

# SOME FORMULAE FOR PARTITIONS INTO SQUARES

MICHAEL D. HIRSCHHORN

## Abstract.

We consider the new problem of determining the number of partitions of a number into a fixed number  $k$  of squares, and find explicit formulae in the cases  $k = 2, 3, 4$ .

**Classification:** 11P81, 05A17

**Keywords:** partitions, squares.

## 1. Introduction and statement of results

We consider the problem of determining the number of partitions of a number into a fixed number of squares. This problem is distinct from, but not unrelated to, the corresponding classical problem of determining the number of representations of a number as the sum of a fixed number of squares. Whereas the classical problem has received an enormous amount of attention over the years, see Dickson [3, Vol. II, Chapters VI–IX], it appears that this (restricted) partition problem has not previously been investigated. Hardy and Ramanujan [4, p.305] and Baxter [1] have looked at the problem of partitions of a number into (an unrestricted number of) squares, which finds application in statistical mechanics.

To see why one might want to examine the partition problem, consider the following.

Jacobi's four square theorem states that the number of representations of  $n > 1$  as the sum of four squares is given by

$$r_4(n) = 8 \sum_{d|n, 4 \nmid d} d.$$

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$

This is in fact the number of representations of  $n > 1$  as the sum of four squares of integers positive, negative or zero, with order relevant. Thus, for example,  $r_4(30) = 8(1 + 2 + 3 + 5 + 6 + 10 + 15 + 30) = 576$ . The expression

$$30 = 4^2 + 3^2 + 2^2 + 1^2$$

accounts for  $4! \times 2^4 = 384$  of the representations (permute the four numbers in  $4!$  ways, place a  $\pm$  sign in front of each of the four numbers independently in  $2^4$  ways), while the expression

$$30 = 5^2 + 2^2 + 1^2 + 0^2$$

accounts for  $4! \times 2^3 = 192$  of the representations. But 30 has only 2 partitions into four squares of non-negative integers, the two representations given above.

Thus we see that counting the number of partitions avoids an enormous amount of “undesirable” duplication.

In this note we shall find expressions for  $p_{2\Box}(n)$ ,  $p_{3\Box}(n)$  and  $p_{4\Box}(n)$ , the numbers of partitions of  $n$  into two, three and four squares of non-negative integers, in terms of divisors of  $n$  and of numbers less than  $n$ .

In order to state our results, we need to define a few terms.

Let  $\delta(n)$  be given for  $n \geq 1$  by

$$\delta(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\phi(q) = \sum_{-\infty}^{\infty} q^{n^2} = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty} (-q^2; q^2)_{\infty}} = 1 + 2 \sum_{n \geq 1} \delta(n) q^n.$$

Let  $d_{r,m}(n)$  denote the number of divisors  $d$  of  $n$  with  $d \equiv r \pmod{m}$ , and let

$$e_{r,s,m}(n) = d_{r,m}(n) - d_{s,m}(n)$$

be the excess of divisors  $d$  of  $n$  with  $d \equiv r \pmod{m}$  over those with  $d \equiv s \pmod{m}$ .

We shall show that

$$\sum_{n \geq 0} p_{2\Box}(n) q^n = \sum_{i \geq j \geq 0} q^{i^2 + j^2} = \frac{1}{8} \left( \phi(q)^2 + 2\phi(q) + 2\phi(q^2) + 3 \right), \quad (1)$$

$$\begin{aligned} \sum_{n \geq 0} p_{3\Box}(n)q^n &= \sum_{i \geq j \geq k \geq 0} q^{i^2+j^2+k^2} \\ &= \frac{1}{48} \left( \phi(q)^3 + 3\phi(q)^2 + 6\phi(q)\phi(q^2) + 9\phi(q) + 6\phi(q^2) + 8\phi(q^3) + 15 \right) \end{aligned} \quad (2)$$

and

$$\begin{aligned} \sum_{n \geq 0} p_{4\Box}(n)q^n &= \sum_{i \geq j \geq k \geq l \geq 0} q^{i^2+j^2+k^2+l^2} \\ &= \frac{1}{384} \left( \phi(q)^4 + 4\phi(q)^3 + 12\phi(q)^2\phi(q^2) + 18\phi(q)^2 + 24\phi(q)\phi(q^2) + 12\phi(q^2)^2 \right. \\ &\quad \left. + 32\phi(q)\phi(q^3) + 60\phi(q) + 36\phi(q^2) + 32\phi(q^3) + 48\phi(q^4) + 105 \right). \end{aligned} \quad (3)$$

If we now use the facts [2, (3.2.23), (9.1.9)], [5, Theorems 312, 385], [6] that

$$\phi(q) = 1 + 2 \sum_{n \geq 1} \delta(n)q^n,$$

$$\phi(q)^2 = 1 + 4 \sum_{n \geq 1} e_{1,3,4}(n)q^n, \quad (\text{Jacobi, 1828})$$

$$\phi(q)\phi(q^2) = 1 + 2 \sum_{n \geq 1} \left( e_{1,7,8}(n) + e_{3,5,8}(n) \right) q^n, \quad (\text{Dirichlet, 1840})$$

$$\phi(q)\phi(q^3) = 1 + \sum_{n \geq 1} \left( 2e_{1,2,3}(n) + 4e_{1,2,3}(n/4) \right) q^n, \quad (\text{Lorenz, 1871})$$

$$\begin{aligned} \phi(q)^3 &= \left( 1 + 4 \sum_{n \geq 1} e_{1,3,4}(n)q^n \right) \left( 1 + 2 \sum_{n \geq 1} \delta(n)q^n \right) \\ &= 1 + \sum_{n \geq 1} \left( 4e_{1,3,4}(n) + 8 \sum_{1 \leq k^2 < n} e_{1,3,4}(n - k^2) + 2\delta(n) \right) q^n, \end{aligned}$$

$$\begin{aligned} \phi(q)^2\phi(q^2) &= \left( 1 + 4 \sum_{n \geq 1} e_{1,3,4}(n)q^n \right) \left( 1 + 2 \sum_{n \geq 1} \delta(n)q^{2n} \right) \\ &= 1 + \sum_{n \geq 1} \left( 4e_{1,3,4}(n) + 8 \sum_{1 \leq 2k^2 < n} e_{1,3,4}(n - 2k^2) + 2\delta(n/2) \right) q^n \end{aligned}$$

and

$$\phi(q)^4 = 1 + 8 \sum_{n \geq 1} (\sigma(n) - 4\sigma(n/4)) q^n \quad (\text{Jacobi, 1829})$$

we find that for  $n \geq 1$

$$p_{2\Box}(n) = \frac{1}{2} (e_{1,3,4}(n) + \delta(n) + \delta(n/2)), \quad (4)$$

$$p_{3\Box}(n) = \frac{1}{12} \left( 4e_{1,3,4}(n) + 2 \sum_{1 \leq k^2 < n} e_{1,3,4}(n - k^2) + 3e_{1,7,8}(n) + 3e_{3,5,8}(n) \right. \\ \left. + 5\delta(n) + 3\delta(n/2) + 4\delta(n/3) \right) \quad (5)$$

and

$$p_{4\Box}(n) = \frac{1}{48} \left( \sigma(n) - 4\sigma(n/4) + 17e_{1,3,4}(n) + 4 \sum_{1 \leq k^2 < n} e_{1,3,4}(n - k^2) \right. \\ \left. + 12 \sum_{1 \leq 2k^2 < n} e_{1,3,4}(n - 2k^2) + 6e_{1,7,8}(n) + 6e_{3,5,8}(n) + 6e_{1,3,4}(n/2) \right. \\ \left. + 8e_{1,2,3}(n) + 16e_{1,2,3}(n/4) + 16\delta(n) + 12\delta(n/2) + 8\delta(n/3) + 12\delta(n/4) \right). \quad (6)$$

## 2. Proofs

Let  $x_i$ ,  $i = 0, \dots, N$  be arbitrary.

It is easy to verify that

$$2 \sum_{i \geq j} x_i x_j = \left( \sum_{i=0}^N x_i \right)^2 + \sum_{i=0}^N x_i^2, \quad (7)$$

$$6 \sum_{i \geq j \geq k} x_i x_j x_k = \left( \sum_{i=0}^N x_i \right)^3 + 3 \sum_{i=0}^N x_i \sum_{i=0}^N x_i^2 + 2 \sum_{i=0}^N x_i^3 \quad (8)$$

and

$$24 \sum_{i \geq j \geq k \geq l} x_i x_j x_k x_l \quad (9)$$

$$= \left( \sum_{i=0}^N x_i \right)^4 + 6 \left( \sum_{i=0}^N x_i \right)^2 \sum_{i=0}^N x_i^2 + 3 \left( \sum_{i=0}^N x_i^2 \right)^2 + 8 \sum_{i=0}^N x_i \sum_{i=0}^N x_i^3 + 6 \sum_{i=0}^N x_i^4.$$

If in (7), (8) and (9) we set  $x_i = q^{i^2}$ , let  $N \rightarrow \infty$  and use the fact that

$$\sum_{i \geq 0} q^{i^2} = \frac{1}{2}(\phi(q) + 1)$$

we obtain (1), (2) and (3).

### References

- [1] R. J. Baxter, A direct proof of Kim's identities, *J. Phys. A: Math. Gen.* 31 No 3 (1998), 1105–1108.
- [2] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Canadian Mathematical Society, John Wiley & Sons Inc., New York, 1986.
- [3] L. E. Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1966.
- [4] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Collected Papers of S. Ramanujan, Chelsea, New York, 1962.
- [5] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth edition, OUP, 1979.
- [6] M. D. Hirschhorn, Partial fractions and four classical theorems of number theory, *Amer. Math. Monthly*, to appear.

M. Hirschhorn  
School of Mathematics  
UNSW  
Sydney 2052  
NSW  
Australia  
m.hirschhorn@unsw.edu.au