.$$

(8) The *hypergroup character table* is the matrix $\chi_i(c_j)$. Its rows are orthogonal with respect to the inner product (2.1) and its columns are orthogonal with respect to the corresponding inner product on $\mathcal{K}^\wedge$. The character table allows an explicit realization of $\mathcal{K}$ as column vectors or of $\mathcal{K}^\wedge$ as row vectors.
(9) If the structure constants of $\mathcal{K}^\wedge$ are given by m_{ij}^k, then the n_{ij}^k and m_{ij}^k can be obtained from the character table by

$$n_{ij}^k \omega(c_k)^{-1} = \frac{1}{\omega(\mathcal{K})} \sum_l \omega(\chi_l) \chi_l(c_i) \chi_l(c_j) \overline{\chi_l(c_k)}$$
$$m_{ij}^k \omega(\chi_k)^{-1} = \frac{1}{\omega(\mathcal{K})} \sum_l \omega(c_l) \chi_i(c_l) \chi_j(c_l) \overline{\chi_k(c_l)}.$$

(10) Define an *observable* of $\mathcal{K}$ to be any element of $\mathcal{A}$ of the form $c_{i_1} c_{i_2} \cdots c_{i_l}$. We say a signed hypergroup has *bounded observables* if all observables are contained in a bounded subset of $\mathcal{A}$. It is not hard to see that this is equivalent to requiring that $|\chi_j(c_i)| \leq 1$ for all i, j. This set is thus closed under duality, and includes all hypergroups, for which the observables all lie in the compact set

$$\text{conv}\ (\mathcal{K}) = \{\sum_i a_i c_i \mid a_i \geq 0 \quad \sum_i a_i = 1\} \subset \mathcal{A}.$$

(11) If a group H acts by automorphisms on a finite abelian group G, then it also acts on the dual group $G^\wedge$. Then we have a canonical isomorphism of hypergroups

$$\mathcal{K}(G; H)^\wedge \simeq \mathcal{K}(G^\wedge; H).$$

§3. Connections with group theory.

We have already seen that to any finite group there are canonically associated two hypergroups, $\mathcal{K}(G)$ and $\mathcal{K}(G^\wedge)$. The fundamental fact here is

$$\mathcal{K}(G)^\wedge \simeq \mathcal{K}(G^\wedge).$$

This means the character theory of the non-abelian group G is completely determined by the character theory of the commutative hypergroup $\mathcal{K}(G)$. So-called 'non-abelian' harmonic analysis is thus not so 'non-abelian' after all.

As the familiar example with groups of order eight shows, $\mathcal{K}(G)$ does not determine G. However it *does* determine G if G is simple. Unfortunately this beautiful fact is only known via the classification.

Problem 1. *Find a structural explanation (that is, independent of the classification) for the fact that* $G \to \mathcal{K}(G)$ *is* $1:1$ *on the set of simple groups.*

Let us illustrate with the simplest example. The symmetric group S_3 has conjugacy classes

$$C_0 = \{e\}, \quad C_1 = \{(12),(23),(31)\} \quad \text{and} \quad C_2 = \{(123),(132)\}$$

with the structure equations of $\mathcal{K}(S_3) = \{c_0, c_1, c_2\}$ easily seen to be

$$\begin{aligned} c_1^2 &= \frac{1}{3}c_0 + \frac{2}{3}c_2 \\ c_1c_2 &= c_1 \\ c_2^2 &= \frac{1}{2}c_0 + \frac{1}{2}c_2. \end{aligned} \tag{3.1}$$

The characters can be determined by simply staring at (3.1). We get the hypergroup character table

	c_0	c_1	c_2
χ_0	1	1	1
χ_1	1	-1	1
χ_2	1	0	$-\frac{1}{2}$

For reference, here is the group character table, obtained by multiplying the j^{th} row of S by $d_j = \sqrt{\omega(\chi_j)}$ (or, if you prefer, by computing traces of representations).

	C_0	C_1	C_2
$d_0\chi_0$	1	1	1
$d_1\chi_1$	1	-1	1
$d_2\chi_2$	2	0	-1

Note the pleasant symmetry of the hypergroup character table. The story is not finished yet however, as it remains to actually write down the structure of $\mathcal{K}(S_3^\wedge)$. This is easily done in this case:

$$\begin{aligned}\chi_1^2 &= \chi_0 \\ \chi_1\chi_2 &= \chi_2 \\ \chi_2^2 &= \frac{1}{4}\chi_0 + \frac{1}{4}\chi_1 + \frac{1}{2}\chi_2.\end{aligned}$$

Now let us mention some results that were motivated by the character theory of finite groups and shed interesting light on particle interactions quite generally. Let $\mathcal{K} = \{c_0, \cdots, c_n\}$ be a hypergroup. $\mathcal{K}$ is called *simple* if it contains no subhypergroups in the obvious sense. For an element $a = \sum_i a_i c_i$ of $\mathrm{conv}(\mathcal{K})$, define

$$\mathrm{supp}(a) = \{c_i | a_i \neq 0\}.$$

Following Arad and Blau [AB], we define the *covering number* $cn(\mathcal{K})$ of $\mathcal{K}$ to be the smallest positive integer m such that $\mathrm{supp}(c_i^m) = \mathcal{K}$ for all $i \neq 0$, if it exists.

Theorem. *(Arad and Blau [AB]) Suppose that $\mathcal{K}$ is simple and not an abelian group. Let $r = \#\{c_i | i \neq 0, c_i^* = c_i\}$. Then $cn(\mathcal{K})$ exists and*

$$cn(\mathcal{K}) \leq \frac{1}{2}((n+1)^2 - (r-1)^2).$$

Note that in the special case when $\mathcal{K}$ is hermitian, the right hand side becomes simply $2n$. In the same direction we mention the following result.

Theorem. *(Arad , Fisman and Blau [AFB]) Assume there exists an m such that $supp(c_i^m) = \mathcal{K}$ for some i. Then*

$$c_i \in supp(\prod_j c_j).$$

Finally here are some problems:

Problem 2. *Describe $\mathcal{K}(S_n)$; that is, determine the structure constants as a function of triples of partitions of n. The structure of $\mathcal{K}(S_n^\wedge)$ is known - this is of course also a function of triples of partitions. Explicit tables for the structure constants of $\mathcal{K}(S_n)$ for low n can be found in James and Kerber [JK].*

Problem 3. *Complete the 'atlas' of finite simple groups by writing down the structural equations of $\mathcal{K}(G)$ and $\mathcal{K}(G^\wedge)$ for each simple group G.*

§4. Connections with random walks and graphs.

We illustrate with an example. What is the probability of a random walk on the edge - vertex graph X of an icosahedron returning to the starting point after n steps? The graph X is illustrated in Fig. 1, where all the outside vertices (marked $\circ$) are to be identified- so there are a total of 12 vertices and 30 edges.

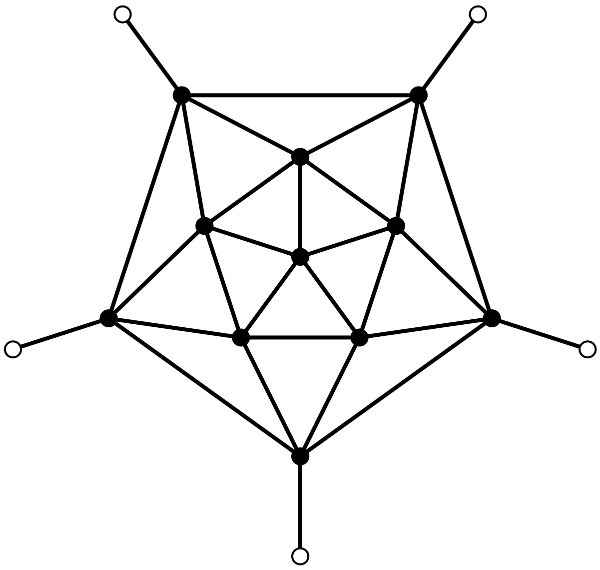

Figure 1

This problem is a standard application of group theoretic ideas, since the automorphism group G of the graph acts distance transitively. This means it commutes with the Laplacian of the graph and so the Laplacian may be diagonalized by decomposing the regular representation of G on $L^2(X)$ into irreducible subspaces. It seems that we need to know some of the representation theory of G.

In fact considerably less work is necessary from a hypergroup-theoretic view point. Let $C_i,\ i = 0, \cdots, 3$ be the sphere of radius i about the central point which we denote by x_0. We may convolve two such spheres in the same way one convolves spheres in $\mathbb{R}^n$. Let y be a random point of distance i from x_0. Choose a random point z of distance j from y. Let n_{ij}^k be the probability that z is of distance k from x_0 - a quantity in this case independent of the choice of base point x_0. This defines a hermitian hypergroup $\mathcal{K}(X) = \{c_0, c_1, c_2, c_3\}$ by

$$c_i c_j \ = \ \sum_k n_{ij}^k c_k.$$

In our particular case the structure equations are easily seen to be

$$\begin{aligned} c_1^2 &= \frac{1}{5}c_0 + \frac{2}{5}c_1 + \frac{2}{5}c_2 & \qquad c_2^2 &= \frac{1}{5}c_0 + \frac{2}{5}c_1 + \frac{2}{5}c_2 \\ c_1 c_2 &= \frac{2}{5}c_1 + \frac{2}{5}c_2 + \frac{1}{5}c_3 & \qquad c_2 c_3 &= c_1 \\ c_1 c_3 &= c_2 & \qquad c_3^2 &= c_0 \end{aligned}$$

The character table is easy to evaluate since $c_3^2 = c_0$ forces $\chi_j(c_3) = \pm 1$ and each of these choices quickly determines a quadratic equation for the values of

$\chi_j(c_i), i = 1, 2$. We get

	c_0	c_1	c_2	c_3
χ_0	1	1	1	1
χ_1	1	$-\frac{1}{5}$	$-\frac{1}{5}$	1
χ_2	1	$-\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	-1
χ_3	1	$\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	-1

Somewhat more work using §2 Fact 9 gives us the structure equations for $\mathcal{K}(X)^\wedge$.

$$\begin{aligned} \chi_1^2 &= \frac{1}{5}\chi_0 + \frac{4}{5}\chi_1 & \chi_2^2 &= \frac{1}{3}\chi_0 + \frac{2}{3}\chi_1 \\ \chi_1\chi_2 &= \frac{2}{5}\chi_2 + \frac{3}{5}\chi_3 & \chi_2\chi_3 &= \chi_1 \\ \chi_1\chi_3 &= \frac{3}{5}\chi_2 + \frac{2}{5}\chi_3 & \chi_3^2 &= \frac{1}{3}\chi_0 + \frac{2}{3}\chi_1. \end{aligned}$$

Note therefore that $\omega(\chi_1) = 5,\ \omega(\chi_2) = \omega(\chi_3) = 3.$

We use these weights and the character table to solve our original problem. The probability p we require is the coefficient of c_0 in c_1^n, which by the orthogonality of the columns of the character table is

$$p = \frac{1}{12}(1 + 5(-\frac{1}{5})^n + 3(-\frac{1}{\sqrt{5}})^n + 3(\frac{1}{\sqrt{5}})^n).$$

A similiar analysis may be made with any of the Platonic solids (see [Wil5]) and in fact the associated hypergroup to another of them has already appeared in the previous section in a different context.

Note that the element c_1 represents the Laplacian of the graph and that the character values $\chi_j(c_1)$ are essentially the eigenvalues of the graph X (see Schwenk and Wilson [SW]). Even if this were our only interest, use of the full set of hypergroup equations simplifies the task of evaluating these eigenvalues.

The hypergroup $\mathcal{K}(X)$ exists for any distance transitive graph X. More generally we say a graph X is *distance regular* if the same procedure of convolving spheres is well defined and produces a hypergroup $\mathcal{K}(X)$ independent of base point x_0. Not all distance regular graphs are distance transitive. Distance regular graphs exist in profusion and arise in many combinatorial situations.

When the diameter of X is greater than two, there has been considerable progress toward a classification. When the diameter is two, the situation is also very interesting and leads to the theory of *strongly regular* graphs (see Hubaut [Hu], Cameron [C]).

A graph X is strongly regular if there exist integers k, λ, μ such that

i) X is regular of valency k

ii) given any 2 distinct vertices x and y, the number of vertices adjacent to both x and y is λ if x and y are adjacent, and μ otherwise.

Such a graph is distance regular (it need not be distance transitive however) with the corresponding hypergroup $\mathcal{K}(X)$ consisting of exactly three elements $\mathcal{K}(X) = \{c_0, c_1, c_2\}$. It is not hard to see that the structure equations are

$$\begin{aligned} c_1^2 &= \frac{1}{k}c_0 + \frac{\lambda}{k}c_1 + \frac{k-\lambda-1}{k}c_2 \\ c_1c_2 &= \frac{\mu}{k}c_1 + \frac{k-\mu}{k}c_2 \\ c_2^2 &= \frac{1}{\ell}c_0 + \frac{k-\mu}{\ell}c_1 + \frac{\ell+\mu-k-1}{\ell}c_2 \end{aligned}$$

where $\ell = p - k - 1$.

The simplest nontrivial strongly regular graph is the pentagon for which the corresponding hypergroup is

$$\begin{aligned} c_1^2 &= \frac{1}{2}c_0 + \frac{1}{2}c_2 \\ c_1c_2 &= \frac{1}{2}c_1 + \frac{1}{2}c_2 \\ c_2^2 &= \frac{1}{2}c_0 + \frac{1}{2}c_1 \end{aligned}$$

This simple but beautiful hypergroup deserves the name '*Golden hypergroup*' – its unique position in the family of all order three hypergroups is described in [Wil7].

Not all parameters k, λ, μ correspond to strongly regular graphs. The two most important necessary conditions are known in the theory as the Integrality Conditions and the Krein Conditions (see Scott [Sc].) It is here that the hypergroup point of view shows its usefulness – the Integrality Conditions turn out to be just that the weights of the dual signed hypergroup $\mathcal{K}(X)^\wedge$ are integral, and the Krein Condition is simply that the dual $\mathcal{K}(X)^\wedge$ is actually a hypergroup.

Let us now proceed to a less trivial example involving 2-planes in F_q^n (or lines in $PG(n-1,q)$.) Declare two planes to be adjacent if they intersect nontrivially. One may then verify that the resulting graph X is strongly regular. The resulting hypergroup we denote by $\mathcal{K}(X) = \{c_o, c_1, c_2\}$. To be specific, consider the case $q = 5$ and $n = 3$ and set $C_i = \omega(c_i)c_i$. Then we get the following generalized hypergroup

$$\begin{aligned} C_1^2 &= 180C_0 + 54C_1 + 36C_2 \\ C_1C_2 &= 125C_1 + 144C_2 \\ C_2^2 &= 625C_0 + 500C_1 + 480C_2 \end{aligned}$$

The dual hypergroup $\mathcal{K}(X)^\wedge = \{\chi_0, \chi_1, \chi_2\}$ may be renormalized by setting $X_j = \omega(\chi_j)\chi_j$ to obtain

$$\begin{aligned} X_1^2 &= 155X_0 + \frac{124}{3}X_1 + \frac{403}{15}X_2 \\ X_1X_2 &= \frac{338}{3}X_1 + \frac{1922}{15}X_2 \\ X_2^2 &= 650X_0 + \frac{1612}{3}X_1 + \frac{7813}{15}X_2 \end{aligned}$$

The corresponding hypergroup character table is

	c_0	c_1	c_2
χ_0	1	1	1
χ_1	1	$\frac{2}{15}$	$-\frac{1}{25}$
χ_2	1	$-\frac{1}{30}$	$\frac{1}{125}$

Note the rather remarkable economy of this table.

The hypergroups $\mathcal{K}(X)$ associated to a distance regular graph X have an important property. Clearly the element c_1, the sphere of radius 1, plays a distinguished role - partly because of its connection with the Laplacian but also purely algebraically since the element c_1c_i can only contain terms involving c_{i-1}, c_i or c_{i+1}. This implies more generally that a product c_ic_j contains only terms c_k with

$$|i-j| \leq k \leq i+j.$$

It further means that associated to the hypergroup there is a family of orthogonal polynomials on a finite set in $\mathbb{R}$ with respect to some measure.

Conversely, given a measure μ on $\mathbb{R}$ supported on a finite set $Y \subset \mathbb{R}$ with $|Y| = n+1$, we may construct in the classical way using Gram Schmidt orthogonalization a finite set of orthogonal polynomials say $\{P_0, P_1, \cdots, P_n\}$ (see Szego [Sz]). Since we may ensure that the degree of P_i is i, the span of the P_i will be closed under multiplication on the set Y. This means that the set automatically forms a generalized hypergroup which is clearly real. We may make it normalized if all of the polynomials are nonzero at some $y \in Y$. In this case we have obtained a hermitian signed hypergroup. This construction is quite general.

A natural question to ask at this point is the following.

Problem 4. *What conditions on μ will ensure that the resulting signed hypergroup is in fact a hypergroup?*

For the (infinite) case of Jacobi polynomials, the exact range of positivity and the dual structure has been determined by Gasper [Ga1], [Ga2].

Let us remark however that the harmonic analysis of §2 holds whether or not positivity is present.

§5. Connections with Cyclotomy.

The classical theory of cyclotomy concerns itself with generalizations of quadratic residues and nonresidues and has been an important subject in number theory since Gauss. The connection with hypergroup theory has been described in [Wil3], [Wil4], which we follow here.

Recall that if p is a prime, a quadratic residue is simply a nonzero square in $\mathbb{Z}_p = \{0, \ldots, p-1\}$ i.e. an element of the form $x = y^2$. The nonzero elements are divided equally into residues and nonresidues, and Gauss realized that it was interesting to compute the probabilities that two residues sum to a residue, or to a nonresidue etc. In other words, there is an order three hypergroup here whose coefficients are normalizations of the classical cyclotomic constants found by Gauss.

More generally if $p-1 = kn$, we may divide $\mathbb{Z}_p/\{0\}$ into n classes, one of which consists of the n^{th} powers. That a hypergroup arises as a result can be seen by noting that the automorphism group of $\mathbb{Z}_p$ is cyclic of order $p-1$, so it has a unique subgroup H_n of index n. The orbits of H_n on $\mathbb{Z}_p$ consist of $C_0 = \{0\}$ together with classes $C_1, \ldots, C_n$, where C_1 consists of all the nonzero n^{th} powers of $\mathbb{Z}_p$ and where we may choose a labelling such that $\gamma(C_i) = C_{\sigma(i)}$ for some generator $\gamma \in \mathrm{Aut}(\mathbb{Z}_p)$ and where σ is the permutation of $\{0, \cdots, n\}$ which fixes 0 and rotates $1, \cdots, n$ cyclically.

The corresponding hypergroup $\mathcal{K} = \mathcal{K}(\mathbb{Z}_p; H_n) = \mathcal{K}(p, n)$ is then as described in §1 and will furthermore exhibit an important cyclic symmetry in its structure constants; namely

$$n_{\sigma(i)\sigma(j)}^{\sigma(k)} = n_{ij}^{k}.$$

As an example consider $p = 13$ and $n = 3$. Then

$$C_1 = \{1, 5, 8, 12\} \quad C_2 = \{2, 3, 10, 11\} \quad C_3 = \{4, 6, 7, 9\}$$

and the associated generalized hypergroup $\widetilde{\mathcal{K}} = \{C_0, C_1, C_2, C_3\}$ is determined by the equations

$$\begin{aligned} C_1^2 &= 4C_0 + C_2 + 2C_3 \\ C_1 C_2 &= C_1 + 2C_2 + C_3 \end{aligned}$$

together with the cyclic symmetry noted above.

The structure constants are closely connected to the numbers of solutions of Diophantine equations of 'Fermat' type. For example the fact that the coefficient of C_1 in C_1^2 is 0 is simply a statement of the fact that the equation $x^3 + y^3 = z^3$ has no nontrivial solutions in $\mathbb{Z}_{13}$ - where by nontrivial we mean that none of the variables are allowed to be zero (warning: this is *not* standard usage of this term!)

Along the same lines, consider trying to find the number of nontrivial solutions to the equation

$$x^3 + 4y^3 + 6z^3 = 3w^3$$

in $\mathbb{Z}_{13}$. Since $4, 6 \in C_3$ and $3 \in C_2$, it is not hard to see that the required number is nothing other than 4 times the coefficient of C_2 in the product $C_1 C_3^2$. Now from the structure equations we may compute

$$C_1 C_3^2 = C_1(4C_0 + C_1 + 2C_2) = 4C_0 + 6C_1 + 5C_2 + 4C_3$$

and so there are 20 nontrivial solutions.

Our main result is a rigidity theorem.

Theorem. *(Wildberger [Wil4]) Let p be a prime and suppose that $n|(p-1)$. Let $\mathcal{K} = \{c_0, \ldots, c_n\}$ be a hypergroup of total weight p that satisfies the cyclic symmetry condition and has the property that the renormalized generalized hypergroup given by $C_i = \omega(c_i)c_i$ has integral structure constants. Then $\mathcal{K}$ is isomorphic to $\mathcal{K}(p,n)$.*

Problem 5. *What happens when p is not a prime?*

It is known that the Theorem no longer holds in this case (see the counterexample in [Wil4].)

§6. Association schemes

Almost all of the previous examples, with the notable exception of the hypergroups $\mathcal{K}(G^\wedge)$, are examples of combinatorial structures known as *association schemes*. These have been much studied by combinatorialists, coding theorists and group theorists. We mention Bose and Mesner [BM], Delsarte [De], Wielandt [Wie] and the books of Bannai and Ito [BI] and Brouwer, Cohen and Neumaier [BCN] for an introduction to this subject. Here we must make do with only the barest of definitions and examples.

An *association scheme* with n classes is a collection of $n+1$ subsets R_i, $i = 0, \cdots, n$ of $X \times X$ for a finite set X such that

(1) $\{R_0, R_1, \cdots, R_n\}$ is a partition of $X \times X$.
(2) $R_0 = \{(x,x)|x \in X\}$.
(3) for each i there is a j such that $R_i^T = R_j$.
(4) there are numbers N_{ij}^k (the *intersection numbers* of the scheme) such that for any pair $(x,y) \in R_k$ the number of $z \in X$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ equals N_{ij}^k.

The number w_i of $z \in X$ with $(x,z) \in R_i$ (which is independent of $x \in X$) is called the *valency* of R_i. One has

$$\sum_i w_i = |X|.$$

Now define for each i a matrix A_i by $(A_i)_{xy} = 1$ if $(x,y) \in R_i$ and 0 otherwise. Then

$$\sum_i A_i = J$$

where J is the matrix of all 1's and

$$A_i A_j = \sum_k N_{ij}^k A_k.$$

Thus the collection of matrices $A_0, \cdots, A_n$ forms a positive generalized hypergroup with integral coefficients and if we renormalize by dividing A_i by w_i , we get a hypergroup. We may thus think of an association scheme as a positive generalized hypergroup that may be realized by $0,1$ matrices.

The algebra spanned by the A_i is called the *Bose Mesner* algebra of the scheme. A pleasant fact in this situation is that the dual (signed) hypergroup may be realized by introducing the Hadamard or Schur (or pointwise) product of matrices. Unfortunately there is no reason to suppose that the dual signed hypergroup of an association scheme is also an association scheme- and in fact it generally is not.

A main example of an association scheme can arise with a transitive action of a group G on a set X. Fix a point $x_o \in X$ and let H be the stabilizer of x_0 in G. Then $X \cong G/H$. The *Hecke algebra* is the subalgebra of $\mathbb{C}G$ generated by the probability measures on the H double cosets. It is also naturally identified with the space of functions on X which are invariant under H, and is commutative if $Ind_H^G 1$ is multiplicity free (in the Lie group case such a situation is referred to as a *Gelfand pair*). The characters of the Hecke algebra are then known as the *spherical functions*. In such a situation the orbits of G on $X \times X$ form an association scheme, and the spherical functions can equally be viewed as the characters of the corresponding hypergroup. Group theorists refer to the Bose Mesner algebra as the *centralizer* algebra of the group action.

If X carries a G-invariant adjacency structure, then X is a distance transitive graph and the Hecke algebra is automatically commutative. Higman [H] has studied when a finite homogeneous space can be given such a structure. See also Biggs [Bi] for a discussion of distance transitive graphs, and Dunkl [Du2] for a discussion of classical orthogonal functions which arise as spherical functions.

As an example, let W be a finite set and X be the collection of all order n subsets of W, where we suppose that $|W| = m \geq n$. Define $(x, y) \in R_i$ if and only if $x \cap y$ has cardinality $n - i$, for $i = 0, 1, \cdots, n$. This is called the *Johnson* scheme and is denoted by $J(m, n)$. The special cases $J(m, 2)$ are known as the *triangular* graphs.

There are many other interesting and wonderful examples connected with coding theory, group theory and other combinatorial structures- see the references above.

§7. Connection with Information Theory.

The probabilistic nature of a hypergroup makes it reasonable that information theoretic ideas may play a role in the theory. We show here briefly that they do, and state a Second Law of Thermodynamics for particle collisions in this context.

For a particle constrained to $[0, 1]$, we may say the information in knowing it is in some particular subinterval of length m is $-\log m$ (we could use base 2 here but prefer natural logs). What is then the information $I(m_1, \ldots, m_\ell;\ p_1, \ldots, p_\ell)$ of knowing the particle is in one of ℓ disjoint intervals of lengths $m_1, \ldots, m_\ell$ with probabilities $p_1, \ldots, p_\ell$? Subject to some reasonable hypotheses on I, we may obtain

$$I(m_1, \ldots, m_\ell; p_1, \ldots, p_\ell) \;=\; -\log \sum_{i=1}^{\ell} \frac{p_i^2}{m}.$$

This motivates us to define the entropy of an element $a = \sum_i a_i c_i \;\in\; \mathrm{conv}(\mathcal{K})$ by

$$E(a) \;=\; -\log \sum_i \frac{|a_i|^2}{\omega(c_i)}.$$

(In fact this definition makes sense for all $a \in \mathcal{A}$.) As a main result, we have

Theorem. *(Wildberger [Wil1]) Let $a, b \in$ conv$(\mathcal{K})$ for a hypergroup $\mathcal{K}$. Then*

$$\min(E(a), E(b)) \;\leq\; E(ab) \;\leq\; E(a) + E(b).$$

Furthermore the maximum possible value of $E(a)$, $a \in$ conv$(\mathcal{K})$, is $\log \omega(\mathcal{K})$, obtained when $a = e_0$.

As an application, we may unravel this theorem for the case $\mathcal{K} = \mathcal{K}(G^\wedge)$ to find

Corollary. *Let G be a finite group and*

$$\rho_i \otimes \rho_j \;=\; \sum_k M_{ij}^k \rho_k$$

the decomposition of the tensor product of two irreducible representations ρ_i, ρ_j. Then

$$\sum_k (M_{ij}^k)^2 \;\leq\; \min(d_i^2, d_j^2)$$

where $d_i = \dim(\rho_i)$.

Note: this result appears to be new, but can also be established by use of the classical orthogonality relations. In particular it implies that the tensor product of a two dimensional representation with an arbitrary irreducible is either multiplicity free with at most four constituents, or is twice an irreducible.

Finally we should mention that notions from the theory of fuzzy systems could be potentially useful to hypergroup theory and vice versa. This might be an interesting connection to pursue.

§8. Hypergroups arising from inclusions of II_1 factors.

Most of the hypergroups discussed so far have arisen from the subdivision of an underlying set into pieces that allow a convolution - thus the general theory of association schemes. Not all hypergroups that occur in applications are of this type however. One example has already been mentioned - $\mathcal{K}(G^\wedge)$. In this section and the next we deal with others.

One of the basic results of Jones [Jo] was to associate to a pair of II_1 factors $M \subset N$ an index and to limit the possible set of values which such an index might assume to the set

$$\{4\cos^2(\frac{\pi}{N})\}_{N=3,4,\cdots} \cup [4, \infty).$$

Ocneanu [O] showed that by considering families of bimodules for M and N and their structure under tensor products one could associate to the inclusion $M \subset N$ a finer invariant than the index - an algebraic object which he called a *paragroup*- a certain type of generalized hypergroup in which the structure constants satisfy an integrality condition and come equipped with additional data called a connection. It turns out that there often exists a special element in the hypergroup whose multiplication matrix is the adjacency matrix of a graph. The graphs that arise in this way are intimately related to the Bratteli diagram associated to the inclusion and under reasonable conditions are constrained in a very strong fashion by their possible norms.

As a result, the graphs that appear in this theory are closely related to the classical Dynkin diagrams or Coxeter graphs and their affine generalizations. Recently Sunder and Vijayarajan have introduced the notion of an M_2 graded hypergroup and shown how to use it to further clarify the possible graphs that can arise (see [Vi] and the article by Sunder in this volume).

It turns out that the hypergroups studied here are very close to objects arising in conformal field theory which physicists call *fusion rule algebras.*

§9. Connections with fusion rule algebras and conformal field theory.

By a (finite dimensional) *fusion rule algebra* we mean a generalized hypergroup $\widetilde{\mathcal{K}} = \{C_0, C_1, \ldots, C_n\}$ with structure constants N_{ij}^k that satisfy the additional axioms

(A4) N_{ij}^k is a nonnegative integer for all i, j, k

(A5) $N_{ij}^0 = 1$ if $C_j = C_i^*$.

An example is the set of irreducible representations $\{\rho_0, \ldots, \rho_n\}$ of a finite group G as in §3. Recall that there we renormalized by use of the positive character $\rho_i \to d_i$. That this works generally is contained in the following (see Sunder [Su], Fröhlich and Kerler [FK])

Theorem. *Let $\widetilde{\mathcal{K}}$ be a fusion rule algebra as above. Then there exists a unique function $d : \widetilde{\mathcal{K}} \to \mathbb{R}^+$ such that $d(C_i) = d_i$ satisfies*

$$d_i d_j = \sum_k N_{ij}^k d_k.$$

Furthermore the function d takes values in the set

$$\{2\cos\frac{\pi}{N}\}_{N=3,4,\ldots} \cup [2, \infty).$$

Note the relationship between this set and the set of possible values of the Jones index described in the previous section.

By replacing C_i with $d_i c_i$ we obtain a hypergroup $\mathcal{K} = \{c_0, c_1, \ldots, c_n\}$ in which $\omega(c_i) = d_i^2$ (recall this happened already with $\mathcal{K}(G^\wedge)$) and furthermore $\widetilde{\mathcal{K}}$ can be recovered from $\mathcal{K}$.

The simplest class of examples arising in conformal field theory are related to the A_n Dynkin diagrams. This fusion rule algebra is $\widetilde{\mathcal{K}} = \{C_0, \ldots, C_{n-1}\}$ with structure constants

$$N_{ij}^k = \begin{cases} 1 & \text{if} \quad |i-j| \le k \le \min(i+j, 2(n-1)-(i+j)) \\ & \qquad\qquad \text{and } k \equiv i+j \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le i, j, k \le n-1$. The operation of multiplication by C_1 has matrix the adjacency matrix of A_n (with nodes 0 to $n-1$.)

The corresponding function d is given by

$$d_i = \sin\frac{(i+1)\pi}{n+1} \Big/ \sin\frac{\pi}{n+1}.$$

The corresponding hypergroup character table has entries

$$\chi_i(c_j) = \sin\frac{(i+1)(j+1)\pi}{n+1} \sin\frac{\pi}{n+1} \Big/ \sin\frac{(i+1)\pi}{n+1} \sin\frac{(j+1)\pi}{n+1}$$

for $0 \le i, j \le n-1$.

This shows that the associated hypergroup $\mathcal{K}$ is isomorphic to its dual in this case. It also reveals the usefulness of the hypergroup character table. This particular fusion rule algebra is related to the Lie algebra $sl(2)$. More general examples are related to other semisimple Lie algebras, and are considerably less well understood.

We may try to understand fusion rules mathematically in a variety of ways.

i) Affine Lie algebras

If $\mathfrak{g}$ is a semisimple (finite dimensional) Lie algebra with Killing form $<,>$ we may consider the affine Lie algebra of $\mathfrak{g}$

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\,[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where $\mathbb{C}\,[t, t^{-1}]$ is the algebra of Laurent polynomials in an indeterminate t, c is a nonzero central element of $\tilde{\mathfrak{g}}$, and $\tilde{\mathfrak{g}}$ has Lie brackets given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y\rangle m \delta_{m+n,0} c$$

$$[d, x \otimes t^m] = mx \otimes t^m$$

In analogy with the finite dimensional theory, one may define Cartan subalgebras, root systems, positive roots and weights and a Weyl group $\tilde{W}$ (essentially the affine Weyl group associated to the Weyl group W of $\mathfrak{g}$) (see for example Kac [Kac]). Of particular interest are the so called integrable highest weight modules of $\tilde{\mathfrak{g}}$.

A highest weight $\tilde{\lambda}$ of $\tilde{\mathfrak{g}}$ is determined by a highest weight λ of $\mathfrak{g}$ and a real number k called the *level* , which is the scalar by which the central element c acts. Integrality ensures that $k \geq 0$ must be a nonnegative integer and that λ must lie in a generalized 'alcove' associated to a subgroup of $\tilde{W}$ determined by k. In particular there are only a finite number of such modules of a given level.

It is this set of modules, associated to a given level k, which forms a fusion rule algebra under a modified tensor product (the ordinary tensor product would be additive on the levels).

From a physical point of view, highest weight modules may be realized as 'operator-valued fields' on $\mathbb{C}$ which transform in a specified way under the conformal group (or infinitesimally, under the Virasoro algebra V). This is due to the fact that such a module for $\tilde{\mathfrak{g}}$ can be extended to a semi-direct product of $\tilde{\mathfrak{g}}$ and V (this is known as the Sugawara construction- see for example Fuchs [Fu] , Kaku [Kak]). These fields, called primary, may be 'multiplied' together by a suitable operator product calculus, and the result is a combination of primary fields with some other 'secondary' fields. When these secondary fields are ignored, one obtains the fusion rules amongst the primary fields. When applied to $\mathfrak{g} = sl(2)$ and $k = n$, we obtain the example above.

ii) Quantum groups

For $\mathfrak{g}$ a finite dimensional semisimple Lie algebra, the universal enveloping algebra $U(\mathfrak{g})$ may be q-deformed to obtain $U_q(\mathfrak{g})$, the *quantum universal enveloping algebra* (Drinfel'd [Dr], Jimbo [Ji]). For generic q, the finite dimensional representation theories of $\mathfrak{g}$ and $U_q(\mathfrak{g})$ are isomorphic; namely all finite dimensional modules of $U_q(\mathfrak{g})$ are fully reducible, the irreducible finite dimensional modules of $U_q(\mathfrak{g})$ are parametrized by the dominant integral weights of $\mathfrak{g}$, and the tensor product of modules decomposes into irreducibles with the same multiplicities as do the corresponding modules of $\mathfrak{g}$ (see Rosso [Ro]).

For applications to physics however, one is interested in the *non-generic* situation when q is a root of unity. In this case, $U_q(\mathfrak{g})$ is no longer semisimple, and its representation theory becomes suddenly more complicated. The situation is simplified by passing somehow to a 'semisimple quotient'- in practice one way of doing this is by introducing the notion of the *quantum dimension* of a module. The character of an irreducible module is given by a q-version of the Weyl character formula and it makes sense to evaluate it at a certain point in the Cartan subgroup to obtain the quantum dimension

$$D_\lambda = \chi_\lambda(\ln qH_\rho)$$

for a certain distinguished element H_ρ.

Now by considering only those modules such that $D_\lambda > 0$, we obtain a finite collection of irreducibles. When we form the tensor product of two such modules and only count constituents appearing from this same set, we get a fusion rule algebra. This operation is known as the *truncated* Kronecker product. Furthermore if the root of unity is related in a certain way to the level k discussed previously, the fusion rule algebra obtained from this quantum group approach coincides with the earlier one obtained from the affine Lie algebra and the level k. In the case of $U_q(sl(2))$, the required relation is

$$q = e^{\frac{\pi i}{k+2}}.$$

iii) Superselection sectors

From a C^*-algebra point of view, fusion rule algebras arise very briefly as follows (for more detail see Doplicher, Haag and Roberts [DHR1], [DHR2], Longo [Lo]). Let G be the gauge group of a quantum field theory on space time and $\mathcal{H}$ the Hilbert space of physical states. The C^*-algebra of observables $\mathcal{A}$ acting on $\mathcal{H}$ decomposes $\mathcal{H}$ into a direct sum of subspaces called *superselection sectors*, carrying inequivalent representations of $\mathcal{A}$. There is then a 'composition' of sectors which is akin to the tensor product of group representations and with this composition and a 'charge conjugation', the set of sectors becomes a fusion rule algebra.

iv) Representations of Lie groups

Finally we mention another way of looking at fusion rule algebras. Consider the fusion rule algebra associated with $sl(2)$ and the Dynkin diagram A_n described above. Recall that the irreducible representations of $SU(2)$ may be listed as $\{\rho_0, \rho_1, \cdots\}$ with

$$\rho_i\rho_j = \rho_{|i-j|} + \rho_{|i-j|+2} + \cdots + \rho_{i+j}.$$

At first sight it might appear that the A_n fusion rule algebra is obtained from this infinite one by a suitable truncation, but closer examination reveals the connection to be more of a reflection - cancellation.

More precisely, there is an ideal $I_n \subset \mathcal{K}(SU(2)^\wedge)$ which is associated to a subgroup of the affine Weyl group such that the quotient of $\mathcal{K}(SU(2)^\wedge)$ by I_n is the fusion rule algebra.

There is in fact an analogous construction that works for general semisimple $\mathfrak{g}$ and any level k and provides a mathematically straightforward construction of the fusion rules described above. We hope to describe this elsewhere.

Problem 6. *Clarify the relationships between these various approaches to fusion rule algebras.*

References

[AB] Z. Arad and H. Blau, *On table algebras and applications to finite group theory*, J. of Algebra **138** (1991), 137–185.

[AFB] Z. Arad, E. Fisman and H. Blau, *Table algebras and applications to products of characters in finite groups*, J. of Algebra **138** (1991), 186–194.

[BI] E. Bannai and T. Ito, *Algebraic Combinatorics I - Association Schemes*, Benjamin and Cummings, Menlo Park, 1984.

[BK] Y. M. Berezansky and A. A. Kalyushnyi, *Hypercomplex systems with locally compact bases*, Sel.Math.Sov. **4** (1985), 151–200.

[Bi] N. Biggs, *Algebraic Graph Theory*, Cambridge Tracts in Mathematics No. 67, Cambridge University Press, Cambridge, 1974.

[BM] R. C. Bose and D. M. Mesner, *On linear associative algebras corresponding to association schemes of partially balanced designs*, Ann. Math. Statist. **30** (1959), 21–38.

[Bo] R. C. Bose and K. R. Nair, *Partially balanced incomplete block designs*, Sankhya **4** (1939), 337–372.

[Br] R. Brauer, *On pseudo groups*, J. Math. Soc. Japan **20** (1968), 13–22.

[BCN] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete no 3. Folge . Band 18, Springer- Verlag, Berlin, 1989.

[C] P. J. Cameron, *Strongly regular graphs*, Selected Topics in Graph Theory, Academic Press, New York, 1978, pp. 377–360.

[De] P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Research Reports Suppl. **10** (1973).

[DHR1] S. Doplicher, R. Haag and J. E. Roberts, *Local observables and particle statistics I*, Commun. Math. Phys. **23** (1971), 199–230.

[DHR2] S. Doplicher, R. Haag and J. E. Roberts, *Local observables and particle statistics II*, Commun. Math. Phys. **35** (1974), 49–85.

[Dr] V. G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. (1985), no. 32(1).

[Du1] C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Math. Soc. **179** (1973), 331–348.

[Du2] C. F. Dunkl, *Orthogonal functions on some permutations groups*, Proceedings of Symposia in Pure Mathematics 34, American Math Society, Providence R.I., 1979.

[Fe] R. P. Feynmann, *Negative Probability*, Quantum implications, Editors B. J. Hiley and F. D. Peat, Routledge & Kegan Paul Ltd, London, 1987.

[Fr] G. Frobenius, *Über Gruppencharaktere*, Gesammelte Abhandlungen, Vol.III, Springer-Verlag, pp. 1–37.

[FK] J. Fröhlich and T. Kerler, *Quantum groups, Quantum Categories and Quantum Field Theory*, LNM 1542, Springer-Verlag, Berlin, 1993.

[Fu] J. Fuchs, *Affine Lie algebras and Quantum Groups*, Cambridge University Press, Cambridge, 1992.

[Ga1] G. Gasper, *Linearization of the product of Jacobi polynomials, II*, Can. J. Math. **22** (1970), 582–593.

[Ga2] G. Gasper, *Positivity and the convolution structure for Jacobi series*, Ann. of Math. **93** (1971), 112–118.

[Hi] D. G. Higman, *Intersection matrices for finite permutation groups*, J. of Algebra **6** (1967), 22–42.

[Hu] X. L. Hubaut, *Strongly regular graphs*, Discrete Math. **13** (1975), 357–381.

[JK] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications Vol. 16, Addison- Wesley, Reading, MA, 1981.

[Je] R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. Math. **18** (1975), 1–101.

[Ji] M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang- Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.

[Jo] V. F. R. Jones, *Index of subfactors*, Invent. Math. **72** (1983), 1–25.

[Kac] V. G. Kac, *Infinite-dimensional Lie algebras*, 2nd ed., Cambridge University Press, Cambridge, 1990.

[Kaw] Y. Kawada, *Über den Dualitatssatz der Charaktere nichtcommutative Gruppen*, Proc. Phys. Math. Soc. Japan (3) **24** (1942), 97–109.

[Kak] M. Kaku, *Strings, Conformal fields and Topology*, GTCP, Springer-Verlag, New York, 1991.

[Le] B. M. Levitan, *Generalized Translation Operators and Some of their Applications*, Israel Program for Scientific Translations, Jerusalem, 1962.

[Li] G. L. Litvinov, *Hypergroups and hypergroup algebras*, J.Soviet.Math. **38** (1987), 1734–1761.

[Lo] R. Longo, *Index of subfactors and statistics of quantum fields I*, Commun. Math. Phys. **126** (1989), 217–247.

[M] J. R. McMullen, *An algebraic theory of hypergroups*, Bull.Austral.Math.Soc. **20** (1979), 35–55.

[MP] J. R. McMullen and J. F. Price, *Duality for finite abelian hypergroups over splitting fields*, Bull. Austral. Math. Soc. **20** (1979), 57–70.

[MS] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Commun. Math. Phys. **123** (1989), 177–254.

[O] A. Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*, Operator algebras and applications, vol. 136, London Math. Soc. Lecture Note Series, 1988, pp. 119–172.

[R] K. A. Ross, *Hypergroups and centers of measure algebras*, Symposia Math. **22** (1977), 189–203.

[Ro] M. Rosso, *Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra*, Commun. Math.Phys. **117** (1988), 581–593.

[SW] A. J. Schwenk and R. J. Wilson, *Eigenvalues of graphs*, Selected topics in Graph Theory, Academic Press, New York, 1978.

[Sc] L. L. Scott, *Some properties of character products*, J. of Algebra **45**, 259–265.

[Sh] W. T. Sharp, *Racah Algebra and the contraction of groups*, Mathematics Dept., University of Toronto, Toronto, 1984.

[Sp] R. Spector, *Mesures invariantes sur les hypergroupes*, Trans. Amer. Math. Soc. **239** (1978), 147–165.

[Su] V. S. Sunder, II_1 *factors, their bimodules and hypergroups*, Trans. Amer. Math. Soc. (1992), no. 330(1), 227–256.

[Sz] G. Szego, *Orthogonal Polynomials*, Am.Math.Soc.Coll.Publ. 23, Providence, R.I., 1959.

[Va] L. I. Vainerman, *Harmonic analysis on hypercomplex systems with a compact and discete basis*, Selecta Math. Sovietica (1991), no. 10(2).

[Ve] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B **300** (1988), 360–376.

[Vi] A. K. Vijayarajan, *On M_2-Graded Hypergroups*, Proc. Indian Acad. Sci. **105** (1993), 167–180.

[Wie] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.

[Wil1] N. J. Wildberger, *Duality and entropy for finite abelian hypergroups*, Preprint, Univ. of NSW (1989).

[Wil2] N. J. Wildberger, *Hypergroups and Harmonic Analysis*, Centre Math Anal. (ANU), vol. 29, 1992, pp. 238–253.

[Wil3] N. J. Wildberger, *Hypergroups and Cyclotomy*, Preprint, Univ. of NSW (1990).

[Wil4] N. J. Wildberger, *On the algebraic structure of Gaussian periods*, Preprint, Univ. of NSW (1991).

[Wil5] N. J. Wildberger, *Hypergroups associated to random walks on Platonic solids*, Preprint, Univ. of NSW (1994).

[Wil6] N. J. Wildberger, *Lagrange's theorem and integrality for hypergroups*, Preprint, Univ. of NSW (1993).

[Wil7] N. J. Wildberger, *Hypergroups of order three*, Preprint, Univ. of NSW (1994).

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