PARTIAL FRACTIONS AND FOUR CLASSICAL THEOREMS OF NUMBER THEORY

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For some years Jacobi's two and four squares theorems (stated below) have been sources of considerable fascination to me, so much so that I have continually sought the simplest, most direct proofs of them. Indeed, I have in the past presented two proofs of each [1,2,3,4]. It is my intention to give now what I regard as even simpler proofs of these two theorems as well as results due to Dirichlet and Lorenz on representations by a square plus twice a square and a square plus three times a square respectively. We shall see that all four classical theorems follow from just one partial fractions expansion together with special cases of Jacobi's triple product identity. This work is my attempt to make rigorous, to simplify and to present to a wider readership the work of Lorenz [5].

With $d_{r,m}(n)$ denoting the number of divisors d of n with $d \equiv r \pmod{m}$, the theorems we prove are

Theorem (Jacobi, 1828). *The number of representations of the number $n \geq 1$ as the sum of two squares is*

$$4\Big(d_{1,4}(n) - d_{3,4}(n)\Big),$$

Theorem (Dirichlet, 1840). *The number of representations of $n \geq 1$ as the sum of a square and twice a square is*

$$2\Big(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)\Big),$$

Theorem (Lorenz, 1871). *The number of representations of $n \geq 1$ as the sum of a square and three times a square is*

$$2\Big(d_{1,3}(n) - d_{2,3}(n)\Big) + 4\Big(d_{4,12}(n) - d_{8,12}(n)\Big)$$

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and

Theorem (Jacobi, 1829). *The number of representations of $n \geq 1$ as the sum of four squares is*

$$8 \sum_{d|n,\ 4\nmid d} d.$$

The proofs

Before presenting the partial fractions expansion referred to in the opening paragraph, let us look at the special cases of Jacobi's triple product identity we shall require. Jacobi's triple product identity says that if $a \neq 0,\ |q| < 1$ then

$$\prod_{n\geq 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

If we set $a = -1$ we find

$$\begin{aligned}\sum_{-\infty}^{\infty}(-1)^n q^{n^2} &= \prod_{n\geq 1}(1-q^{2n-1})^2(1-q^{2n}) = \prod_{n\geq 1}\frac{(1-q^n)^2}{(1-q^{2n})}\\ &= \prod_{n\geq 1}\frac{1-q^n}{1+q^n} = \prod_{n\geq 1}\frac{(1-q^{2n-1})}{(1+q^{2n})}\frac{(1-q^{2n})}{(1+q^{2n-1})},\end{aligned}$$

and, if we put $-q^2$ for q in this, we find

$$\sum_{-\infty}^{\infty} q^{2n^2} = \prod_{n\geq 1}\frac{(1+q^{4n-2})(1-q^{4n})}{(1+q^{4n})(1-q^{4n-2})} = \prod_{n\geq 1}\frac{(1+q^{4n-2})(1-q^{2n})}{(1+q^{4n})(1+q^{2n-1})}\frac{(1+q^{2n})}{(1-q^{2n-1})}.$$

Now, it is an easy exercise in partial fractions to show that, provided no term in the denominator is zero,

(1)
$$\frac{1}{2-x}\prod_{n=1}^{N}\frac{1-aq^{n-1}x+a^2q^{2n-2}}{1-q^nx+q^{2n}} = \frac{A_0}{2-x} + \sum_{n=1}^{N}\frac{A_n}{1-q^nx+q^{2n}}$$

where, with $(a,q)_n = (1-a)(1-aq)\ \cdots\ (1-aq^{n-1}),\ (a;q)_0 = 1$,

$$A_0 = \frac{(a;q)_N^2}{(q;q)_N^2}$$

and for $n \geq 1$

$$A_n = \frac{a^n(a^{-1}q;q)_n(1+q^n)}{(a;q)_n} \frac{(a;q)_{N+n}(a;q)_{N-n}}{(q;q)_{N+n}(q;q)_{N-n}}.$$

If we divide (1) by A_0, suppose $0 < |a| < 1,\ |q| < 1$ and let $N \to \infty$ we obtain

(2)

$$\frac{1}{2-x} \prod_{n\geq 1} \frac{(1 - aq^{n-1}x + a^2q^{2n-2})}{(1 - q^n x + q^{2n})} \frac{(1-q^n)^2}{(1-aq^{n-1})^2} = \frac{1}{2-x} + \sum_{n\geq 1} \frac{a^n(a^{-1}q;q)_n(1+q^n)}{(a;q)_n(1-q^n x + q^{2n})},$$

again provided no term in the denominator is zero.

If in (2) we set $a = -q,\ x = 0$ we find

$$\prod_{n\geq 1} \frac{(1-q^n)^2}{(1+q^n)^2} = 1 + 4 \sum_{n\geq 1} \frac{(-1)^n q^n}{1+q^{2n}}$$

or,

$$\left(\sum_{-\infty}^{\infty} (-1)^n q^{n^2} \right)^2 = 1 + 4 \sum_{n\geq 1} \frac{(-1)^n q^n}{1+q^{2n}}. \tag{3}$$

If in (3) we put $-q$ for q we obtain

$$\begin{aligned}
\left(\sum_{-\infty}^{\infty} q^{n^2} \right)^2 &= 1 + 4 \sum_{n\geq 1} \frac{q^n}{1+q^{2n}} \\
&= 1 + 4 \sum_{n\geq 1} \left(\frac{q^n}{1-q^{4n}} - \frac{q^{3n}}{1-q^{4n}} \right) \\
&= 1 + 4 \sum_{k\geq 1,\ l\geq 0} \left(q^{k(4l+1)} - q^{k(4l+3)} \right) \\
&= 1 + 4 \sum_{n\geq 1} \Big(d_{1,4}(n) - d_{3,4}(n) \Big) q^n,
\end{aligned}$$

from which Jacobi's two square theorem follows.

If in (2) we set $a = -q,\ x = -2$ we find

$$\prod_{n\geq 1} \frac{(1-q^n)^4}{(1+q^n)^4} = 1 + 8 \sum_{n\geq 1} \frac{(-1)^n q^n}{(1+q^n)^2}$$

or,

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^4 = 1 + 8\sum_{n\geq 1}\frac{(-1)^n q^n}{(1+q^n)^2}. \tag{4}$$

If in (4) we put $-q$ for q we obtain

$$\begin{aligned}
\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^4 &= 1 + 8\sum_{n \text{ even}}\frac{q^n}{(1+q^n)^2} + 8\sum_{n \text{ odd}}\frac{q^n}{(1-q^n)^2} \\
&= 1 + 8\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2} - 8\sum_{n \text{ even}}\left(\frac{q^n}{(1-q^n)^2} - \frac{q^n}{(1+q^n)^2}\right) \\
&= 1 + 8\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2} - 32\sum_{n\geq 1}\frac{q^{4n}}{(1-q^{4n})^2} \\
&= 1 + 8\sum_{n\geq 1}\left(\sigma(n) - 4\sigma\left(\frac{n}{4}\right)\right) q^n \\
&= 1 + 8\sum_{n\geq 1}\left(\sum_{d|n,\ 4\nmid d} d\right) q^n,
\end{aligned}$$

from which Jacobi's four square theorem follows.

If in (2) we set $a = -q,\ x = 1$ we find

$$\prod_{n\geq 1}\frac{(1-q^n)}{(1+q^n)}\frac{(1-q^{3n})}{(1+q^{3n})} = 1 + 2\sum_{n\geq 1}\frac{(-1)^n q^n}{1-q^n+q^{2n}}$$

or,

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)\left(\sum_{-\infty}^{\infty}(-1)^n q^{3n^2}\right) = 1 + 2\sum_{n\geq 1}\frac{(-1)^n q^n}{1-q^n+q^{2n}}. \tag{5}$$

If in (5) we put $-q$ for q we obtain

$$\begin{aligned}
\left(\sum_{-\infty}^{\infty} q^{n^2}\right)\left(\sum_{-\infty}^{\infty} q^{3n^2}\right) &= 1 + 2\sum_{n\geq 1}\frac{q^n}{1-(-1)^n q^n + q^{2n}} \\
&= 1 + 2\sum_{n \text{ even}}\frac{q^n(1+q^n)}{1+q^{3n}} + 2\sum_{n \text{ odd}}\frac{q^n(1-q^n)}{1-q^{3n}}
\end{aligned}$$

$$= 1 + 2\sum_{n\geq 1} \frac{q^n(1-q^n)}{1-q^{3n}} + 2\sum_{n \text{ even}} \left(\frac{q^n(1+q^n)}{1+q^{3n}} - \frac{q^n(1-q^n)}{1-q^{3n}} \right)$$

$$= 1 + 2\sum_{n\geq 1} \frac{q^n(1-q^n)}{1-q^{3n}} + 4\sum_{n \text{ even}} \frac{q^{2n}(1-q^{2n})}{1-q^{6n}}$$

$$= 1 + 2\sum_{n\geq 1} \frac{q^n(1-q^n)}{1-q^{3n}} + 4\sum_{n\geq 1} \frac{q^{4n}(1-q^{4n})}{1-q^{12n}}$$

$$= 1 + 2\sum_{n\geq 1} \Big(d_{1,3}(n) - d_{2,3}(n)\Big)q^n + 4\sum_{n\geq 1} \Big(d_{4,12}(n) - d_{8,12}(n)\Big)q^n,$$

from which Lorenz's theorem follows.

If in (2) we put q^2 for q, $a = -q$, $x = 0$, we find

$$\prod_{n\geq 0} \frac{(1+q^{4n-2})}{(1+q^{4n})} \frac{(1-q^{2n})^2}{(1+q^{2n-1})^2} = 1 + 2\sum_{n\geq 1} \frac{(-1)^n q^n (1+q^{2n})}{1+q^{4n}}$$

or,

$$\left(\sum_{-\infty}^{\infty} q^{2n^2}\right)\left(\sum_{-\infty}^{\infty} (-1)^n q^{n^2}\right) = 1 + 2\sum_{n\geq 1} \frac{(-1)^n q^n(1+q^{2n})}{1+q^{4n}}. \tag{6}$$

If in (6) we put $-q$ for q we obtain

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)\left(\sum_{-\infty}^{\infty} q^{2n^2}\right) = 1 + 2\sum_{n\geq 1} \frac{q^n(1+q^{2n})}{1+q^{4n}}$$

$$= 1 + 2\sum_{n\geq 1} \left(\frac{q^n}{1-q^{8n}} + \frac{q^{3n}}{1-q^{8n}} - \frac{q^{5n}}{1-q^{8n}} - \frac{q^{7n}}{1-q^{8n}} \right)$$

$$= 1 + 2\sum_{n\geq 1} \Big(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)\Big)q^n,$$

from which Dirichlet's theorem follows.

References

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