MORE ON *COOTIE*

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Min Deng and Mary T. Whalen [1] consider the game of *Cootie*, and show that the expected length L of a one-person game is given by $L = 12 + E(X)$ where X is a random variable with $Pr\{X = n\} = p_n$ given by

$$\begin{aligned} p_n = 2 &\sum_{\substack{i\ge 2,\ j\ge 1,\ k\ge 6,\ l\ge 0 \\ i+j+k+l=n-2}} \frac{(n-1)!}{1!i!j!k!l!} \left(\frac{1}{6}\right)^{2+i+j+k} \left(\frac{1}{3}\right)^{l} \\ &+ \sum_{\substack{i\ge 2,\ j\ge 2,\ k\ge 6,\ l\ge 0 \\ i+j+k+l=n-1}} \frac{(n-1)!}{i!j!k!l!} \left(\frac{1}{6}\right)^{1+i+j+k} \left(\frac{1}{3}\right)^{l} \\ &+ \sum_{\substack{i\ge 2,\ j\ge 2,\ k\ge 1,\ l\ge 0 \\ i+j+k+l=n-6}} \frac{(n-1)!}{5!i!j!k!l!} \left(\frac{1}{6}\right)^{6+i+j+k} \left(\frac{1}{3}\right)^{l}. \end{aligned}$$

They evaluate L using *Mathematica*, and state that to eight significant figures, $L = 48.953478$.

The purpose of this note is to report that the exact value of L is

$$L = \frac{584684533}{11943936}$$

whose decimal begins 48.952416774503815... .

We shall use two completely different methods to arrive at this result. The first is to use the probability generating function $P(t) = \sum_{n\ge 0} p_n t^n$ and the fact that $E(X) = P'(1)$. The second is to use an inclusion-exclusion argument to find an alternative formula for p_n and then to use the fact that $E(X) = \sum_{n\ge 0} np_n$.

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We have

$$
\begin{aligned}
P(t) &= \sum_{n\geq 0} p_n t^n \\
&= 2\left(\frac{t}{6}\right)^2 \sum_{k\geq 6,\ i\geq 2,\ j\geq 1,\ l\geq 0} \binom{1+k}{k}\left(\frac{t}{6}\right)^k \binom{1+k+i}{i}\left(\frac{t}{6}\right)^i \binom{1+k+i+j}{j}\left(\frac{t}{6}\right)^j \times \\
&\qquad\qquad \times \binom{1+k+i+j+l}{l}\left(\frac{t}{3}\right)^l \\
&+ \left(\frac{t}{6}\right) \sum_{k\geq 6,\ i\geq 2,\ j\geq 2,\ l\geq 0} \left(\frac{t}{6}\right)^k \binom{k+i}{i}\left(\frac{t}{6}\right)^i \binom{k+i+j}{j}\left(\frac{t}{6}\right)^j \binom{k+i+j+l}{l}\left(\frac{t}{3}\right)^l \\
&+ \left(\frac{t}{6}\right)^6 \sum_{i\geq 2,\ j\geq 2,\ k\geq 1,\ l\geq 0} \binom{5+i}{i}\left(\frac{t}{6}\right)^i \binom{5+i+j}{j}\left(\frac{t}{6}\right)^j \binom{5+i+j+k}{k}\left(\frac{t}{6}\right)^k \times \\
&\qquad\qquad \times \binom{5+i+j+k+l}{l}\left(\frac{t}{3}\right)^l .
\end{aligned}
$$

We now make use of the binomial theorem,

$$\sum_{l\geq 0}\binom{N+l}{l} t^l = (1-t)^{-(N+1)}$$

first to sum on l, then on the remaining three variables, and eventually we find

$$
\begin{aligned}
P(t) &= (w+2w^2+w^6) - (3x+9x^2+9x^3+7x^4+9x^5+14x^6+24x^7) + 3y + 12y^2 \\
&+ 24y^3 + 32y^4 + 50y^5 + 75y^6 + 120y^7 + 126y^8 - (z+5z^2+15z^3+31z^4+53z^5 \\
&+ 82z^6 + 126z^7 + 168z^8)
\end{aligned}
$$

where

$$z = \frac{t/6}{1-t/3},\quad y = \frac{t/6}{1-t/2},\quad x = \frac{t/6}{1-2t/3},\quad w = \frac{t/6}{1-5t/6}.$$

It follows that

$$
\begin{aligned}
P'(t) &= (1+4w+6w^5)w' - (3+18x+27x^2+28x^3+45x^4+84x^5+168x^6)x' + (3+24y \\
&+ 72y^2 + 128y^3 + 250y^4 + 450y^5 + 840y^6 + 1008y^7)y' - (1+10z+45z^2+124z^3 \\
&+ 265z^4 + 492z^5 + 882z^6 + 1344z^7)z'.
\end{aligned}
$$

If we now set $t = 1$, then $w = 1,\ x = 1/2,\ y = 1/3,\ z = 1/4,\ w' = 6,\ x' = 3/2,\ y' = 2/3,\ z' = 3/8$ and we find

$$E(X) = \frac{441357301}{11943936}, \quad L = 12 + E(X) = \frac{584684533}{11943936},$$

as claimed. ■

Our second approach starts by observing that $Pr\{X = n\} = p_n$ is given for $n > 0$ by

$$\begin{aligned} p_n = \frac{1}{6^n} \Bigg\{ & 5^{n-1} + 2\binom{n-1}{1}5^{n-2} + \binom{n-1}{5}5^{n-6} - 3 \times 4^{n-1} \\ & -9\binom{n-1}{1}4^{n-2} - 9\binom{n-2}{2}4^{n-3} - 7\binom{n-1}{3}4^{n-4} - 9\binom{n-1}{4}4^{n-5} \\ & -14\binom{n-1}{5}4^{n-6} - 24\binom{n-1}{6}4^{n-7} + 3 \times 3^{n-1} + 12\binom{n-1}{1}3^{n-2} \\ & +24\binom{n-1}{2}3^{n-3} + 32\binom{n-1}{3}3^{n-4} + 50\binom{n-1}{4}3^{n-5} + 75\binom{n-1}{5}3^{n-6} \\ & +120\binom{n-1}{6}3^{n-7} + 126\binom{n-1}{7}3^{n-8} - 2^{n-1} - 5\binom{n-1}{1}2^{n-2} \\ & -15\binom{n-1}{2}2^{n-3} - 31\binom{n-1}{3}2^{n-4} - 53\binom{n-1}{4}2^{n-5} - 82\binom{n-1}{5}2^{n-6} \\ & -126\binom{n-1}{6}2^{n-7} - 168\binom{n-1}{7}2^{n-8} \Bigg\}. \end{aligned}$$

This formula is established for $n \geq 11$ by an inclusion–exclusion argument (details available from the author), together with the facts that

$$\binom{n-1}{1}\binom{n-2}{k} = \binom{k+1}{1}\binom{n-1}{k+1} \text{ and } \binom{n-1}{5}\binom{n-6}{k} = \binom{k+5}{5}\binom{n-1}{k+5}$$

and then checked to be true for $1 \leq n \leq 10$ also.

If we multiply by n, use the fact that

$$n\binom{n-1}{k} = (k+1)\binom{n}{k+1}$$

and sum over $n \geq 0$, we find that

$$E(X) = \sum_{n\geq 0} np_n = \frac{441357301}{11943936}, \quad L = 12 + E(X) = \frac{584684533}{11943936}. \blacksquare$$

[1] Min Deng and Mary T. Whalen, The mathematics of *Cootie*, The College Mathematics Journal 29 (1998), 222-224.

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