Quarks, diamonds, and representations of $\mathfrak{sl}_3$.

N. J. Wildberger
School of Mathematics UNSW
Sydney 2052 Australia
email:n.wildberger@unsw.edu.au

October 15, 2003

Abstract

A new model for the irreducible representations of $\mathfrak{sl}_3$ is presented which is constructed over the integers. This model utilizes the combinatorial geometry of certain polytopes in three dimensional space which we call *diamonds*. These are not Gelfand-Tsetlin polytopes, but share some of their properties. Matrix coefficients are directly computable in terms of maximal ladders of edges of given directions and type in the diamonds. We show that the generic diamond is the vector sum of dilates of the fundamental diamonds associated to quark and anti-quark triples, and is simultaneously both a classical and quantum object.

1 Introduction

The representation theory of the two smallest simple Lie algebras, $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, plays an important role in the Standard Model of particle physics. So while a mathematician tries to understand these two examples because of the light they necessarily shed on the more general theory of simple finite dimensional (or even Kac-Moody) Lie algebras, a physicist tries to understand them in order to gain a deeper insight into the behavior of quarks, gluons, and their composite particles (see for example [St].)

Physicists like to determine bases of representation spaces, and find models for raising and lowering operators that allow explicit calculation of matrix coefficients—which often correspond to expectation values. With the increasing interest in *combinatorial representation theory,* mathematicians have also come around to this point of view, and try to construct models in which character and tensor product formulae, as well as matrix coefficients, are related to explicit combinatorial data (for a general introduction to the subject see [BR].)

The main forerunner here is the celebrated Gelfand-Tsetlin construction [GT] in 1950 of the irreducible representations of $\mathfrak{sl}_n$ and its generalizations. For any irreducible representation of $\mathfrak{sl}_n$ we may describe an explicit basis consisting of Gelfand-Tsetlin patterns, or equivalently semistandard Young tableau, and explicit matrix coefficients for all the raising and lowering operators in this basis. These matrix coefficients are in general rational numbers, and are computed as ratios of products of integers determined from the tableau.

In the past decade or so various new and remarkable bases have appeared, such as Lusztig's canonical bases [Lu], Kashiwara's crystal basis [Ka] and Littelmann's path model [Li]. These are often closely connected, as are the Gelfand-Tsetlin patterns, with convex polytopes, as in the works [BZ], [BGR],[KTT],[Ki], [KT] and [NZ].

In this paper we describe another model, valid as yet only for $\mathfrak{sl}_3$, that has the advantage over Gelfand-Tsetlin that all matrix coefficients are *integers* which can be easily computed, even by hand for large representations. It allows us to work over any field. In upcoming work with Arnal and Bel-Baraka, we will show that it also allows an extension to the quantum group $U_q(\mathfrak{sl}_3)$. Perhaps this new approach may also find some application to our understanding of particle theory.

The model also is intimately related to the combinatorial geometry of certain interesting polytopes in three dimensional space, which we call *diamonds*. There is some reason for this particular choice of terminology, since most cut diamonds have a distinguished plane, called the *girdle,* which also plays an important role in our theory.

For any positive integers a and b there is a diamond $D(a,b)$ whose 'integral nodes' $D_I(a,b)$ index weight vectors for the representation $\pi_{[a,b]}$ of $\mathfrak{sl}_3$. If we project these nodes onto the $x-y$ plane we recover the weight multiplicity diagram. All the integer matrix entries for both raising and lowering operators have a direct geometric interpretation in terms of numbers of paths in a given direction between nodes in the diamond. The integral diamond $D_I(a,b)$ may be conveniently represented by a two dimensional diagram which records the height (in the z direction) above a given weight in the $x-y$ plane.

So the philosophy being promoted here is that a weight diagram in the Cartan subalgebra of a simple Lie algebra may be understood as a projection of a diamond in some higher dimensional space.

The two simplest diamonds $D(1,0)$ and $D(0,1)$ are both triangles, which we call the **quark triple** and the **anti-quark triple** respectively. Their diagrams are respectively

	1	and	-1	-1
0	1		0	

We show that the general diamond is obtained as a vector sum

$$D(a,b) = D(a,0) + D(0,b)$$

of multiples of the quark and anti-quark triples. This corresponds to the fact that $\pi_{[a,b]}$ is a constituent of the tensor product of suitable copies of the two fundamental three dimensional representations $\pi_{[1,0]}$ and $\pi_{[0,1]}$. It means that in fact all of the combinatorial information for representations of $\mathfrak{sl}_3$ is ultimately incoded in these two simple diagrams.

In addition the diamonds $D(a,b)$ are simultaneously *classical* and *quantum* objects, a property that they share with Gelfand-Tsetlin polytopes. By this we mean that the projection onto the $x-y$ plane of the volume of $D\,(a,b)$ is a classical object, namely the Duistermaat–Heckman measure associated to the corresponding symplectic co-adjoint orbit in the dual of the Lie algebra $\mathfrak{su}_3$, while the projection of the integral nodes is a quantum object, namely the weight multiplicity diagram of the corresponding unitary representation.

After a brief section in which we recall some of the basic facts about the representations of $\mathfrak{sl}_3$, we give a complete description of the diamonds and the raising and lowering operators which determine the representations. This is sufficiently self contained to enable the reader to explicitly create such models, by hand, even for reasonably large representations.

It is not an unpleasant exercise to create three dimensional physical models of diamonds, as can be done reasonably easily using commercially available construction kits. This is briefly explained in the last section of the paper. In any case we provide a complete list of algebraic expressions for all matrix coefficients, since they are needed for us in the course of the proofs of the above results. In practise we require only the two dimensional realization which may be easily implemented using ordinary graph paper.

The remainder of the paper will clarify and augment this discussion and provide proofs of the main results. These are mostly technical, and rely on somewhat miraculous manipulations and cancellations. It would be interesting to have geometric, or combinatorial, proofs for the commutation relations we derive. My own feeling is that the best proofs will only become apparent if and when the appropriate generalization of this construction has been found for the general simple Lie algebra. Such a development will likely reveal that weight diagrams are projections onto the Cartan subalgebra of 'diamonds' in a higher dimensional space (probably of dimension the number of positive roots).

This present work is part of a larger program to explicitly construct combinatorial models for the irreducible representations of the general simple Lie algebra. It should be compared to the constructions of the simply laced Lie algebras in [Wi1] (see also [Wi2]) and of G_2 in [Wi3]. As in these other papers, we want to show that the combinatorial approach may be introduced without any sophisticated knowledge of Lie theory. As a result, we have not always taken the most direct (abstract) path to verifying commutation relations, but rather one which any patient reader may follow.

2 Representations of $\mathfrak{sl}_3$

In this preliminary section we review some basic facts about the representations of the Lie algebra $\mathfrak{sl}_3$ of trace zero three by three matrices, as can be found in [Hu]. In this section only, the symbols X_η, Y_η, H_η, where η is one of $\alpha, \beta, \alpha+\beta$, represent specific matrices. In all the remaining sections these same symbols will be used to denote operators corresponding to these matrices in more general representation spaces. This overlapping of notation is intentional, but we draw attention to it here to minimize possible confusion. After all, the three dimensional representation is just one of an infinite number of ways to specify the Lie algebra, which in fact is more properly defined by listing its structure constants with respect to a linear basis (or perhaps spanning set), an approach however that perhaps provides too much information for human readers.

The Lie algebra $\mathfrak{sl}_3$ has a basis consisting of two diagonal matrices H_α, H_β together with three upper triangular matrices (raising operators) $X_\alpha, X_\beta, X_{\alpha+\beta}$ and three lower triangular matrices (lowering operators) $Y_\alpha, Y_\beta, Y_{\alpha+\beta}$, one for each of the positive roots of the Lie algebra. Our conventions for these matrices may be represented by the following schematic diagram

$$\begin{bmatrix} H_\alpha & X_\alpha & X_{\alpha+\beta} \\ Y_\alpha & H_\beta - H_\alpha & X_\beta \\ Y_{\alpha+\beta} & Y_\beta & -H_\beta \end{bmatrix}$$

so that $X_\alpha = E_{12}, Y_\alpha = E_{21}$, $H_\alpha = E_{11} - E_{22}$ etc where E_{ij} represents the matrix with one in the ij position and zeros elsewhere. For convenience we introduce the term $H_{\alpha+\beta} = H_\alpha + H_\beta$. Thus we have the commutation relations $[X_\alpha, Y_\alpha] = H_\alpha, [X_\beta, Y_\beta] = H_\beta, [X_{\alpha+\beta}, Y_{\alpha+\beta}] = H_{\alpha+\beta}$ and so on.

Irreducible finite dimensional representations π over the complex numbers are parametrized by an ordered pair $[a, b]$ of positive (greater than or equal to zero) integers. Such a pair is called a **highest weight**. The dimension of the irreducible representation $\pi_{[a,b]}$ is

$$\frac{1}{6}\left(a+1\right)\left(b+1\right)\left(a+b+2\right).$$

Relative to a two dimensional Cartan subalgebra, usually taken to be the diagonal subalgebra, the weights of $\pi_{[a,b]}$ form in general a hexagonal pattern in the plane which has S_3 symmetry, acting on the plane by reflections and rotations. This pattern of weights has a ring of ones around the perimeter of the hexagon, with multiplicities increasing linearly into the interior of the hexagon until it reaches a central triangle, at which point it is becomes stable. Knowledge of this weight multiplicity pattern is equivalent to knowing the character of the representation, which is also given by the classical formulae of Weyl, Freudenthal, Kostant and Demazure. While elements of the Cartan subalgebra act on weight vectors by scalars, indeed by scalar functions which are restrictions to the weight diagram of linear functions in the plane, the raising and lowering operators send the elements of one weight space to the elements of the weight space obtained by adding or subtracting the corresponding root from the initial weight.

So the problem of an explicit description of a particular representation amounts to choosing a basis for each weight space and showing how the raising and lowering operators act on each of these basis elements in this basis. Since the weight spaces are not one dimensional, there is a great deal of scope for doing this. A systematic exploration of the space of possibilities for such choices, together with an identification of particular key configurations with important minimizing properties, has been initiated by Donnelly [Do].

The Gelfand-Tsetlin construction is a particular explicit solution to this problem, and was originally stated (without proof) for the case of $\mathfrak{sl}_n$. The construction can be stated in terms of either semistandard Young tableux or Gelfand-Tsetlin patterns, and matrix coefficients are generally rational numbers.

Our new model, which we now describe, has integral matrix coefficients for all root vectors. By using Donnelly's work [Do], we can easily see that the diamond module bases are not renormalizations of the Gelfand-Tsetlin bases.

3 Description of the diamond modules

In this section we will give a brief but complete description of the diamond model of representations of $\mathfrak{sl}_3$, with all the definitions we need, but no proofs. It is intended to be largely independent of the rest of the paper, and to allow the reader to proceed directly to construction of representations.

We work in three dimensional Euclidean space with ordered basis $\{\alpha, \beta, \gamma\}$ and typical element $v = [x, y, z]$. The z coordinate of v will be called the **height** of v and denoted by $h(v)$. We consider the convex cone D spanned by the four vectors

$$[1, 0, 1] \quad [0, 1, -1] \quad [1, 1, 1] \quad [1, 1, -1]$$

and called the **diamond cone**. An integral point $[m, n, l]$ lying inside or on D will be called a **node** if it satisfies

$$l \equiv \max(m, n) \operatorname{mod}(2) .$$

The collection D_I of nodes is called the **integral diamond cone**. It is made into a directed graph by joining a node v to a node u by a directed edge e if the z coordinates differ by at most one and if the $[x, y]$ coordinates of $u - v$ are one of $[-1, 0], [0, -1]$ or $[-1, -1]$. In such a case the **direction** of e is declared to be $-\alpha, -\beta, -(\alpha + \beta)$ respectively. If e has direction $-\eta$ then we call it an X_η**-edge**.

The **type** of e is declared to be **up** if $h(u) \geq h(v)$, **down** if $h(u) \leq h(v)$, and **both** up and down if $h(u) = h(v)$. The types up and down will be referred to as **opposite types**.

Every edge e in addition receives an integral weight $w(e)$ defined as follows. Let r be the maximal number of successive edges in a ladder of edges beginning with and including e and of the same direction and type as e. If e has direction $-\alpha$ or $-\beta$ and either type up or type down (or both), or direction $-(\alpha + \beta)$ and type down, then we define

$$w(e) = r$$

while if e has direction $-(\alpha+\beta)$ and type up we define

$$w(e) = -r.$$

Its not hard to see that edges of direction $-(\alpha+\beta)$ can never be both up and down, so this is well defined.

Let V be a vector space with basis $\{w_v | v \text{ a node in } D_I\}$ and define three operators X_α, X_β and $X_{\alpha+\beta}$ on V by the rule

$$X_\eta w_v = \sum_u w(e) w_u$$

sum over all nodes u for which there is an X_η-edge e from v to u.

Theorem 1 *The operators X_α, X_β, and $X_{\alpha+\beta}$ on V satisfy the commutation relations $[X_\alpha, X_\beta] = X_{\alpha+\beta}, [X_\alpha, X_{\alpha+\beta}] = [X_{\alpha+\beta}, X_\beta] = 0$.*

The module V of the Heisenberg Lie algebra thus established is not irreducible. If we regard D_I as a poset with the directed edges denoting covering relations (so that moving in the direction of an edge lowers us in the partial order), then any ideal defines a submodule. In particular, for any natural numbers $a, b \geq 0$ the principal ideal generated by the node $[a+b, a+b, a-b]$ lying over the main diagonal is a submodule which we call a **diamond** and denote by $D_I(a,b)$.

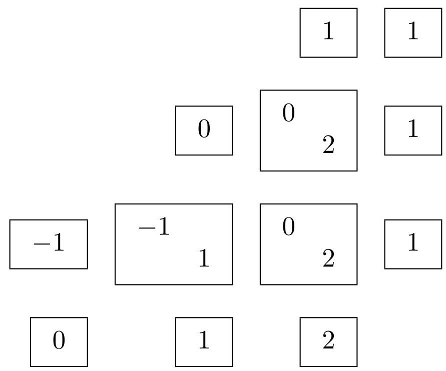
It turns out that the number of nodes in $D_I(a,b)$ is exactly the dimension of the irreducible representation $\pi_{[a,b]}$ of $\mathfrak{sl}_3$, and that there is an additional, somewhat unexpected, symmetry in the polytopes $D_I(a,b)$ that allows one to extend the module from the Heisenberg Lie algebra to $\mathfrak{sl}_3$.

The nodes in $D_I(a,b)$ may be conveniently represented on a two dimensional diagram, by displaying the heights of all those nodes lying over a point $[m,n]$ in a box located at the point $[m,n]$ in the plane. Placing these heights down the diagonals of the boxes simplifies the drawing of a two dimensional picture of the directed graph D_I. The **girdle** of the diamond consists of those nodes lying above the main diagonal $x = y$.

Example 2 *The diamond $D_I(2,1)$ has the following 15 nodes*

$$\begin{gathered}[0,0,0], [1,0,1], [2,0,2], [0,1,-1], [1,1,1], \\ [1,1,-1], [2,1,2], [2,1,0], [3,1,1], [1,2,0], \\ [2,2,2], [2,2,0], [3,2,1], [2,3,1], [3,3,1]\end{gathered}$$

with two dimensional diagram



The girdle here contains six nodes.

Define the **diagonal involution** $\tau = \tau(a,b)$ on nodes of $D_I(a,b)$ by the rule

$$\tau[x,y,z] = [a+b-y, a+b-x, a-b+x-y-z].$$

There is also a more geometric description of this involution in terms of diagonal slices (parallel to the girdle) of the diamond $D_I(a,b)$.

We now augment the graph $D_I(a,b)$ by introducing new edges which are the images under the diagonal involution of the X_η-edges. Specifically, for each existing edge e from a node v to a node u in $D_I(a,b)$, define a corresponding new edge $f = \tau e$ from τv to τu. If the $[x,y]$ coordinates of $\tau u - \tau v$ are η then we say that f is a **Y_η-edge**. The form of the diagonal involution ensures that then η is one of $\alpha, \beta, \alpha+\beta$. It is to be noted that in general these new Y_η-edges do *not* have the property that they connect only vertices whose heights differ by at most one.

Define the weight of a Y_η-edge $f = \tau e$ by the rule

$$w(f) = w(e).$$

For η one of α, β or $\alpha+\beta$, define an operator Y_η on $V(a,b)$ by

$$Y_\eta w_u = \sum_v w(f) w_v$$

where the sum is over all nodes v for which there is a Y_η-edge f from u to v. Equivalently we have

$$\begin{aligned} Y_\alpha &= \tau X_\beta \tau \\ Y_\beta &= \tau X_\alpha \tau \\ Y_{\alpha+\beta} &= \tau X_{\alpha+\beta} \tau. \end{aligned}$$

Define, in addition, the operators H_η by the rule

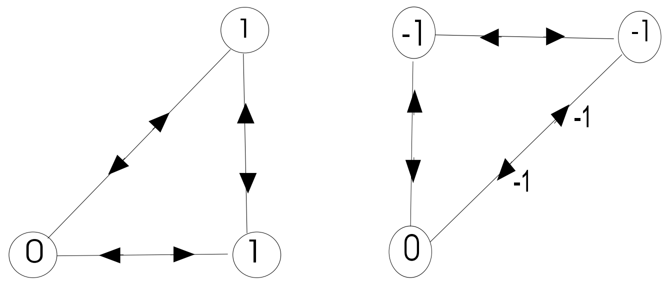
$$H_\eta = [X_\eta, Y_\eta].$$

Proposition 3 *Each of the operators H_η acts diagonally on the nodes of the diamond, so that $H_\eta w_v = \lambda_\eta(v) w_v$ with $\lambda_\eta(v)$ an affine function of v that depends only on the first two coordinates of v. Furthermore $H_{\alpha+\beta} = H_\alpha + H_\beta$.*

Theorem 4 *The span of the nine operators X_η, Y_η, and H_η on $V(a,b)$, where η is one of α, β or $\alpha+\beta$, is closed under brackets and forms an eight dimensional Lie algebra isomorphic to $\mathfrak{sl}_3$. This representation $\pi_{[a,b]}$ of $\mathfrak{sl}_3$ on $V(a,b)$ has highest weight $[a,b]$.*

In the examples that follow, operators $X_\alpha, X_\beta, X_{\alpha+\beta}$ are represented by arrows pointing W,S,SW respectively, and operators $Y_\alpha, Y_\beta, Y_{\alpha+\beta}$ are represented by arrows pointing E,N,NE respectively on the diagram of the diamond. Integers besides arrows denote weights of edges, with the default weight being one.

Example 5 *The quark representation $\pi_{[1,0]}$ and the anti-quark representation $\pi_{[0,1]}$ have respective diagrams*



Note that all the operators take any of the nodes to either plus or minus one times another node. Thus for example in $\pi_{[1,0]}$ we see that

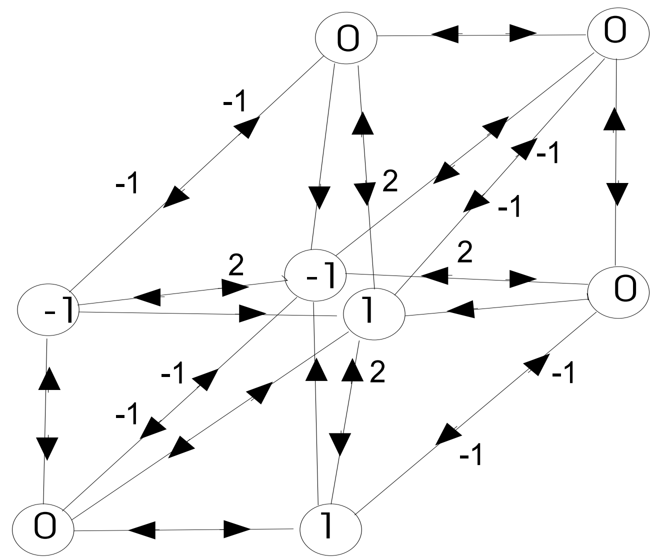
$$X_\alpha w_{[0,0,0]} = X_\alpha w_{[1,1,1]} = 0$$
$$X_\alpha w_{[1,0,1]} = w_{[0,0,0]}$$

while in $\pi_{[0,1]}$ we see that

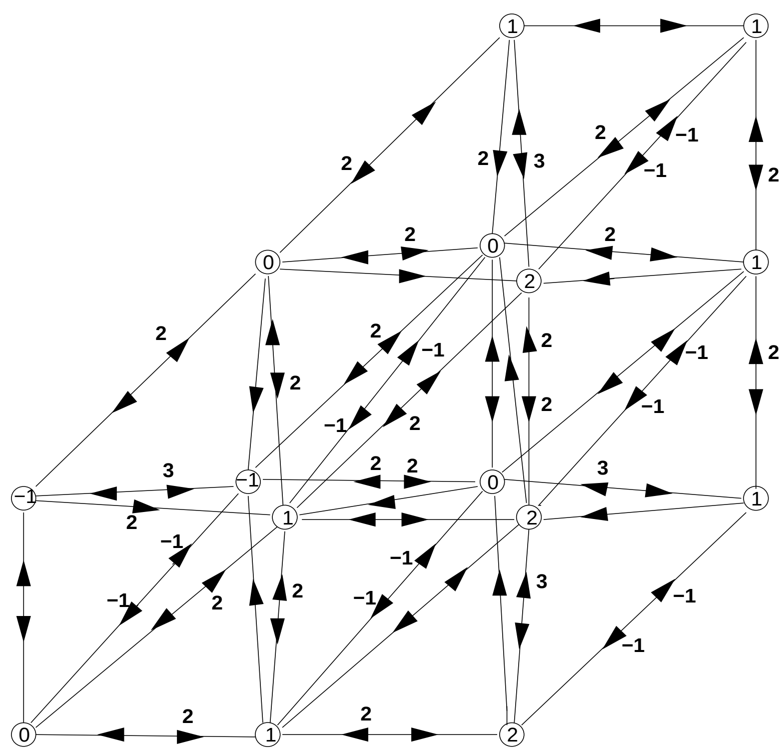
$$Y_{\alpha+\beta} w_{[0,0,0]} = -w_{[1,1,-1]}.$$

It is remarkable that all of the higher dimensional representations are derivable from these two simple pictures.

Example 6 *Here is the eight-dimensional adjoint or 'gluon' representation $\pi_{[1,1]}$, also known by physicists as the 'eight-fold way'. Note that we see edges which have weights assigned in only one direction. Thus for example there is no X_α edge from $[1,1,1]$ to $[0,1,-1]$ which means that the corresponding X_α operator sends the former node to 0.*



Example 7 *Here is the fifteen-dimensional representation* $\pi_{[2,1]}$.



π(2,1)

As $[a, b]$ varies over all ordered pairs of positive (greater than or equal to zero) integers, we get in this fashion a model of each finite dimensional irreducible representations $\pi_{[a,b]}$ of $\mathfrak{sl}_3$ with the property that each of the nine operators X_η, Y_η, and H_η is represented by a matrix with integer entries which has at most two non-zero elements in each row and column. The form of the model makes the calculation of these integers relatively simple; even by hand for large representations. One of course does not require three dimensional models of the diamonds themselves, as all the relevant information can be encoded in the two dimensional diagrams containing the heights of the nodes, augmented by the edges of the diamonds and their weights.

4 The diamond cone

We will now proceed to explain this construction more fully, providing details and proofs. Three dimensional Euclidean space E^3 has elements $[x, y, z]$ and basis vectors $\alpha = [1, 0, 0], \beta = [0, 1, 0], \gamma = [0, 0, 1]$. The projection p onto the $x - y$ plane E^2, the span of α and β, is $p[x, y, z] = [x, y]$. We refer to $p(v)$ as the **weight** of the vector v, and say that v **lies over** $p(v)$. The z coordinate of a vector v will be called the **height** of v and denoted by $h(v)$.

Let D, the **diamond cone**, be the convex hull of the rays in the four directions

$$\left[[1,0,1] \quad [0,1,-1] \quad [1,1,1] \quad [1,1,-1]\right].$$

Thus

$$D = \{\alpha\,[1,0,1] + \beta\,[0,1,-1] + \gamma\,[1,1,1] + \delta\,[1,1,-1]\,|\alpha, \beta, \gamma, \delta \geq 0\}.$$

Then D is bounded by the four planes

$$\begin{aligned} z &= x \\ z &= x - 2y \\ z &= -y \\ z &= 2x - y. \end{aligned}$$

Clearly $p(D)$ is the positive quadrant in E^2. If $0 \leq x \leq y$ then those points in D lying over $[x, y]$ have height from $-y$ to $2x - y$, while if $0 \leq y \leq x$ then those points in D lying over $[x, y]$ have height from $x - 2y$ to x. So D is given by the inequalities

$$\begin{aligned} 0 &\leq x, y \\ -y \leq z &\leq 2x - y \\ x - 2y \leq z &\leq x. \end{aligned}$$

Let D_I consist of those points in D with integral coordinates $[m, n, l]$ satisfying

$$l \equiv \max(m, n) \bmod(2).$$

This is equivalent to $[m, n, l]$ differing by a multiple of $[0, 0, 2]$ from a point on the boundary of the cone. We refer to the elements of D_I as **integral nodes**, or just **nodes**.

D_I can be described by specifying the range of heights of those nodes having a given weight $[m, n]$ for non-negative integers m and n. For this purpose, if l_1 and l_2 are integers which satisfy $l_1 \leq l_2$ and differ by a multiple of two, we will use the notation

$$\lfloor l_1, l_2 \rfloor = \{l_1, l_1 + 2, .., l_2\}\,.$$

Here is a table showing the heights of nodes in D_I with weights $[m, n]$ where $0 \leq m, n \leq 6$.

$$\left[\begin{array}{ccccccc}
-6 & \lfloor -6,-4 \rfloor & \lfloor -6,-2 \rfloor & \lfloor -6,0 \rfloor & \lfloor -6,2 \rfloor & \lfloor -6,4 \rfloor & \lfloor -6,6 \rfloor \\
-5 & \lfloor -5,-3 \rfloor & \lfloor -5,-1 \rfloor & \lfloor -5,1 \rfloor & \lfloor -5,3 \rfloor & \lfloor -5,5 \rfloor & \lfloor -4,6 \rfloor \\
-4 & \lfloor -4,-2 \rfloor & \lfloor -4,0 \rfloor & \lfloor -4,2 \rfloor & \lfloor -4,4 \rfloor & \lfloor -3,5 \rfloor & \lfloor -2,6 \rfloor \\
-3 & \lfloor -3,-1 \rfloor & \lfloor -3,1 \rfloor & \lfloor -3,3 \rfloor & \lfloor -2,4 \rfloor & \lfloor -1,5 \rfloor & \lfloor 0,6 \rfloor \\
-2 & \lfloor -2,0 \rfloor & \lfloor -2,2 \rfloor & \lfloor -1,3 \rfloor & \lfloor 0,4 \rfloor & \lfloor 1,5 \rfloor & \lfloor 2,6 \rfloor \\
-1 & \lfloor -1,1 \rfloor & \lfloor 0,2 \rfloor & \lfloor 1,3 \rfloor & \lfloor 2,4 \rfloor & \lfloor 3,5 \rfloor & \lfloor 4,6 \rfloor \\
0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}\right]$$

The matrix entry in the position $[m, n]$ is $\lfloor m - 2n, m \rfloor$ if $n \leq m$ and is $\lfloor -n, -n + 2m \rfloor$ if $n \geq m$. If $m = n$ then both of these expressions are equal to $\lfloor -m, m \rfloor = \lfloor -n, n \rfloor$.

Note that both D and D_I are invariant under the involution $[m, n, l] \to [n, m, -l]$, which will play a role in the subsequent proofs.

Let the nodes in D_I be the vertices of a directed graph, also denoted by D_I, with edges e labelled by a *name*, a *direction*, a *type*, and a *weight,* described as follows.

For η one of α, β, and $\alpha + \beta$, create an **X_η-edge** from a vertex v to a vertex u in D_I precisely when both

$$p(v) - p(u) = \eta$$

and

$$|h(v) - h(u)| \leq 1.$$

Such an X_η-edge will be said to have **name** X_η and **direction** $-\eta$, so that one of these determines the other. It will be said to be of **up type** if $h(v) \leq h(u)$ and of **down type** if $h(v) \geq h(u)$. Note that if $h(v) = h(u)$ then the edge is to be considered to be *both* of up type and down type; we call such an edge **horizontal**.

The final ingredient to complete our graph is to assign to each edge e an integer **weight** $w\,(e)$. The rule to compute weights is the following. Count the number r of edges occurring in the maximal ladder of successive edges beginning at and including the specified edge e, and of the same direction and type as e. For X_α-edges and X_β-edges the weight $w(e)$ is defined to be this integer r. In particular weights of all X_α-edges and X_β-edges are positive integers. For $X_{\alpha+\beta}$-edges the weight $w(e)$ is defined to be r if e is of down type and is defined to be $-r$ if e is of up type.

D_I is then a directed, edge-weighted graph (with edges having direction one of $-\alpha, -\beta$, or $-(\alpha+\beta)$, of type either up or down or both, and with integral weights). In addition, we make D_I into a partially ordered set by taking the directed edges to represent covering relations. In other words, we define $u < v$ when there is a directed path from v to u. In this partial order, there is then a unique minimal element $(0, 0, 0)$. Note that all the nodes u less than or equal to a node $[m, n, l]$ have projection $p(u)$ contained in the rectangle $0 \leq x \leq m, 0 \leq y \leq n$.

5 Diamonds

For any node v, the set of nodes $\{u \in D_I | \, u \leq v\}$ is a principal ideal of the poset D_I. For any pair of positive integers (a, b) the principle ideal

$$D_I(a, b) = \{u \in D_I | \, u \leq [a+b, a+b, a-b]\}$$

will be called a **diamond**.

We say further that the diamond $D_I(a, b)$ has **highest weight** (a, b). The **highest node** $[a+b, a+b, a-b]$ lies over the main diagonal and is maximal amongst all the nodes of $D_I(a, b)$.

Let $D(a, b)$ denote the convex hull of $D_I(a, b)$. By a **vertex** of the diamond $D_I(a, b)$ we mean a vertex of the convex hull $D(a, b)$.

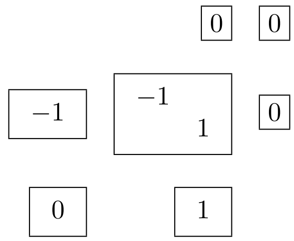
By the **diagram** of the diamond $D_I(a, b)$ we mean the two-dimensional representation of the heights of the nodes above the corresponding weights in the $x - y$ plane E^2. We generally use the convention that the heights above a point $[m, n]$ will be displayed in a diagonal fashion to facilitate the addition of edges.

Example 8 *The diamonds $D_I(1, 0)$ and $D_I(0, 1)$ both have three nodes (all are vertices) and have respective diagrams*

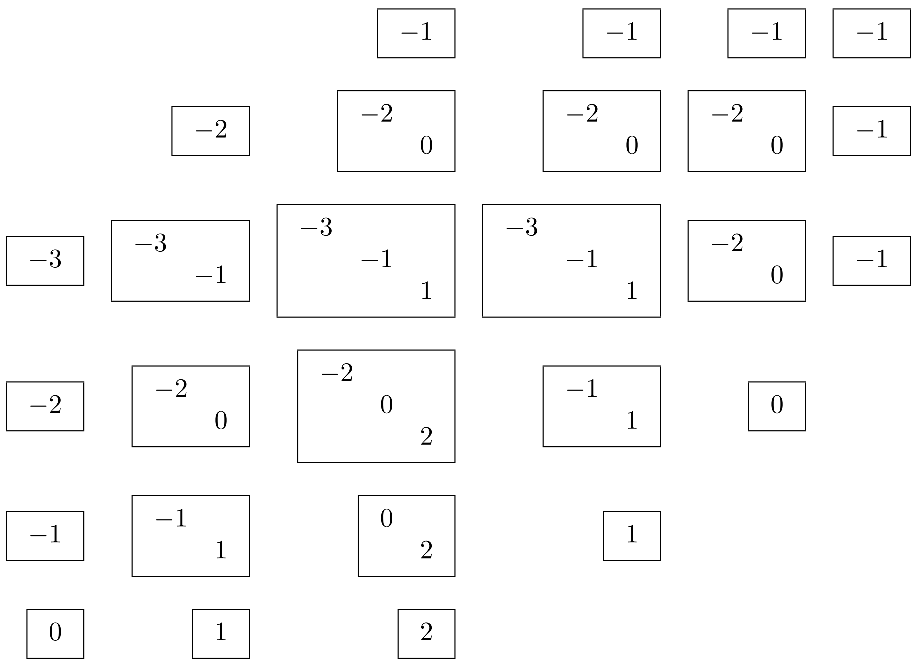
	1	*and*	-1	-1
0	1		0	

Example 9 *The diamond $D_I(1, 1)$ has eight nodes (all are vertices) and dia-*

gram



Example 10 *The diamond $D_I(2,3)$ has 42 nodes (eight are vertices) and diagram*



The faces of $D(a,b)$ are given by the six equations

$$\begin{array}{cc} -y \le z \le 2x - y & x - 2b \le z \le 2a - x \\ x - 2y \le z \le x & y - 2b \le z \le 2x - y \\ x - 2y \le z \le 2a - x & y - 2b \le z \le 2a - y \end{array}$$

These generally amount to eight independent equations, namely

$$\begin{array}{rl} -y \le z & z \le x \\ x - 2y \le z & z \le 2x - y \\ x - 2b \le z & z \le 2a - x \\ y - 2b \le z & z \le 2a - y \end{array}$$

In general each of these equations contributes to a face, so that the resulting polytope is not a Gelfand-Tsetlin polytope, since for $\mathfrak{sl}_3$ the latter generally has six sides.

6 The diagonal involution

It will be useful to declare a line l in the plane E^2 to be **diagonal** if it has the form $y = x + k$ for some k.

Lemma 11 *If $u < v$ are nodes in D_I lying over the same diagonal line l, then there is a directed path from v to u all of whose nodes lie above l.*

Proof. One of the properties of the graph D_I is that if we consider any vertex v, then for at least one of the directions $\eta = \alpha$ or β, there is only one X_η-edge leaving v, and it is horizontal. In fact if $p(v) = [m, n]$ with $m < n$ then there is only one X_α-edge leaving v and it is horizontal, and if $m > n$ then there is only one X_β-edge leaving v and it is horizontal, while if $m = n$, then there is only one X_α-edge leaving v and there is only one X_β-edge leaving v, both of which are horizontal. It follows that if $p\,(v - u) = [r, r]$ then any directed path in D_I from v to u can create a change in height of at most r. But that means that we could get from v to u by using only $X_{\alpha+\beta}$-edges. ■

If we consider subposets of D_I lying over diagonal lines in E^2, we thus get infinite inverted binary trees (as posets they are all isomorphic.) Of particular importance is the tree of nodes lying over the main diagonal $y = x$; any such node has the form $[a + b, a + b, a - b]$ for some unique pair of positive integers $[a, b]$.

If we consider a line in E^2 with equation of the form $y = k$ or $x = k$, then the subposet of D_I lying over it is an infinite ranked poset with all levels eventually containing $k + 1$ elements. This poset has $k + 1$ minimal elements, one of each rank $1, 2, .., (k + 1)$.

Lemma 12 *The subposet of $D_I(a, b)$ lying above a diagonal line l is a (possibly empty) rectangle, that is, a poset isomorphic to a product of chains $\mathbf{p} \times \mathbf{q}$. Here $\mathbf{r}$ represents the chain $\{0, 1, ..., r\}$.*

Proof. The previous Lemma showed that the subposet of D_I lying above a diagonal line l is just the poset determined by the directed subgraph formed from the nodes and edges lying above l, namely an inverted binary tree. If $v = [a + b, a + b, a - b]$, then any vertex u of $D_I(a, b)$ lying over one of the lines $x = a + b$ or $y = a + b$ is reached from v by a sequence of horizontal X_β-edges or a sequence of horizontal X_α-edges respectively, so in particular has height $h(u) = a - b$.

Such a node u is the maximal element in the rectangle of which it is a member, and every rectangle lying above some diagonal line l has maximal element u of this form, that is, with either x or y coordinate equal to $a + b$. In the proof of the preceeding Lemma we saw that for any path in D_I from

v, edges are not horizontal exactly when those edges are X_α-edges lying over the region $x \leq y$ or X_β-edges lying over the region $x \geq y$. This implies that if w is any vertex in $D_I(a,b)$ above the diagonal line l through $p(u)$, so that $p(u) - p(w) = (s,s)$, then $|h(u) - h(w)| \leq s$. But then there is a path from u to w. Thus the subposet of $D_I(a,b)$ lying above a diagonal line $y = x + k$, for some integer k, is a principal ideal of an inverted binary tree and hence a rectangle. ■

In the example of $D_I(2,3)$ above, the subposet lying over the line $x - y = 1$ is isomorphic to the product $\mathbf{1} \times \mathbf{3}$. As a rectangle, its corners are at the nodes $[1,0,1], [5,4,-1], [2,1,2]$, and $[4,3,-2]$. The first and second are the minimum and maximum corners (with respect to the partial order) respectively. The third and fourth corners are at the greatest and least heights respectively.

More generally in the diamond $D_I(a,b)$ the rectangle lying above the diagonal line $x - y = k$ can be described as follows.

Case 1. For $0 \leq k \leq a$, the minimum corner is $[k,0,k]$, and the maximum corner is $[a+b, a+b-k, a-b]$. The other corners are at $[a, a-k, a]$, and $[b+k, b-b+k]$, the former having the greatest height, and the latter having the least height.

Case 2. For $-b \leq k \leq 0$, the minimum corner is $[0,-k,k]$, and the maximum corner is $[a+b+k, a+b, a-b]$. The other corners are at $[b+k, b, -b]$ and $[a, a-k, a+k]$, the former having the least height and the latter having the greatest height.

Proposition 13 *The vertices of the diamond $D_I(a,b)$ are the points*

$$[0,0,0], [a,0,a], [0,b,-b], [b,b,-b], [a,a,a]$$
$$[a+b, b, a-b], [a, a+b, a-b], [a+b, a+b, a-b]$$

Proof. The vertices of the diamond are contained among the vertices of all the diagonal rectangles established above. Note that in either Case as we vary k the relevant corner lies on a straight line—this is just the statement that k appears linearly in all the expressions above. It follows that we need consider only those points on these lines where k takes on an extremal value, namely 0 or a in Case 1. or $-n_2$ or 0 in Case 2. This results in the eight points mentioned. ■

If both a and b are greater than zero, it is easy to see that these eight points are distinct. However if $a = 0$ and b is non-zero, then there are only three distinct points in this list, namely $[0,0,0], [0,b,-b]$, and $[b,b,-b]$, so that the diamond is a two dimensional triangle. Similarly if $b = 0$ and a is non-zero, then there are exactly three distinct points, namely $[0,0,0], [a,0,a]$, and $[a,a,a]$. If both a and b are zero, then the diamond reduces to the single point $[0,0,0]$.

Theorem 14 *For any non-negative integers a and b,*

$$D(a,b) = D(a,0) + D(0,b).$$

Proof. The cases when one or more of a, b are zero are immediate. So suppose both a and b are non-zero. The triangle $D(a, 0)$ has vertices $[0, 0, 0], [a, 0, a]$, and $[a, a, a]$, while the triangle $D(0, b)$ has vertices $[0, 0, 0], [0, b, -b]$, and $[b, b, -b]$. Thus the vector sum $D(a, 0) + D(0, b)$ has vertices contained in the sum of the two triangle vertex sets, namely the nine points

$$[0, 0, 0], [0, b, -b], [b, b, -b], [a, 0, a], [a, b, a - b],$$
$$[a + b, b, a - b], [a, a, a], [a, a + b, a - b], [a + b, a + b, a - b]$$

This latter set consists of the eight vertices of $D(a, b)$, together with the node $[a, b, a - b]$. Suppose without loss of generality that $b \leq a$. Then

$$[a, b, a - b] = \frac{b}{2a + b}[0, b, -b] + \frac{a}{2a + b}[a + b, b, a - b] + \frac{a - b}{2a + b}[a, 0, a] + \frac{b}{2a + b}[a, a, a]$$

so $[a, b, a - b]$ is in the interior of $D(a, b)$. Thus we see that

$$D(a, b) = D(a, 0) + D(0, b).$$

■

Now any chain $\mathbf{p}$ has a canonical involution $a \to \tau a$ which inverts the order of the elements, sending an element a to $\tau a = p - a$ for $a = 1, 2, ...p$. Thus any rectangle $\mathbf{p} \times \mathbf{q}$, consisting of ordered pairs $[a, b], a \in \mathbf{p}, b \in \mathbf{q}$ also has a canonical involution, sending $[a, b]$ to $[p - a, q - b]$. Since each diagonal slice of a diamond is a rectangle, we may define for any diamond a canonical involution, $v \to \tau v$ which we call the diamond involution, by applying the canonical involution to every diagonal slice. This involution acts affinely on any diagonal slice.

Lemma 15 *If $[x, y, z]$ lies on the diamond $D_I(a, b)$, then*

$$\tau[x, y, z] = [a + b - y, a + b - x, a - b + x - y - z].$$

Proof. Consider the transformation T, which depends on a and b, given by

$$T[x, y, z] = [a + b - y, a + b - x, a - b + x - y - z].$$

Then we calculate that

$$\begin{aligned} T[k, 0, k] &= [a + b, a + b - k, a - b] \\ T[a, a - k, a] &= [b + k, b, -b + k] \\ T[0, -k, k] &= [a + b + k, a + b, a - b] \\ T[b + k, b, -b] &= [a, a - k, a + k]. \end{aligned}$$

Thus G interchanges each of the minimum and maximum corners of each diagonal slice. It also interchanges the corners of greatest and least heights. Since T is in addition an affine transformation of E^3, it must restrict to the diagonal involution on any diagonal slice. Thus T agrees with the diagonal involution τ.
■

7 The operators X_η

Let V be a vector space with basis $\{w_v | v \text{ a node in } D_I\}$. Since all subsequent operators in this space will be defined using only integer multiples of w_v it suffices to take the field to be the rational numbers.

For η one of α, β or $\alpha + \beta$, define an operator X_η on V by the rule

$$X_\eta w_v = \sum_u w(e) w_u$$

sum over all nodes u for which there is an X_η-edge e from v to u. For simplicity, we will identify the vector w_v in V with the node v in D_I, so that our rule can be more simply written

$$X_\eta v = \sum_u w(e) u.$$

This gives us three operators X_α, X_β, and $X_{\alpha+\beta}$ on V. We are now going to determine the brackets of these operators. Our convention is that

$$[X, Y]s = XYs - YXs$$

which is consistent with the bracket of matrices when considered as operators, acting on the left on column vectors.

Theorem 16 *The operators X_α, X_β, and $X_{\alpha+\beta}$ on V satisfy the commutation relations $[X_\alpha, X_\beta] = X_{\alpha+\beta}, [X_\alpha, X_{\alpha+\beta}] = [X_{\alpha+\beta}, X_\beta] = 0$.*

Proof. We begin by writing down explicitly the actions of the three operators on generic nodes in D_I. To verify these equations, we need to check the weights of the appropriate edges.

Case 1. $v = [m, n, m - 2r]$ where $n < m$ and $r \leq n$.

$$\begin{aligned}
X_\alpha [m, n, m - 2r] &= (m - n + r) [m - 1, n, m - 2r - 1] \\
&\quad + r [m - 1, n, m - 2r + 1] \\
X_\beta [m, n, m - 2r] &= (n - r) [m, n - 1, m - 2r] \\
X_{\alpha+\beta} [m, n, m - 2r] &= (n - r) [m - 1, n - 1, m - 2r - 1] \\
&\quad - r [m - 1, n - 1, m - 2r + 1]
\end{aligned}$$

The X_α equation says that the edge e from $[m, n, m - 2r]$ to $[m - 1, n, m - 2r - 1]$ has weight $w(e) = m - n + r$, while the edge e' joining $[m, n, m - 2r]$ to $[m - 1, n, m - 2r + 1]$ has weight $w(e') = r$. Indeed the edge e is an X_α-edge of down type and is the first of the ladder

$$\begin{aligned}
&[m, n, m - 2r] \to [m - 1, n, m - 2r - 1] \to \cdots \to [n, n, m - 2r - (m - n)] \\
&\qquad = [n, n, n - 2r] \to [n - 1, n, n - 2r] \to \cdots \to [n - r, n, n - 2r]
\end{aligned}$$

with a total of $(m-n)+r$ edges, while the edge e' is an X_α-edge of up type and is the first of the ladder

$$[m,n,m-2r] \to [m-1,n,m-2r+1] \to \cdots \to [m-r,n,m-r]$$

with a total of r edges.
The X_β equation follows from the ladder of X_β-edges of horizontal type given by

$$[m,n,m-2r] \to [m,n-1,m-2r] \to \cdots \to [m,r,m-2r]\,.$$

It has $n-r$ edges.
The $X_{\alpha+\beta}$ equation follows from two ladders of $X_{\alpha+\beta}$-edges. The first consists of $n-r$ down type edges given by

$$[m,n,m-2r] \to [m-1,n-1,m-2r-1] \to \cdots \to [m-n+r,r,m-n-r]\,.$$

The second consists of r up type edges given by

$$[m,n,m-2r] \to [m-1,n-1,m-2r+1] \to \cdots \to [m-r,n-r,m-r]\,.$$

Case 2. $v=[m,n,-n+2r]$ where $m<n$ and $r \le m$.

$$\begin{aligned}
X_\alpha\,[m,n,-n+2r] &= (m-r)\,[m-1,n,-n+2r]\\
X_\beta\,[m,n,-n+2r] &= (n-m+r)\,[m,n-1,-n+2r+1]\\
&\quad + r\,[m,n-1,-n+2r-1]\\
X_{\alpha+\beta}\,[m,n,-n+2r] &= -(m-r)\,[m-1,n-1,-n+2r+1]\\
&\quad + r\,[m-1,n-1,-n+2r-1]
\end{aligned}$$

The equations of Case 2 need not be treated separately; they follow from those of Case 1 by using the involution $[m,n,l] \to [n,m,-l]$.

Case 3. $v=[m,m,m-2r]$ where $r \le m$.

$$\begin{aligned}
X_\alpha\,[m,m,m-2r] &= r\,[m-1,m,m-2r]\\
X_\beta\,[m,m,m-2r] &= (m-r)\,[m,m-1,m-2r]\\
X_{\alpha+\beta}\,[m,m,m-2r] &= (m-r)\,[m-1,m-1,m-2r-1]\\
&\quad - r\,[m-1,m-1,m-2r+1]
\end{aligned}$$

The equations of Case 3 involve special cases of the ladders occurring in Cases 1 and 2. We prefer however to list them separately to avoid possible confusion. The X_α equation follows from the ladder

$$[m,m,m-2r] \to [m-1,m,m-2r] \to ... \to [m-r,m,m-2r]$$

with r X_α-edges of horizontal type. The X_β equation follows from the ladder

$$[m,m,m-2r] \to [m,m-1,m-2r] \to ... \to [m,r,m-2r]$$

with $(m-r)$ X_β-edges of horizontal type. The $X_{\alpha+\beta}$ equation follows from the ladder

$$[m,m,m-2r] \to [m-1,m-1,m-2r-1] \to ... \to [r,r,-r]$$

with $(m-r)$ $\alpha+\beta$ edges of down type and the ladder

$$[m,m,m-2r] \to [m-1,m-1,m-2r+1] \to ... \to [m-r,m-r,m-r]$$

with r $\alpha+\beta$ edges of up type.

Having established the form of the X_η operators, let us now check the commutation relations satisfied by them.

Case 1 For $v = [m,n,m-2r]$ where $r \le n < m$,

$$\begin{aligned}[X_\alpha, X_\beta]v &= X_\alpha\left((n-r)\,[m,n-1,m-2r]\right)\\ &- X_\beta((m-n+r)\,[m-1,n,m-2r-1]\\ &+ r\,[m-1,n,m-2r+1])\\ &= (n-r)((m-n+1+r)\,[m-1,n-1,m-2r-1]\\ &+ r\,[m-1,n-1,m-2r+1])\\ &- (m-n+r)(n-r)\,[m-1,n-1,m-2r-1]\\ &- r(n-r+1)\,[m-1,n-1,m-2r+1]\\ &= (n-r)\,[m-1,n-1,m-2r-1]\\ &- r\,[m-1,n-1,m-2r+1]\\ &= X_{\alpha+\beta}v\end{aligned}$$

$$\begin{aligned}[X_\alpha, X_{\alpha+\beta}]v &= X_\alpha((n-r)\,[m-1,n-1,m-2r-1] - r\,[m-1,n-1,m-2r+1])\\ &- X_{\alpha+\beta}((m-n+r)\,[m-1,n,m-2r-1] + r\,[m-1,n,m-2r+1])\\ &= (n-r)((m-n+r)\,[m-2,n-1,m-2r-2] + r\,[m-2,n-1,m-2r])\\ &- r((m-n+r-1)\,[m-2,n-1,m-2r] + (r-1)\,[m-2,n-1,m-2r+2])\\ &- (m-n+r)((n-r)\,[m-2,n-1,m-2r-2] - r\,[m-2,n-1,m-2r])\\ &- r((n-r+1)\,[m-2,n-1,m-2r] - (r-1)\,[m-2,n-1,m-2r+2])\\ &= 0\end{aligned}$$

$$\begin{aligned}[X_\beta, X_{\alpha+\beta}]v &= X_\beta((n-r)\,[m-1,n-1,m-2r-1] - r\,[m-1,n-1,m-2r+1])\\ &- X_{\alpha+\beta}(n-r)\,[m,n-1,m-2r]\\ &= (n-r)(n-1-r)\,[m-1,n-2,m-2r-1]\\ &- r(n-r)\,[m-1,n-2,m-2r+1]\\ &- (n-r)((n-1-r)\,[m-1,n-2,m-2r-1]\\ &- r\,[m-1,n-2,m-2r+1])\\ &= 0.\end{aligned}$$

Case 2. For $v = [m, n, -n+2r]$ where $m < n$ and $r \leq m$ we use the involution $[m, n, l] \to [n, m, -l]$, which takes X_α-edges of up/down types to X_β-edges of down/up types respectively. Similiarly it takes $X_{\alpha+\beta}$-edges of up/down types to down/up types respectively, which changes then the sign of the corresponding weights. This means that the commutation relation $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ will hold when applied to any such node $v = [m, n, -n+2r]$, since the order of X_α and X_β has been switched, and $X_{\alpha+\beta}$ multiplied by -1 when compared to $[n, m, n-2r]$.

Case 3. For $v = [m, m, m-2r]$ where $r \leq m$,

$$
\begin{aligned}
[X_\alpha, X_\beta]v &= X_\alpha (m-r)\,[m, m-1, m-2r] - X_\beta r\,[m-1, m, m-2r] \\
&= (m-r)((1+r)\,[m-1, m-1, m-2r-1] + r\,[m-1, m-1, m-2r+1]) \\
&- r((1+m-r)\,[m-1, m-1, m-2r+1] + (m-r)\,[m-1, m-1, m-2r-1]) \\
&= (m-r)\,[m-1, m-1, m-2r-1] - r\,[m-1, m-1, m-2r+1] \\
&= X_{\alpha+\beta} v
\end{aligned}
$$

$$
\begin{aligned}
[X_\alpha, X_{\alpha+\beta}]v &= X_\alpha((m-r)\,[m-1, m-1, m-2r-1] - r\,[m-1, m-1, m-2r+1]) \\
&- X_{\alpha+\beta}(r\,[m-1, m, m-2r]) \\
&= (m-r)r\,[m-2, m-1, m-2r-1] - r(r-1)\,[m-2, m-1, m-2r+1] \\
&- r(-(r-1)\,[m-2, m-1, m-2r+1] + (m-r)\,[m-2, m-1, m-2r-1]) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[X_\beta, X_{\alpha+\beta}]v &= X_\beta\,((m-r)\,[m-1, m-1, m-2r-1] - r\,[m-1, m-1, m-2r+1]) \\
&- X_{\alpha+\beta}\,((m-r)\,[m, m-1, m-2r]) \\
&= (m-r)(m-1-r)\,[m-1, m-2, m-2r-1] \\
&- r(m-r)\,[m-1, m-2, m-2r+1] \\
&- (m-r)(m-1-r)\,[m-1, m-2, m-2r-1] \\
&+ (m-r)r\,[m-1, m-2, m-2r+1] \\
&= 0
\end{aligned}
$$

This completes the proof of the Theorem.

■

We are now going to rewrite the X_η operators in terms of the more general coordinates $[m, n, l]$.

Proposition 17 *i) For $m > n$*

$$
\begin{aligned}
X_\alpha\,[m, n, l] &= (m - n + \frac{m-l}{2})\,[m-1, n, l-1] + \frac{m-l}{2}\,[m-1, n, l+1] \\
X_\beta\,[m, n, l] &= (n - \frac{m-l}{2})\,[m, n-1, l] \\
X_{\alpha+\beta}\,[m, n, l] &= (n - \frac{m-l}{2})\,[m-1, n-1, l-1] - \frac{m-l}{2}\,[m-1, n-1, l+1]
\end{aligned}
$$

ii) For $m < n$

$$X_\alpha [m,n,l] = (m - \frac{l+n}{2}) [m-1,n,l]$$
$$X_\beta [m,n,l] = (n-m+\frac{l+n}{2}) [m,n-1,l+1] + \frac{l+n}{2} [m,n-1,l-1]$$
$$X_{\alpha+\beta} [m,n,l] = -(m-\frac{l+n}{2}) [m-1,n-1,l+1] + \frac{l+n}{2} [m-1,n-1,l-1]$$

iii) For $m = n$

$$X_\alpha [m,m,l] = \frac{m-l}{2} [m-1,m,l]$$
$$X_\beta [m,m,l] = \frac{m+l}{2} [m,m-1,l]$$
$$X_{\alpha+\beta} [m,m,l] = \frac{m+l}{2} [m-1,m-1,l-1] - \frac{m-l}{2} [m-1,m-1,l+1] .$$

Proof. These formulae follow from the expressions derived in the proof of the previous theorem by making the obvious change of variables. ■

We will call V the **cone module** of the Heisenberg Lie algebra. The cone module is clearly not irreducible, as the operators X_α, X_β and $X_{\alpha+\beta}$ send any node v to a linear combination of those nodes directly below v and hence any ideal of the poset D_I spans a submodule. In particular, any diamond defines a submodule of the cone module, called a **diamond module**. Now we extend such a diamond module to a module for $\mathfrak{sl}_3$.

8 The operators Y_η

We are going to use the diamond involution to create a second set of directed edges on $D_I(a,b)$, and an associated second set of operators Y_η, for η one of α, β or $\alpha+\beta$. The new edges are defined to be the images of the existing edges under the diagonal involution $\tau = \tau(a,b)$, where we recall that

$$\tau [x,y,z] = [a+b-y, a+b-x, a-b+x-y-z] .$$

More precisely, if e is a directed edge from the node v to the node u then we define a directed edge $f = \tau e$ from τv to τu. We see that

$$p(v) - p(u) = \alpha \text{ if and only if } p(\tau v) - p(\tau u) = -\beta$$
$$p(v) - p(u) = \beta \text{ if and only if } p(\tau v) - p(\tau u) = -\alpha$$
$$p(v) - p(u) = \alpha+\beta \text{ if and only if } p(\tau v) - p(\tau u) = -(\alpha+\beta).$$

This motivates us to call $f = \tau e$ a Y_β**-edge** if e is an X_α-edge, a Y_α**-edge** if e is an X_β-edge, and a $Y_{\alpha+\beta}$**-edge** if e is an $X_{\alpha+\beta}$-edge. Thus $f = \tau e$ from v to u is a Y_η-edge if and only if

$$p(u) - p(v) = \eta$$

and we say that f has **direction** η. Thus e is an edge of direction $-\alpha$,$-\beta$, or $-(\alpha+\beta)$, if and only if $f = \tau e$ is an edge of direction β, α, or $\alpha+\beta$ respectively.

Note however that while an X_η-edge always has height of absolute value at most one, this is not generally true of the corresponding Y_η-edges unless $\eta = \alpha + \beta$.

We define the **weight**

$$w(f) = w(\tau f)$$

for all Y_η-edges f.

For η one of α, β or $\alpha + \beta$, define an operator Y_η on $V(a,b)$ by the rule

$$Y_\eta u = \sum_f w(f)v$$

where the sum is over all Y_η-edges f from u to some node v.

Proposition 18 *We have the identities*

$$\begin{aligned} Y_\alpha &= \tau X_\beta \tau \\ Y_\beta &= \tau X_\alpha \tau \\ Y_{\alpha+\beta} &= \tau X_{\alpha+\beta} \tau. \end{aligned}$$

Proof. *Immediate from the definition.* ■

Remark 19 *We could have used this as the definition of the Y_η but prefer to construct the appropriate edges on the graph first.*

Proposition 20 *The operators Y_α, Y_β, and $Y_{\alpha+\beta}$ on $V(a,b)$ satisfy the commutation relations $[Y_\beta, Y_\alpha] = Y_{\alpha+\beta}, [Y_\alpha, Y_{\alpha+\beta}] = [Y_{\alpha+\beta}, Y_\beta] = 0$.*

Proof. Since the operators $Y_\alpha, Y_\beta, Y_{\alpha+\beta}$ are conjugates of the operators $X_\beta, X_\alpha, X_{\alpha+\beta}$ respectively under the diagonal involution τ, the commutation relations of the Y_η operators follow from the commutation relations of the X_η operators given in Theorem 1, with the single change that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ becomes $[Y_\beta, Y_\alpha] = Y_{\alpha+\beta}$.
■

Proposition 21 *i) For $m > n$*

$$\begin{aligned} Y_\alpha [m,n,l] &= (a - \frac{m+l}{2})\,[m+1,n,l+1] \\ Y_\beta [m,n,l] &= \left(-n + b + \frac{m+l}{2}\right)[m,n+1,l] + \left(b - \frac{m-l}{2}\right)[m,n+1,l-2] \\ Y_{\alpha+\beta} [m,n,l] &= \left(a - \frac{m+l}{2}\right)[m+1,n+1,l+1] - \left(b - \frac{m-l}{2}\right)[m+1,n+1,l-1] \end{aligned}$$

ii) For $m < n$

$$Y_\alpha [m,n,l] = \left(a - m + \frac{n-l}{2}\right)[m+1,n,l] + (a - \frac{l+n}{2})[m+1,n,l+2]$$

$$Y_\beta [m,n,l] = \left(b + \frac{l-n}{2}\right)[m,n+1,l-1]$$

$$Y_{\alpha+\beta} [m,n,l] = \left(-b + \frac{n-l}{2}\right)[m+1,n+1,l-1] + \left(a - \frac{l+n}{2}\right)[m+1,n+1,l+1]$$

iii) For $m = n$

$$Y_\alpha [m,m,l] = \left(a - \frac{m+l}{2}\right)[m+1,m,l+1]$$

$$Y_\beta [m,m,l] = \left(b - \frac{m-l}{2}\right)[m,m+1,l-1]$$

$$Y_{\alpha+\beta} [m,m,l] = \left(a - \frac{m+l}{2}\right)[m+1,m+1,l+1] - \left(b - \frac{m-l}{2}\right)[m+1,m+1,l-1].$$

Proof. We use the form of the X_η operators conjugated by the involution τ.

Case 1. $v = [m,n,l]$ where $m > n$

$$\begin{aligned}Y_\alpha v &= \tau X_\beta \tau [m,n,l] = \tau X_\beta [a+b-n, a+b-m, a-b+m-n-l] \\ &= \left(a+b-m-\frac{2b-m+l}{2}\right)\tau[a+b-n, a+b-m-1, a-b+m-n-l] \\ &= \left(a - \frac{m+l}{2}\right)[m+1,n,l+1]\end{aligned}$$

$$\begin{aligned}Y_\beta v &= \tau X_\alpha \tau [m,n,l] = \tau X_\alpha [a+b-n, a+b-m, a-b+m-n-l] \\ &= \left(m-n+\frac{2b-m+l}{2}\right)\tau[a+b-n-1, a+b-m, a-b+m-n-l-1] \\ &+ \frac{2b-m+l}{2}\tau[a+b-n-1, a+b-m, a-b+m-n-l+1] \\ &= \left(b-n+\frac{m+l}{2}\right)[m,n+1,l] + \left(b - \frac{m-l}{2}\right)[m,n+1,l-2]\end{aligned}$$

$$\begin{aligned}Y_{\alpha+\beta} v &= \tau X_{\alpha+\beta} \tau [m,n,l] = \tau X_{\alpha+\beta} [a+b-n, a+b-m, a-b+m-n-l] \\ &= \left(a - \frac{m+l}{2}\right)\tau[a+b-n-1, a+b-m-1, a-b+m-n-l-1] \\ &- \frac{2b-m+l}{2}\tau[a+b-n-1, a+b-m-1, a-b+m-n-l+1] \\ &= \left(a - \frac{m+l}{2}\right)[m+1,n+1,l+1] - \left(b - \frac{m-l}{2}\right)[m+1,n+1,l-1]\end{aligned}$$

Case 2. $v = [m, n, l]$ where $m < n$

$$\begin{aligned}
Y_\alpha v &= \tau X_\beta \tau [m,n,l] = \tau X_\beta [a+b-n, a+b-m, a-b+m-n-l] \\
&= \left(n - m + \frac{2a-n-l}{2}\right)\tau [a+b-n, a+b-m-1, a-b+m-n-l+1] \\
&+ \frac{2a-n-l}{2}\tau [a+b-n, a+b-m-1, a-b+m-n-l-1] \\
&= \left(a - m + \frac{n-l}{2}\right)[m+1, n, l] + \left(a - \frac{n+l}{2}\right)[m+1, n, l+2]
\end{aligned}$$

$$\begin{aligned}
Y_\beta v &= \tau X_\alpha \tau [m,n,l] = \tau X_\alpha [a+b-n, a+b-m, a-b+m-n-l] \\
&= \left(a + b - n - \frac{2a-n-l}{2}\right)\tau [a+b-n-1, a+b-m, a-b+m-n-l] \\
&= \left(b - \frac{n-l}{2}\right)[m, n+1, l-1]
\end{aligned}$$

$$\begin{aligned}
Y_{\alpha+\beta} v &= \tau X_{\alpha+\beta} \tau [m,n,l] = \tau X_{\alpha+\beta} [a+b-n, a+b-m, a-b+m-n-l] \\
&= -\left(a + b - n - \frac{2a-n-l}{2}\right)\tau [a+b-n-1, a+b-m-1, a-b+m-n-l+1] \\
&+ \frac{2a-n-l}{2}\tau [a+b-n-1, a+b-m-1, a-b+m-n-l-1] \\
&= \left(-b + \frac{n-l}{2}\right)[m+1, n+1, l-1] + \left(a - \frac{n+l}{2}\right)[m+1, n+1, l+1]
\end{aligned}$$

Case 3. $v = [m, n, l]$ where $m = n$

$$\begin{aligned}
Y_\alpha v &= \tau X_\beta \tau [m,m,l] = \tau X_\beta [a+b-m, a+b-m, a-b-l] \\
&= \frac{2a-m-l}{2}\tau [a+b-m, a+b-m-1, a-b-l] \\
&= \left(a - \frac{m+l}{2}\right)[m+1, m, l+1]
\end{aligned}$$

$$\begin{aligned}
Y_\beta v &= \tau X_\alpha \tau [m,m,l] = \tau X_\alpha [a+b-m, a+b-m, a-b-l] \\
&= \frac{2b-m+l}{2}\tau [a+b-m-1, a+b-m, a-b-l] \\
&= \left(b - \frac{m-l}{2}\right)[m, m+1, l-1]
\end{aligned}$$

$$
\begin{aligned}
Y_{\alpha+\beta}v &= \tau X_{\alpha+\beta}\tau\,[m,m,l] = \tau X_{\alpha+\beta}\,[a+b-m,a+b-m,a-b-l] \\
&= \frac{2a-m-l}{2}\tau\,[a+b-m-1,a+b-m-1,a-b-l-1] \\
&- \frac{2b-m+l}{2}\tau\,[a+b-m-1,a+b-m-1,a-b-l+1] \\
&= \left(a-\frac{m+l}{2}\right)[m+1,m+1,l+1] - \left(b-\frac{m-l}{2}\right)[m+1,m+1,l-1]
\end{aligned}
$$

■

9 Brackets of raising and lowering operators

We are now going to compute the operators $H_\eta = [X_\eta, Y_\eta] = X_\eta Y_\eta - Y_\eta X_\eta$ on $V(a,b)$, for η one of α, β or $\alpha+\beta$.

Proposition 22 *Each of the operators H_η acts diagonally on the nodes of the diamond, in particular*

$$
\begin{aligned}
H_\alpha\,[m,n,l] &= (a-2m+n)\,[m,n,l] \\
H_\beta\,[m,n,l] &= (b-2n+m)\,[m,n,l] \\
H_{\alpha+\beta}\,[m,n,l] &= (a+b-m-n)\,[m,n,l]\,.
\end{aligned}
$$

Thus

$$
H_{\alpha+\beta} = H_\alpha + H_\beta.
$$

Proof. We proceed by treating cases separately.

Case 1. $v = [m,n,l]$ where $m > n$

$$
\begin{aligned}
H_\alpha v &= [X_\alpha, Y_\alpha]\,v = X_\alpha Y_\alpha v - Y_\alpha X_\alpha v \\
&= (a-\frac{m+l}{2})X_\alpha\,[m+1,n,l+1] - (m-n+\frac{m-l}{2})Y_\alpha\,[m-1,n,l-1] \\
&- (\frac{m-l}{2})Y_\alpha\,[m-1,n,l+1] \\
&= (a-\frac{m+l}{2})((m+1-n+\frac{m-l}{2})\,[m,n,l] + \frac{m-l}{2}\,[m,n,l+2]) \\
&- (m-n+\frac{m-l}{2})(a-\frac{m+l}{2}+1)\,[m,n,l] - (\frac{m-l}{2})(a-\frac{m+l}{2})\,[m,n,l+2] \\
&= (a-2m+n)\,[m,n,l]
\end{aligned}
$$

$$
\begin{aligned}
H_\beta v = [X_\beta, Y_\beta]\, v = &\left(-n + n_2 + \frac{m+l}{2}\right) X_\beta\, [m, n+1, l] \\
&+ \left(b - \frac{m-l}{2}\right) X_\beta\, [m, n+1, l-2] - (n - \frac{m-l}{2}) Y_\beta\, [m, n-1, l] \\
&= \left(-n + b + \frac{m+l}{2}\right)(n + 1 - \frac{m-l}{2})\, [m, n, l] + \left(b - \frac{m-l}{2}\right)(n - \frac{m-l}{2})\, [m, n, l-2] \\
&- (n - \frac{m-l}{2})(\left(-n + 1 + b + \frac{m+l}{2}\right)[m, n, l] + \left(b - \frac{m-l}{2}\right)[m, n, l-2]) \\
&= (b - 2n + m)\, [m, n, l]
\end{aligned}
$$

$$
\begin{aligned}
H_{\alpha+\beta} v = [X_{\alpha+\beta}, Y_{\alpha+\beta}]\, v = &\left(a - \frac{m+l}{2}\right) X_{\alpha+\beta}\, [m+1, n+1, l+1] \\
&- \left(b - \frac{m-l}{2}\right) X_{\alpha+\beta}\, [m+1, n+1, l-1] \\
&- Y_{\alpha+\beta}((n - \frac{m-l}{2})\, [m-1, n-1, l-1] - \frac{m-l}{2}\, [m-1, n-1, l+1]) \\
&= \left(a - \frac{m+l}{2}\right)\left((n + 1 - \frac{m-l}{2})\, [m, n, l] - \frac{m-l}{2}\, [m, n, l+2]\right) \\
&- \left(b - \frac{m-l}{2}\right)\left((n - \frac{m-l}{2})\, [m, n, l-2] - (\frac{m-l}{2} + 1)\, [m, n, l]\right) \\
&- (n - \frac{m-l}{2})\left(\left(a - \frac{m+l}{2} + 1\right)[m, n, l] - \left(b - \frac{m-l}{2}\right)[m, n, l-2]\right) \\
&+ \left(\frac{m-l}{2}\right)(\left(a - \frac{m+l}{2}\right)[m, n, l+2] - \left(b - \frac{m-l}{2} + 1\right)[m, n, l]) \\
&= (a + b - m - n)\, [m, n, l]
\end{aligned}
$$

Case 2. $v = [m, n, l]$ where $m < n$

$$
\begin{aligned}
H_\alpha v = [X_\alpha, Y_\alpha]\, v = &\left(a - m + \frac{n-l}{2}\right) X_\alpha\, [m+1, n, l] \\
&+ (a - \frac{l+n}{2}) X_\alpha\, [m+1, n, l+2] - (m - \frac{l+n}{2}) Y_\alpha\, [m-1, n, l] \\
&= \left(a - m + \frac{n-l}{2}\right)(m + 1 - \frac{l+n}{2})\, [m, n, l] \\
&+ (a - \frac{l+n}{2})(m - \frac{l+n}{2})\, [m, n, l+2] \\
&- (m - \frac{l+n}{2})\left(\left(a - m + 1 + \frac{n-l}{2}\right)[m, n, l] + (a - \frac{l+n}{2})\, [m, n, l+2]\right) \\
&= (a - 2m + n)\, [m, n, l]
\end{aligned}
$$

$$
\begin{aligned}
H_\beta v = [X_\beta, Y_\beta]\, v &= \left(b + \frac{l-n}{2}\right) X_\beta\, [m, n+1, l-1] \\
&- ((n - m + \frac{l+n}{2}) Y_\beta\, [m, n-1, l+1] + (\frac{l+n}{2}) Y_\beta\, [m, n-1, l-1]) \\
&= \left(b + \frac{l-n}{2}\right) ((n + 1 - m + \frac{l+n}{2})\, [m, n, l] + (\frac{l+n}{2})\, [m, n, l-2]) \\
&- (n - m + \frac{l+n}{2}) \left(b + \frac{l-n}{2} + 1\right) [m, n, l] - (\frac{l+n}{2}) \left(b + \frac{l-n}{2}\right) [m, n, l-2] \\
&= (b - 2n + m)\, [m, n, l]
\end{aligned}
$$

$$
\begin{aligned}
H_{\alpha+\beta} v = [X_{\alpha+\beta}, Y_{\alpha+\beta}]\, v &= \left(-b + \frac{n-l}{2}\right) X_{\alpha+\beta}\, [m+1, n+1, l-1] \\
&+ \left(a - \frac{l+n}{2}\right) X_{\alpha+\beta}\, [m+1, n+1, l+1] \\
&+ (m - \frac{l+n}{2}) Y_{\alpha+\beta}\, [m-1, n-1, l+1] - \frac{l+n}{2} Y_{\alpha+\beta}\, [m-1, n-1, l-1] \\
&= \left(-b + \frac{n-l}{2}\right) (-(m + 1 - \frac{l+n}{2})\, [m, n, l] + \frac{l+n}{2}\, [m, n, l-2]) \\
&+ \left(a - \frac{l+n}{2}\right) (-(m - \frac{l+n}{2})\, [m, n, l+2] + (\frac{l+n}{2} + 1)\, [m, n, l]) \\
&+ (m - \frac{l+n}{2}) (\left(-b + \frac{n-l}{2} - 1\right) [m, n, l] + \left(a - \frac{l+n}{2}\right) [m, n, l+2]) \\
&- \frac{l+n}{2} (\left(-b + \frac{n-l}{2}\right) [m, n, l-2] + \left(a - \frac{l+n}{2} + 1\right) [m, n, l]) \\
&= (a + b - m - n)\, [m, n, l]
\end{aligned}
$$

Case 3. $v = [m, n, l]$ where $m = n$

$$
\begin{aligned}
H_\alpha(v) &= [X_\alpha, Y_\alpha]\, v \\
&= \left(a - m + \frac{m-l}{2}\right) X_\alpha\, [m+1, m, l+1] - \frac{m-l}{2} Y_\alpha\, [m-1, m, l] \\
&= \left(a - m + \frac{m-l}{2}\right) ((1 + \frac{m-l}{2})\, [m, m, l] + \frac{m-l}{2}\, [m, m, l+2]) \\
&- \left(\frac{m-l}{2}\right) (\left(a - m + 1 + \frac{m-l}{2}\right) [m, m, l] + (a - \frac{l+m}{2})\, [m, m, l+2]) \\
&= (a - m)\, [m, m, l]
\end{aligned}
$$

$$
\begin{aligned}
H_\beta v &= [X_\beta, Y_\beta]\, v \\
&= \left(b - \frac{m-l}{2}\right) X_\beta \,[m, m+1, l-1] - (\frac{m+l}{2}) Y_\beta \,[m, m-1, l] \\
&= \left(b - \frac{m-l}{2}\right) ((1 + \frac{l+m}{2})\,[m,m,l] + (\frac{l+m}{2})\,[m,m,l-2]) \\
&- (\frac{m+l}{2})(\left(-m+1+b+\frac{m+l}{2}\right)[m,m,l] + \left(b - \frac{m-l}{2}\right)[m,m,l-2]) \\
&= (b-m)\,[m,m,l]
\end{aligned}
$$

$$
\begin{aligned}
H_{\alpha+\beta} v = [X_{\alpha+\beta}, Y_{\alpha+\beta}]\, v &= \left(a - \frac{m+l}{2}\right) X_{\alpha+\beta}\,[m+1, m+1, l+1] \\
&- \left(b - \frac{m-l}{2}\right) X_{\alpha+\beta}\,[m+1, m+1, l-1] \\
&- (\frac{m+l}{2}) Y_{\alpha+\beta}\,[m-1, m-1, l-1] + \frac{m-l}{2} Y_{\alpha+\beta}\,[m-1, m-1, l+1] \\
&= \left(a - \frac{m+l}{2}\right)((\frac{m+l}{2}+1)\,[m,m,l] - \left(\frac{m-l}{2}\right)[m,m,l+2]) \\
&- \left(b - \frac{m-l}{2}\right)((\frac{m+l}{2})\,[m,m,l-2] - \left(\frac{m-l}{2}+1\right)[m,m,l]) \\
&- (\frac{m+l}{2})(\left(a - \frac{m+l}{2}+1\right)[m,m,l] - \left(b - \frac{m-l}{2}\right)[m,m,l-2]) \\
&+ \left(\frac{m-l}{2}\right)(\left(a - \frac{m+l}{2}\right)[m,m,l+2] - \left(b - \frac{m-l}{2}+1\right)[m,m,l]) \\
&= (a+b-2m)\,[m,m,l]
\end{aligned}
$$

■

10 The main theorem

Proposition 23 *On $V(a,b)$ we have $[X_\alpha, Y_\beta] = [X_\beta, Y_\alpha] = 0$.*

Proof. We verify that $[X_\alpha, Y_\beta]\, v = 0$ for all v. The second equation then follows by using the involution $[m,n,l] \to [n,m,-l]$.

Case 1. $v = [m, n, l]$ where $n + 1 < m$

$$
\begin{aligned}
[X_\alpha, Y_\beta]\, v &= X_\alpha Y_\beta v - Y_\beta X_\alpha v \\
&= \left(-n + b + \frac{m+l}{2}\right) X_\alpha\, [m, n+1, l] + \left(b - \frac{m-l}{2}\right) X_\alpha\, [m, n+1, l-2] \\
&- (m - n + \frac{m-l}{2}) Y_\beta\, [m-1, n, l-1] - (\frac{m-l}{2}) Y_\beta\, [m-1, n, l+1] \\
&= \left(-n + b + \frac{m+l}{2}\right) \begin{pmatrix} (m - n - 1 + \frac{m-l}{2})\, [m-1, n+1, l-1] \\ + \frac{m-l}{2}\, [m-1, n+1, l+1] \end{pmatrix} \\
&+ \left(b - \frac{m-l}{2}\right) \begin{pmatrix} (m - n + \frac{m-l}{2})\, [m-1, n+1, l-3] \\ + \frac{m-l+2}{2}\, [m-1, n+1, l-1] \end{pmatrix} \\
&- (m - n + \frac{m-l}{2}) \begin{pmatrix} \left(-n + b + \frac{m+l}{2} - 1\right) [m-1, n+1, l-1] \\ + \left(b - \frac{m-l}{2}\right) [m-1, n+1, l-3] \end{pmatrix} \\
&- (\frac{m-l}{2}) \begin{pmatrix} \left(-n + b + \frac{m+l}{2}\right) [m-1, n+1, l+1] \\ + \left(b - \frac{m-l}{2} + 1\right) [m-1, n+1, l-1] \end{pmatrix} \\
&= 0
\end{aligned}
$$

Case 2. $v = [m, n, l]$ where $n + 1 = m$ and $r \le n$.

$$
\begin{aligned}
[X_\alpha, Y_\beta]\, v &= X_\alpha Y_\beta v - Y_\beta X_\alpha v \\
&= \left(-n + b + \frac{m+l}{2}\right) X_\alpha\, [m, m, l] + \left(b - \frac{m-l}{2}\right) X_\alpha\, [m, m, l-2] \\
&- (1 + \frac{m-l}{2}) Y_\beta\, [m-1, m-1, l-1] - (\frac{m-l}{2}) Y_\beta\, [m-1, m-1, l+1] \\
&= \left(-m + 1 + b + \frac{m+l}{2}\right) \left(\frac{m-l}{2}\right) [m-1, m, l] \\
&+ \left(b - \frac{m-l}{2}\right) \left(\frac{m-l}{2} + 1\right) [m-1, m, l-2] \\
&- (1 + \frac{m-l}{2}) \left(b - \frac{m-l}{2}\right) [m-1, m, l-2] \\
&- (\frac{m-l}{2}) \left(b - \frac{m-l-2}{2}\right) [m-1, m, l] \\
&= 0
\end{aligned}
$$

Case 3. $v = [m, n, l]$ where $m < n$

$$\begin{aligned}[X_\alpha, Y_\beta]\, v &= X_\alpha Y_\beta v - Y_\beta X_\alpha v \\ &= \left(b + \frac{l-n}{2}\right) X_\alpha\, [m, n+1, l-1] - (m - \frac{l+n}{2}) Y_\beta\, [m-1, n, l] \\ &= \left(b + \frac{l-n}{2}\right)\left(m - \frac{l+n}{2}\right) [m-1, n+1, l-1] \\ &- (m - \frac{l+n}{2}) \left(b + \frac{l-n}{2}\right) [m-1, n+1, l-1] \\ &= 0\end{aligned}$$

Case 4. $v = [m, n, l]$ where $m = n$.

$$\begin{aligned}[X_\alpha, Y_\beta]\, v &= X_\alpha Y_\beta v - Y_\beta X_\alpha v \\ &= \left(b - \frac{m-l}{2}\right) X_\alpha\, [m, m+1, l-1] - (\frac{m-l}{2}) Y_\beta\, [m-1, m, l] \\ &= \left(b - \frac{m-l}{2}\right)\left(\frac{m-l}{2}\right) [m-1, m+1, l-1] \\ &- (\frac{m-l}{2}) \left(b + \frac{l-m}{2}\right) [m-1, m+1, l-1] \\ &= 0\end{aligned}$$

■

Theorem 24 *The span of the nine operators X_η, Y_η, and H_η on $V(a, b)$, where η is one of α, β or $\alpha + \beta$, is closed under brackets and forms a Lie algebra isomorphic to $\mathfrak{sl}_3$.*

Proof. We have already verified that the X_η operators are closed under brackets, as are the Y_η operators, and that $[X_\eta, Y_\eta] = H_\eta$. Since each of the H_η operators act diagonally, they commute amongst each other. To obtain the bracket of an H_η operator with one of the X_σ, we observe that if say $X_\sigma(v) = au + bw$, with v, u, w nodes in $D_I(a, b)$, then $p(u) = p(w) = p(v) - \sigma$ so that $\lambda_\eta(u) = \lambda_\eta(w) = \lambda_\eta(v) - A_\eta(\sigma)$, where A_η is the linear part of λ_η. Thus

$$\begin{aligned}[H_\eta, X_\sigma](v) &= H_\eta\,(au + bw) - X_\sigma \lambda_\eta(v) v \\ &= aH_\eta u + bH_\eta w - \lambda_\eta(v)\,(au + bw) \\ &= a\lambda_\eta(u)u + b\lambda_\eta(w)w - \lambda_\eta(v)\,(au + bw) \\ &= (\lambda_\eta(u) - \lambda_\eta(v))\,(au + bw) \\ &= -A_\eta \sigma X_\sigma v.\end{aligned}$$

Now from the proof of the previous Lemma,

$$\begin{aligned}A_\alpha\,[m, n, l] &= -2m + n \\ A_\beta\,[m, n, l] &= -2n + m \\ A_{\alpha+\beta}\,[m, n, l] &= -m - n\end{aligned}$$

Thus

$$\begin{array}{lll} [H_\alpha, X_a] = 2X_\alpha & [H_\beta, X_a] = -X_\alpha & [H_{\alpha+\beta}, X_a] = X_\alpha \\ [H_\alpha, X_\beta] = -X_\alpha & [H_\beta, X_\beta] = 2X_\beta & [H_{\alpha+\beta}, X_\beta] = X_\beta \\ [H_\alpha, X_{\alpha+\beta}] = X_{\alpha+\beta} & [H_\beta, X_{\alpha+\beta}] = X_{\alpha+\beta} & [H_{\alpha+\beta}, X_{a+\beta}] = 2X_{\alpha+\beta} \end{array}$$

Similiarly, by using the involution $[m,n,l] \to [n,m,-l]$ on $V(b,a)$ we deduce that the operators $[H_\eta, Y_\lambda]$ are always multiples of $Y_{\lambda.}$

From the Jacobi identity (valid since we are dealing with operators) and the previous Proposition, we find that

$$\begin{aligned} [X_{\alpha+\beta}, Y_\alpha] &= [[X_\alpha, X_\beta], Y_\alpha] \\ &= [[X_\alpha, Y_\alpha], X_\beta] + [[Y_\alpha, X_\beta], X_\alpha,] \\ &= [H_\alpha, X_\beta] = -X_\beta \end{aligned}$$

and similiarly

$$\begin{aligned} [X_{\alpha+\beta}, Y_\beta] &= -X_\alpha \\ [Y_{\alpha+\beta}, X_\beta] &= -Y_\alpha \\ [Y_{\alpha+\beta}, X_\beta] &= -Y_\beta. \end{aligned}$$

This completes the check of all the commutation relations. ■

11 Construction of physical models of diamonds

A commercially available construction kit which is well adapted to producing models of diamonds is the ZOME system, consisting of yellow, red, and blue struts which connect small white balls with different shaped and positioned holes. To create diamonds, take a number of red struts and blue struts of approximately the same size, together with some yellow struts of approximately half this size. Let vectors $[1,1,1]$ and $[1,1,-1]$ be represented by red struts making an angle of $2\pi/3$, let vectors $[1,0,1]$ and $[0,-1,-1]$ be represented by blue struts which form an angle of $\pi/6$ with the aforementioned red struts, and let vectors $[1,0,0]$ and $[0,1,0]$ be represented by yellow struts, which connect the ends of an adjacent red/blue pair.

Diagonal slices correspond to planes formed by red struts, which thus indicate $X_{\alpha+\beta}$ edges. On one side of the spindle (lying over the main diagonal) X_α-edges are represented by blue edges while X_β-edges are represented by yellow edges. On the other side of the spindle the situation is reversed.

The resulting models, which the reader may easily construct, are not perfect in the sense that the projection onto the $x-y$ plane yields polytopes which are only approximations to hexagons with full S_3 symmetry. This is due to the particular choice of struts used, and could probably be remedied by using the more advanced green system. Nevertheless the approximation is a reasonably good one and allows other interesting features of these polytopes to be investigated concretely.

References

[BR] H. Barcelo and R. Arun, Combinatorial representation theory. New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), 23–90, Math. Sci. Res. Inst. Publ. 38, Cambridge Univ. Press, Cambridge, 1999.

[BZ] A. D. Berenstein and A. V. Zelevinsky, Tensor product multiplicities and convex polytopes in partition space, J. Geom. Phys. 5 (1998), no.3, 453–472.

[BGR] S. Billey, V. Guillemin and E. Rassart, A vector partition function for the multiplicities of sl_k (■), preprint 2003.

[Do] R. Donnelly, Explicit properties of bases for representations of semisimple Lie algebras, to appear, J. Alg. Combin.

[GT] I. M. Gelfand and M. L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices, Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 1017-1020.

[Hu] J. E. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics 9, Springer-Verlag, New York-Berlin, 1978.

[Ka] M. Kashiwara, Crystallizing the q-analogue of universal enveloping algebras, in Proceedings of the INternational Congress of Mathematicians, Kyoto, 1990.

[KTT] R. C. King, C. Tollu and F. Toumazet, Stretched Littlewood-Richardson and Kostka coefficients, to appear in CRM Proceedings and Lecture Notes, Vol. 34, 2003.

[Ki] A. N. Kirillov, Ubiquity of Kostka polynomials, Physics and combinatorics 1999 (Nagoya), 85–200, World Sci. Publishing, River Edge, NJ, 2001

[KT] A. Knutson and T. Tao, The honeycomb model of GL_n (■) tensor products I: Proof of the Saturation Conjecture, J. Amer. Math. Soc. **12** (1999), 1055–1090.

[Li] P. Littelmann, Paths and root operators in representation theory, Ann. Math. (2) **142** (1995), 411–525.

[Lu] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, Boston, 1993.

[NZ] T. Nakashima and A. Zelevinsky, Polyhedral realizations of crystal bases for quantized Kac-Moody algebras, Adv. Math. **131** (1997), no,1, 253–278.

[St] S. Sternberg, Group theory and physics, Cambridge University Press 1994.

[Wi1] N. J. Wildberger, A combinatorial construction for simply-laced Lie algebras, Adv. Appl. Math. **30** (2003) 385–396.

[Wi2] N. J. Wildberger, Minuscule posets from neighbourly graph sequences, Europ. J. Combin. **24** (2003) 741–757.

[Wi3] N. J. Wildberger, A combinatorial construction of G_2, J. Lie Theory **13** (2003) 155–165.