ON THE EXPANSION OF A CONTINUED FRACTION OF GORDON

Michael D. Hirschhorn

m.hirschhorn@unsw.edu.au
School of Mathematics, UNSW, Sydney, Australia 2052

Abstract. We show that when certain infinite products associated with a continued fraction of Basil Gordon are expanded as power series, the sign of the coefficients is periodic, with period 8.

Key words: Gordon's continued fraction, power series expansion, periodicity of sign of coefficients, Jacobi's triple product.

2000 Mathematics Subject Classification: Primary–30B70

1. Introduction and statement of theorem

Basil Gordon [3] showed that

$$1+q+\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\,\frac{q^6}{1+q^7+\ \cdots}=\left(\begin{matrix} q^3, q^5 \\ q,\ q^7 \end{matrix};q^8\right)_\infty . \tag{1}$$

Richmond and Szekeres [4] consider the power series expansion of the infinite product on the right of (1), and of its reciprocal. Thus, let

$$\left(\begin{matrix} q^3, q^5 \\ q,\ q^7 \end{matrix};q^8\right)_\infty=\sum_{n\geq 0}a_nq^n \tag{2}$$

and

$$\left(\begin{matrix} q,\ q^7 \\ q^3, q^5 \end{matrix};q^8\right)_\infty=\sum_{n\geq 0}b_nq^n. \tag{3}$$

Using the Hardy–Ramanujan–Rademacher circle method they obtain series representations for a_n and b_n valid from some point on, and prove the striking result that for $n \geq 0$,

$$a_{4n+3} = 0, \quad b_{4n+2} = 0. \tag{4}$$

This result has inspired two further papers, by Andrews and Bressoud [2] and by Alladi and Gordon [1]. In each of these, the authors consider infinite products more general than those appearing in (2) and (3), and, using Ramanujan's ${}_1\psi_1$ summation, prove general results which include (4) .

The object of this note is not only to give a proof of (4), but also to show that the sign of each of a_n and b_n is periodic with period 8. Indeed, using only Jacobi's triple product identity, we prove the following result.

Theorem

With a_n and b_n defined by (2) and (3), we have, for $n \geq 0$

$$\begin{aligned} a_{8n} > 0,\ a_{8n+1} > 0,\ a_{8n+2} > 0,\ a_{8n+3} = 0,\\ a_{8n+12} < 0,\ a_{8n+5} < 0,\ a_{8n+6} < 0,\ a_{8n+7} = 0,\end{aligned}$$

$$\begin{aligned} b_{8n} > 0,\ b_{8n+1} < 0,\ b_{8n+2} = 0,\ b_{8n+3} > 0,\\ b_{8n+4} < 0,\ b_{8n+5} > 0,\ b_{8n+6} = 0,\ b_{8n+7} < 0.\end{aligned}$$

2. Proof of theorem

We start by proving

$$\begin{aligned} (q^3,q^5,q^8;q^8)_\infty^2 + q(q,q^7,q^8;q^8)_\infty^2 &= \frac{(q^2)_\infty^4}{(q)_\infty(q^4)_\infty},\\ (q^3,q^5,q^8;q^8)_\infty^2 - q(q,q^7,q^8;q^8)_\infty^2 &= \frac{(q)_\infty(q^4)_\infty^6}{(q^2)_\infty^3(q^8)_\infty^2}.\end{aligned} \tag{5}$$

We have

$$\begin{aligned} (q^3,q^5,q^8;q^8)_\infty^2 &= \left\{\sum_{-\infty}^{\infty}(-1)^n q^{4n^2-n}\right\}^2\\ &= \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{4m^2-m+4n^2-n}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\equiv n \pmod 2} q^{4m^2-m+4n^2-n} - \sum_{m\not\equiv n \pmod 2} q^{4m^2-m+4n^2-n} \\
&= \sum_{u,v=-\infty}^{\infty} q^{4(u+v)^2-(u+v)+4(u-v)^2-4(u-v)} \\
&\quad - \sum_{u,v=-\infty}^{\infty} q^{4(u+v+1)^2-4(u+v+1)+4(u-v)^2-(u-v)} \\
&= \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v} \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
(q,q^7,q^8;q^8)_\infty^2 &= \sum_{u,v=-\infty}^{\infty} q^{4(u+v)^2-3(u+v)+4(u-v)^2-3(u-v)} \\
&\quad - \sum_{u,v=-\infty}^{\infty} q^{4(u+v+1)^2-3(u+v+1)+4(u-v)^2-3(u-v)} \\
&= \sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q \sum_{u,v=-\infty}^{\infty} q^{8u^2+2u+8v^2+8v}. \qquad (7)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(q^3,q^5,q^8;q^8)_\infty^2 + q(q,q^7,q^8;q^8)_\infty^2 \\
&\qquad = \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^2 \sum_{u,v=-\infty}^{\infty} q^{8u^2+2u+8v^2+8v} \\
&\qquad\quad + q \sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v} \\
&\qquad = \sum_{-\infty}^{\infty} q^{8u^2-2u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\
&\qquad\quad + q \sum_{-\infty}^{\infty} q^{8u^2-6u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\
&\qquad = \sum_{-\infty}^{\infty} (-1)^v q^{2v^2} \left\{ \sum_{-\infty}^{\infty} q^{8u^2-2u} + \sum_{-\infty}^{\infty} q^{8u^2-6u+1} \right\}
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{-\infty}^{\infty} (-1)^v q^{2v^2} \sum_{-\infty}^{\infty} q^{2u^2-u} \\
&= \prod_{n\geq 1} \frac{1-q^{2n}}{1+q^{2n}} \prod_{n\geq 1} \frac{(1-q^{2n})^2}{1-q^n} \\
&= \frac{(q^2)_\infty^4}{(q)_\infty (q^4)_\infty}
\end{aligned}$$

and

$$\begin{aligned}
&(q^3,q^5,q^8;q^8)_\infty^2 - q(q,q^7,q^8;q^8)_\infty^2 \\
&\qquad = \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} + q^2 \sum_{u,v=-\infty}^{\infty} q^{8u^2+2u+8v^2+8v} \\
&\qquad\qquad - q \sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v} \\
&\qquad = \sum_{-\infty}^{\infty} q^{8u^2-2u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} + \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\
&\qquad\qquad - q \sum_{-\infty}^{\infty} q^{8u^2-6u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} + \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\
&\qquad = \sum_{-\infty}^{\infty} q^{2v^2} \left\{ \sum_{-\infty}^{\infty} q^{8u^2-2u} - \sum_{-\infty}^{\infty} q^{8u^2-6u+1} \right\} \\
&\qquad = \sum_{-\infty}^{\infty} q^{2v^2} \sum_{-\infty}^{\infty} (-1)^u q^{2u^2-u} \\
&\qquad = \frac{(q^4)_\infty^5}{(q^2)_\infty^2 (q^8)_\infty^2} \frac{(q)_\infty (q^4)_\infty}{(q^2)_\infty} \\
&\qquad = \frac{(q)_\infty (q^4)_\infty^6}{(q^2)_\infty^3 (q^8)_\infty^2}.
\end{aligned}$$

If we divide (5) by

$$(q,q^3,q^4,q^5,q^7,q^8;q^8)_\infty = (q,q^3,q^4;q^4)_\infty = \frac{(q)_\infty (q^4)_\infty}{(q^2)_\infty}$$

we obtain

$$\begin{pmatrix} q^3, q^5, q^8 \\ q,\ q^4, q^7 \end{pmatrix} ; q^8 \Bigg)_\infty + q \begin{pmatrix} q,\ q^7, q^8 \\ q^3, q^4, q^5 \end{pmatrix} ; q^8 \Bigg)_\infty = \frac{(q^2)_\infty^5}{(q)_\infty^2 (q^4)_\infty^2} = \sum_{-\infty}^{\infty} q^{n^2}, \tag{8}$$

$$\left(\begin{matrix} q^3, q^5, q^8 \\ q,\ q^4, q^7 \end{matrix} ; q^8\right)_\infty - q\left(\begin{matrix} q,\ q^7, q^8 \\ q^3, q^4, q^5 \end{matrix} ; q^8\right)_\infty = \frac{(q^4)_\infty^5}{(q^2)_\infty^2 (q^8)_\infty^2} = \sum_{-\infty}^{\infty} q^{2n^2}.$$

It follows that

$$\left(\begin{matrix} q^3, q^5 \\ q,\ q^7 \end{matrix} ; q^8\right)_\infty = \frac{(q^4)_\infty}{(q^8)_\infty^2} \left\{ 1 + \sum_{n\geq 1} q^{n^2} + \sum_{n\geq 1} q^{2n^2} \right\}, \tag{9}$$

$$\left(\begin{matrix} q,\ q^7 \\ q^3, q^5 \end{matrix} ; q^8\right)_\infty = \frac{(q^4)_\infty}{(q^8)_\infty^2} \left\{ \sum_{n\geq 1} q^{n^2-1} - \sum_{n\geq 1} q^{2n^2-1} \right\}.$$

Since $n^2 \not\equiv 3 \pmod 4$ and $2n^2 \not\equiv 3 \pmod 4$, it is clear that (4) holds.

If we consider n even, n odd in the sums, we find

$$\left(\begin{matrix} q^3, q^5 \\ q,\ q^7 \end{matrix} ; q^8\right)_\infty = \frac{(q^4)_\infty}{(q^8)_\infty^2} \left\{ 1 + \sum_{n\geq 1} q^{4n^2} + \sum_{n\geq 1} q^{8n^2} + \sum_{n\geq 0} q^{4n^2+4n+1} + \sum_{n\geq 0} q^{8n^2+8n+2} \right\}, \tag{10}$$

$$\left(\begin{matrix} q,\ q^7 \\ q^3, q^5 \end{matrix} ; q^8\right)_\infty = \frac{(q^4)_\infty}{(q^8)_\infty^2} \left\{ \sum_{n\geq 1} q^{4n^2-1} - \sum_{n\geq 1} q^{8n^2-1} + \sum_{n\geq 0} q^{4n^2+4n} - \sum_{n\geq 0} q^{8n^2+8n+1} \right\}.$$

Now by (9) with q^4 for q,

$$1 + \sum_{n\geq 1} q^{4n^2} + \sum_{n\geq 1} q^{8n^2} = \left(\begin{matrix} q^{12}, q^{20}, q^{32} \\ q^4,\ q^{16}, q^{28} \end{matrix} ; q^{32}\right)_\infty,$$

$$\sum_{n\geq 1} q^{4n^2} - \sum_{n\geq 1} q^{8n^2} = q^4 \left(\begin{matrix} q^4,\ q^{28}, q^{32} \\ q^{12}, q^{16}, q^{20} \end{matrix} ; q^{32}\right)_\infty,$$

so (10) becomes

$$\left(\begin{matrix} q^3, q^5 \\ q,\ q^7 \end{matrix} ; q^8\right)_\infty = \left(\begin{matrix} q^{12}, q^{12}, q^{20}, q^{20} \\ q^8,\ q^{16}, q^{16}, q^{24} \end{matrix} ; q^{32}\right)_\infty + q\left(\begin{matrix} q^4,\ q^{12}, q^{20}, q^{28} \\ q^8,\ q^8,\ q^{24}, q^{24} \end{matrix} ; q^{32}\right)_\infty + q^2\left(\begin{matrix} q^4,\ q^{12}, q^{20}, q^{28} \\ q^8,\ q^{16}, q^{16}, q^{24} \end{matrix} ; q^{32}\right)_\infty, \tag{11}$$

$$\left(\begin{matrix} q,\ q^7 \\ q^3, q^5 \end{matrix} ; q^8\right)_\infty = \left(\begin{matrix} q^4,\ q^{12}, q^{20}, q^{28} \\ q^8,\ q^8,\ q^{24}, q^{24} \end{matrix} ; q^{32}\right)_\infty - q\left(\begin{matrix} q^4,\ q^{12}, q^{20}, q^{28} \\ q^8,\ q^{16}, q^{16}, q^{24} \end{matrix} ; q^{32}\right)_\infty + q^3\left(\begin{matrix} q^4,\ q^4,\ q^{28}, q^{28} \\ q^8,\ q^{16}, q^{16}, q^{24} \end{matrix} ; q^{32}\right)_\infty.$$

Each of the products on the right of (11) can be separated into powers congruent to 0 or 4 (mod 8) as follows. For example,

$$\begin{aligned}\left(\begin{matrix} q^{12}, q^{12}, q^{20}, q^{20} \\ q^{8},\ q^{16}, q^{16}, q^{24} \end{matrix}; q^{32}\right)_\infty &= \left(\begin{matrix} q^{12}, q^{12}, q^{20}, q^{20}, q^{32}, q^{32} \\ q^{8},\ q^{16}, q^{16}, q^{24}, q^{32}, q^{32} \end{matrix}; q^{32}\right)_\infty \\ &= \frac{(q^{12}, q^{20}, q^{32}; q^{32})_\infty^2}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty}\end{aligned}$$

which by (7) with q^4 for q becomes

$$\frac{(-q^{24}, -q^{32}, -q^{32}, -q^{40}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} - 2q^{12}\frac{(-q^{8}, -q^{56}, -q^{64}, -q^{64}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty}.$$

It follows that

$$\begin{aligned}&\left(\begin{matrix} q^{3}, q^{5} \\ q,\ q^{7} \end{matrix}; q^{8}\right)_\infty \\ &= \frac{(-q^{24}, -q^{32}, -q^{32}, -q^{40}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} + q\frac{(-q^{16}, -q^{24}, -q^{40}, -q^{48}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{8}, q^{24}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} \\ &+ q^2\frac{(-q^{16}, -q^{24}, -q^{40}, -q^{48}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} - 2q^{12}\frac{(-q^{8}, -q^{56} - q^{64}, -q^{64}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} \\ &- q^5\frac{(-q^{8}, -q^{16}, -q^{48}, -q^{56}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{8}, q^{24}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} - q^6\frac{(-q^{8}, -q^{16}, -q^{48}, -q^{56}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty},\end{aligned}$$

$$\begin{aligned}&\left(\begin{matrix} q,\ q^{7} \\ q^{3}, q^{5} \end{matrix}; q^{8}\right)_\infty \\ &= \frac{(-q^{16}, -q^{24}, -q^{40}, -q^{48}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{8}, q^{24}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} - q\frac{(-q^{16}, -q^{24}, -q^{40}, -q^{48}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} \\ &+ q^3\frac{(-q^{8}, -q^{32}, -q^{32}, -q^{56}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} - q^4\frac{(-q^{8}, -q^{16}, -q^{48}, -q^{56}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{8}, q^{24}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} \\ &+ q^5\frac{(-q^{8}, -q^{16}, -q^{48}, -q^{56}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty} - 2q^7\frac{(-q^{24}, -q^{40}, -q^{64}, -q^{64}, q^{64}, q^{64}; q^{64})_\infty}{(q^8, q^{16}, q^{16}, q^{24}, q^{32}, q^{32}; q^{32})_\infty}.\end{aligned}$$

Hence our theorem.

In fact, it is easy to show by manipulating the products that

$$\sum_{n\geq 0} a_{8n}q^n$$

$$= \sum_{-\infty}^{\infty} q^{4n^2}/(q,q^2,q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6,q^7,q^8,q^9,q^{10},q^{11},q^{11},q^{12},q^{12},q^{13},q^{13},q^{14},q^{14},q^{15},q^{16};q^{16})_\infty,$$

$$\sum_{n\geq 0} a_{8n+1}q^n = \sum_{n\geq 0} b_{8n}q^n$$

$$= 1/(q,q,q^2,q^3,q^3,q^3,q^4,q^5,q^5,q^5,q^7,q^7,q^9,q^9,q^{11},q^{11},q^{11},q^{12},q^{13},q^{13},q^{13},q^{14},q^{15},q^{15};q^{16})_\infty,$$

$$\sum_{n\geq 0} a_{8n+2}q^n = -\sum_{n\geq 0} b_{8n+1}q^n$$

$$= 1/(q,q^2,q^2,q^2,q^3,q^3,q^4,q^5,q^5,q^6,q^6,q^7,q^9,q^{10},q^{10},q^{11},q^{11},q^{12},q^{13},q^{13},q^{14},q^{14},q^{14},q^{15};q^{16})_\infty,$$

$$\sum_{n\geq 0} a_{8n+4}q^n$$

$$= -2q/(q,q,q^2,q^3,q^4,q^4,q^5,q^6,q^6,q^7,q^7,q^8,q^8,q^9,q^9,q^{10},q^{10},q^{11},q^{12},q^{12},q^{13},q^{14},q^{15},q^{15};q^{16})_\infty,$$

$$\sum_{n\geq 0} a_{8n+5}q^n = \sum_{n\geq 0} b_{8n+4}q^n$$

$$= -1/(q,q,q,q^3,q^3,q^4,q^5,q^5,q^6,q^7,q^7,q^7,q^9,q^9,q^9,q^{10},q^{11},q^{11},q^{12},q^{13},q^{13},q^{15},q^{15},q^{15};q^{16})_\infty,$$

$$\sum_{n\geq 0} a_{8n+6}q^n = -\sum_{n\geq 0} b_{8n+5}q^n$$

$$= -1/(q,q,q^2,q^2,q^3,q^4,q^5,q^6,q^6,q^6,q^7,q^7,q^9,q^9,q^{10},q^{10},q^{10},q^{11},q^{12},q^{13},q^{14},q^{14},q^{15},q^{15};q^{16})_\infty,$$

$$\sum_{n\geq 0} b_{8n+3}q^n$$

$$= \sum_{-\infty}^{\infty} q^{4n^2}/(q,q,q^2,q^3,q^4,q^4,q^5,q^6,q^6,q^7,q^7,q^8,q^9,q^9,q^{10},q^{10},q^{11},q^{12},q^{12},q^{13},q^{14},q^{15},q^{15},q^{16};q^{16})_\infty,$$

$$\sum_{n\geq 0} b_{8n+7}q^n$$

$$= -2/(q,q^2,q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6,q^7,q^8,q^8,q^9,q^{10},q^{11},q^{11},q^{12},q^{12},q^{13},q^{13},q^{14},q^{14},q^{15};q^{16})_\infty.$$

References

[1] K. Alladi and B. Gordon, Vanishing coefficients in the expansion of products of Rogers–Ramanujan type, The Rademacher Legacy to Mathematics, Contemporary Mathematics **166**(1994),129–139.

[2] G. E. Andrews and D. Bressoud, Vanishing coefficients in infinite product expansions, J. Austral. Math. Soc. (A) **27**(1979),199–202.

[3] B. Gordon, Some continued fractions of the Rogers–Ramanujan type, Duke Math. J. **32**(1965),741–748.

[4] B. Richmond and G. Szekeres, The Taylor coefficients of certain infinite products, Acta Sci. Math. (Szeged), **40**(1978),347–369.