Powers of Euler's Product and Related Identities

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January 25, 2001

1 Introduction

Ramanujan [17] made the conjectures that

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7 \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11},$$

where $p(n)$ denotes the number of partitions of n (see [7], Chapter 19). He gave (loc. cit.) simple proofs of the mod 5 and mod 7 congruences, and these proofs are reproduced in [7]. They involve showing that if the $a_r(n)$ are defined by

$$(q)_\infty^r = \sum_{n\geq 0} a_r(n)q^n, \quad \text{where} \quad (q)_\infty = \prod_{j=1}^{\infty}(1-q^j),$$

then

$$a_4(5n+4) \equiv 0 \pmod 5,$$

and

$$a_6(7n+5) \equiv 0 \pmod 7.$$

Winquist [18] much later gave a similar proof of the mod 11 congruence by showing that

$$a_{10}(11n+6) \equiv 0 \pmod{11}.$$

Indeed, much more is true. As we shall see,

(1) $$a_4(5n+4) = -5\ a_4\left(\frac{n}{5}\right),$$

(2) $$a_6(7n+5) = 49\ a_6\left(\frac{n-1}{7}\right)$$

and

(3) $$a_{10}(11n+6) = 11^4\ a_{10}\left(\frac{n-4}{11}\right).$$

In these and all subsequent formulas, n is assumed to be an integer and $a_r(m/p)$ is taken to be zero whenever m/p is not an integer.

These results can be generalised further. For example, it can be shown that if $p \equiv 5 \pmod 6$ is prime,

$$a_4\left(pn + \frac{p^2-1}{6}\right) = -p\ a_4\left(\frac{n}{p}\right), \tag{4}$$

if $p \equiv 3 \pmod 4$ is prime,

$$a_6\left(pn + \frac{p^2-1}{4}\right) = p^2\ a_6\left(\frac{n}{p}\right), \tag{5}$$

while if $p \equiv 11 \pmod{12}$ is prime,

$$a_{10}\left(pn + \frac{5p^2-5}{12}\right) = p^4\ a_{10}\left(\frac{n}{p}\right). \tag{6}$$

(1) is simply the case $p = 5$ of (4); (2) follows from (5) by taking p=7 and replacing n with $n-1$, while (3) follows from (6) by taking $p = 11$ and replacing n with $n-4$.

Newman [13] generalised equations (4) – (6) and proved the following result.

Theorem (Newman) *Suppose that* r *is one of the numbers* 2, 4, 6, 8, 10, 14, 26. *Let* p *be a prime* > 3 *such that* $r(p+1) \equiv 0 \pmod{24}$. *Let* $\Delta = (p^2-1)/24$, *and define* $a_r(\alpha) = 0$ *if* α *is not a non-negative integer. Then*

$$a_r(np + r\Delta) = (-p)^{(r/2)-1} a_r\left(\frac{n}{p}\right).$$

Furthermore, there are no other values of r *for which the theorem is true.*

Newman's proof of this theorem uses the theory of elliptic modular functions.

The purposes of this paper are as follows. First, we give an explicit, elementary proof of Newman's theorem for all cases except $r = 26$. Our proofs are explicit in the sense that they utilise explicit series expansions for certain infinite products which arise from Macdonald identities of ranks 1 and 2. They are elementary in that they avoid the theory of elliptic modular functions. Our proof also shows that more is true than Newman claimed for the cases $r = 2, 10$ and 14, and we also obtain results for $r = 1$ and 3. We have not been able to use our methods to prove the case $r = 26$.

Second, we obtain new results for infinite products of the types $(q)_\infty^r (q^2)_\infty^s$,

$(q)_\infty^r(q^3)_\infty^s$ and $(q)_\infty^r(q^4)_\infty^s$ for various integers r and s. These results arise out of considering all of the Macdonald identities of ranks 1 and 2. Results for analogous infinite products arising from Macdonald identities of ranks higher than 2 appear not to exist.

Newman [14, 15] also considered products of the form $(q)_\infty^r(q^j)_\infty^s$, although his results are different. There has also been work on classifying eta-products whose Fourier coefficients are multiplicative. See, for example, [2], [4, pp. 77–85], [5], [6], [11] and [12]; some of these eta products arise in Conway and Norton's [1] "monstrous moonshine".

We conclude with a list of conjectures that were suggested by computer search.

2 Statement of Results

Define the ceiling function by

$$\lceil x \rceil = \min_{k \geq x} k$$

where the minimum is taken over all integers k.

Theorem 1. *Suppose r is an integer and p is prime. Let $\Delta = (p^2-1)/24$, let $j = \lceil r/2 \rceil - 1$ and let $(q)_\infty^r = \sum_{n \geq 0} a(n) q^n$. Then the coefficients $a(n)$ satisfy*

$$a(pn + r\Delta) = \epsilon p^j a\left(\frac{n}{p}\right)$$

for the following values of r, p and ϵ:

r	p	ϵ
1	$p > 3$	$\begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12}, \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12} \end{cases}$
2	$p \equiv 5,\ 7 \text{ or } 11 \pmod{12}$	$\begin{cases} 1 & \text{if } p \equiv 7 \text{ or } 11 \pmod{12}, \\ -1 & \text{if } p \equiv 5 \pmod{12} \end{cases}$
3	$p > 2$	$\begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases}$
4	$p \equiv 5 \pmod{6}$	-1
6	$p \equiv 3 \pmod{4}$	1
8	$p \equiv 2 \pmod{3}$	-1
10	$p \equiv 7 \text{ or } 11 \pmod{12}$	1
14	$p \equiv 5 \pmod{6}$	$\begin{cases} 1 & \text{if } p \equiv 11 \pmod{12}, \\ -1 & \text{if } p \equiv 5 \pmod{12} \end{cases}$

Theorem 2. *Suppose r and s are integers and p is prime. Let $\Delta = (p^2-1)/24$, let $j = \lceil (r+s)/2 \rceil - 1$ and let $(q)_\infty^r (q^2)_\infty^s = \sum_{n \geq 0} a(n) q^n$. Then the coefficients $a(n)$ satisfy*

$$a(pn + (r+2s)\Delta) = \epsilon p^j a\left(\frac{n}{p}\right)$$

for the following values of r, s, p and ϵ:

(r,s)	p	ϵ
$(1,1)$	$p \equiv 3, 5 \text{ or } 7 \pmod 8$	$\begin{cases} 1 & \text{if } p \equiv 5 \text{ or } 7 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$
$(2,2)$	$p \equiv 3 \pmod 4$	-1
$(3,3)$	$p \equiv 5 \text{ or } 7 \pmod 8$	1
$(5,5)$	$p \equiv 3 \pmod 4$	$\begin{cases} 1 & \text{if } p \equiv 7 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$
$(4,2),\ (2,4)$	$p \equiv 7 \text{ or } 11 \pmod{12}$	1
$(3,1),\ (1,3)$	$p \equiv 13, 17, 19 \text{ or } 23 \pmod{24}$	$\begin{cases} 1 & \text{if } p \equiv 13 \text{ or } 19 \pmod{24} \\ -1 & \text{if } p \equiv 17 \text{ or } 23 \pmod{24} \end{cases}$
$(5,1),\ (1,5)$	$p \equiv 13, 17, 19 \text{ or } 23 \pmod{24}$	$\begin{cases} 1 & \text{if } p \equiv 13 \text{ or } 23 \pmod{24} \\ -1 & \text{if } p \equiv 17 \text{ or } 19 \pmod{24} \end{cases}$
$(2,-1),\ (-1,2)$	$p > 2$	1
$(3,-1),\ (-1,3)$	$p \equiv 13, 17, 19 \text{ or } 23 \pmod{24}$	$\begin{cases} 1 & \text{if } p \equiv 13 \text{ or } 23 \pmod{24} \\ -1 & \text{if } p \equiv 17 \text{ or } 19 \pmod{24} \end{cases}$
$(5,-1),\ (-1,5)$	$p \equiv 5 \text{ or } 7 \pmod 8$	$\begin{cases} 1 & \text{if } p \equiv 5 \pmod 8 \\ -1 & \text{if } p \equiv 7 \pmod 8 \end{cases}$
$(4,-2),\ (-2,4)$	$p \equiv 3 \pmod 4$	1
$(5,-2),\ (-2,5)$	$p > 3$	$\begin{cases} 1 & \text{if } p \equiv 1 \pmod 6 \\ -1 & \text{if } p \equiv 5 \pmod 6 \end{cases}$
$(6,-2),\ (-2,6)$	$p \equiv 7 \text{ or } 11 \pmod{12}$	-1
$(8,-2),\ (-2,8)$	$p \equiv 5 \pmod 6$	$\begin{cases} 1 & \text{if } p \equiv 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \pmod{12} \end{cases}$
$(7,-3),\ (-3,7)$	$p \equiv 13, 17, 19 \text{ or } 23 \pmod{24}$	$\begin{cases} 1 & \text{if } p \equiv 13 \text{ or } 19 \pmod{24} \\ -1 & \text{if } p \equiv 17 \text{ or } 23 \pmod{24} \end{cases}$
$(9,-3),\ (-3,9)$	$p \equiv 3 \pmod 4$	$\begin{cases} 1 & \text{if } p \equiv 7 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$
$(10,-4),\ (-4,10)$	$p \equiv 7 \text{ or } 11 \pmod{12}$	1
$(14,-4),\ (-4,14)$	$p \equiv 3 \pmod 4$	1
$(11,-5),\ (-5,11)$	$p \equiv 7 \text{ or } 11 \pmod{12}$	$\begin{cases} 1 & \text{if } p \equiv 7 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$

Theorem 3. *Suppose r and s are integers and p is prime. Let $\Delta = (p^2-1)/24$, let $j = \lceil (r+s)/2 \rceil - 1$ and let $(q)_\infty^r (q^3)_\infty^s = \sum_{n \geq 0} a(n) q^n$. Then the coefficients*

$a(n)$ *satisfy*

$$a(pn + (r + 3s)\Delta) = \epsilon p^j a\left(\frac{n}{p}\right)$$

for the following values of $r,\ s,\ p$ and ϵ:

(r, s)	p	ϵ
$(1, 1)$	$p \equiv 5 \pmod 6$	1
$(3, 3)$	$p \equiv 5 \pmod 6$	1
$(7, 7)$	$p \equiv 5 \pmod 6$	1
$(3, 1),\ (1, 3)$	$p \equiv 7$ or $11 \pmod{12}$	$\begin{cases} 1 & \text{if } p \equiv 7 \pmod{12} \\ -1 & \text{if } p \equiv 11 \pmod{12} \end{cases}$

Theorem 4. *Suppose r and s are integers and p is prime. Let $\Delta = (p^2 - 1)/24$, let $j = \lceil (r + s)/2 \rceil - 1$ and let $(q)_\infty^r (q^4)_\infty^s = \sum_{n\geq 0} a(n)q^n$. Then the coefficients $a(n)$ satisfy*

$$a(pn + (r + 4s)\Delta) = \epsilon p^j a\left(\frac{n}{p}\right)$$

for the following values of $r,\ s,\ p$ and ϵ:

(r, s)	p	ϵ
$(1, 1)$	$p \equiv 7$ or $11 \pmod{12}$	1
$(2, 2)$	$p \equiv 7$ or $11 \pmod{12}$	-1
$(3, 3)$	$p \equiv 3 \pmod 4$	1
$(3, 1),\ (1, 3)$	$p \equiv 5 \pmod 6$	-1
$(5, -1),\ (-1, 5)$	$p \equiv 5 \pmod 6$	-1

3 Proofs

The following specialisations of Macdonald identities, listed in the appendix of [10], will be used. Each formula below is stated together with its associated root system. Thus, for example, the first formula is Macdonald's specialisation (c) for the root system BC_1.

$$f_1(q) := (q)_\infty = \sum_{\alpha\equiv 1 \pmod 6} (-1)^{(\alpha-1)/6} q^{(\alpha^2-1)/24} \qquad BC_1(\mathrm{c})$$

$$f_2(q) := (q)_\infty^3 = \sum_{\alpha\equiv 1 \pmod 4} \alpha q^{(\alpha^2-1)/8} \qquad A_1$$

$$f_3(q) := (q)_\infty^2/(q^2)_\infty = \sum_{\alpha} (-1)^\alpha q^{\alpha^2} \qquad B_1(\mathrm{c})$$

$$f_4(q) := (q^2)_\infty^2/(q)_\infty = \sum_{\alpha\equiv 1 \pmod 4} q^{(\alpha^2-1)/8} \qquad B_1(\mathrm{b})$$

$$f_5(q) := (q)_\infty^5/(q^2)_\infty^2 = \sum_{\alpha\equiv 1 \pmod 6} \alpha q^{(\alpha^2-1)/24} \qquad BC_1(\mathrm{a})$$

$$f_6(q) := (q^2)_\infty^5/(q)_\infty^2 = \sum_{\alpha\equiv 1 \pmod 3} (-1)^{\alpha-1} \alpha q^{(\alpha^2-1)/3} \qquad BC_1(\mathrm{d})$$

$$f_7(q) := (q)_\infty^8 = \frac{1}{2} \sum_{\substack{\alpha\equiv 1 \pmod 3 \\ \beta\equiv 1 \pmod 3}} (\alpha+\beta)(2\alpha-\beta)(2\beta-\alpha) q^{(\alpha^2-\alpha\beta+\beta^2-1)/3} \qquad A_2$$

$$f_8(q) := (q)_\infty^{10} = \frac{1}{6} \sum_{\substack{\alpha\equiv 2 \pmod 6 \\ \beta\equiv 1 \pmod 6}} \alpha\beta(\alpha^2-\beta^2) q^{(\alpha^2+\beta^2-5)/12} \qquad B_2(\mathrm{a})$$

$$f_9(q) := (q)_\infty^5 (q^2)_\infty^5 = \frac{1}{24} \sum_{\substack{\alpha\equiv 3 \pmod 8 \\ \beta\equiv 1 \pmod 8}} \alpha\beta(\alpha^2-\beta^2) q^{(\alpha^2+\beta^2-10)/16} \qquad B_2^\vee(\mathrm{a})$$

$$f_{10}(q) := (q)_\infty^{14}/(q^2)_\infty^4 = \frac{1}{24} \sum_{\substack{\alpha\equiv 3 \pmod{10} \\ \beta\equiv 1 \pmod{10}}} \alpha\beta(\alpha^2-\beta^2) q^{(\alpha^2+\beta^2-10)/40} \qquad BC_2(\mathrm{a})$$

$$f_{11}(q) := (q^2)_\infty^{14}/(q)_\infty^4 = \frac{1}{6} \sum_{\substack{\alpha\equiv 2 \pmod 5 \\ \beta\equiv 1 \pmod 5}} (-1)^{\alpha+\beta+1} \alpha\beta(\alpha^2-\beta^2) q^{(\alpha^2+\beta^2-5)/5} \qquad BC_2(\mathrm{d})$$

$$f_{12}(q) := (q)_\infty^{14}$$
$$= \frac{1}{3240} \sum_{\substack{\alpha\equiv 4 \pmod{12} \\ \beta\equiv 1 \pmod{12}}} \alpha\beta(\alpha+\beta)(\alpha-\beta)(2\alpha+\beta)(\alpha+2\beta) q^{(\alpha^2+\alpha\beta+\beta^2-21)/36} \qquad G_2$$

$$f_{13}(q) := (q)_\infty^7 (q^3)_\infty^7$$
$$= \frac{1}{120} \sum_{\substack{\alpha\equiv 2 \pmod 6 \\ \beta\equiv 1 \pmod 6}} \alpha\beta(\alpha+\beta)(\alpha-\beta)(2\alpha+\beta)(\alpha+2\beta) q^{(\alpha^2+\alpha\beta+\beta^2-7)/6} \qquad G_2^\vee$$

Formulas for $f_1, \cdots, f_4$ are classical consequences of the Jacobi triple product identity. Formulas for f_5 and f_6 are consequences of the quintuple product identity, and are stated explicitly by Ramanujan [16] and Macdonald [10]. As noted by Dyson [3], the identity for f_7 goes back to Klein and Fricke [9, p. 373]. The series for f_8 is originally due to Winquist [18]; also see [8]. The one for f_{12} is due to Atkin (unpublished), and Dyson [3, p. 651] gives Atkin's formula for some of the coefficients $a_{26}(n)$ in the expansion of $(q)_\infty^{26}$.

All of Theorems 1 – 4 may be proved either by appealing directly to one of the specialisations of Macdonald's identities above, or by combining two of the single series identities $f_1, \cdots, f_6$. The following four proofs illustrate our procedure in detail. The proof of the case $r = 2$, $p \equiv 5 \pmod{12}$ of Theorem 1 is more complicated than the others, so is given at the end of this section.

Proof of Theorem 1 in the case $r = 1$.
We have

$$(q)_\infty = f_1(q) = \sum_{\alpha \equiv 1 \pmod 6} (-1)^{(\alpha-1)/6} q^{(\alpha^2-1)/24}.$$

Thus

$$a(n) = \sum_{\substack{\alpha \equiv 1 \pmod 6 \\ (\alpha^2-1)/24 = n}} (-1)^{(\alpha-1)/6} = \sum_{\substack{\alpha \equiv 1 \pmod 6 \\ \alpha^2 = 24n+1}} (-1)^{(\alpha-1)/6}.$$

Therefore

$$a(pn+\Delta) = a\left(pn + \frac{p^2-1}{24}\right) = \sum_{\substack{\alpha \equiv 1 \pmod 6 \\ \alpha^2 = 24pn+p^2}} (-1)^{(\alpha-1)/6}.$$

The condition $\alpha^2 = 24pn + p^2$ implies $\alpha^2 \equiv 0 \pmod p$ and so $\alpha \equiv 0 \pmod p$.
Let

$$\lambda = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 6 \\ -1 & \text{if } p \equiv -1 \pmod 6 \end{cases}$$

Let $\alpha = \lambda p \alpha'$. Then α' is an integer. Also, $\alpha' \equiv 1 \pmod 6$ since $\lambda p \equiv 1 \pmod 6$ and $\alpha \equiv 1 \pmod 6$.
Next, modulo 2,

$$\begin{aligned} \frac{\alpha-1}{6} &= \frac{\lambda p \alpha' - 1}{6} \\ &= \frac{\alpha'-1}{6} + \frac{(\lambda p - 1)\alpha'}{6} \\ &= \frac{\alpha'-1}{6} + \lambda\alpha' \frac{p-\lambda}{6} \\ &= \frac{\alpha'-1}{6} + \frac{p-\lambda}{6} + (\lambda\alpha'-1)\frac{p-\lambda}{6} \\ &\equiv \frac{\alpha'-1}{6} + \frac{p-\lambda}{6}. \end{aligned}$$

Furthermore, the condition $\alpha^2 = 24pn + p^2$ becomes ${\alpha'}^2 = 24n/p + 1$. Therefore

$$a\left(pn + \frac{p^2-1}{24}\right) = \sum_{\substack{\alpha \equiv 1 \pmod 6 \\ \alpha^2 = 24pn+p^2}} (-1)^{(\alpha-1)/6}$$

$$
\begin{aligned}
&= (-1)^{(p-\lambda)/6} \sum_{\substack{\alpha' \equiv 1 \pmod 6 \\ \alpha'^2 = 24n/p+1}} (-1)^{(\alpha'-1)/6} \\
&= (-1)^{(p-\lambda)/6} a\left(\frac{n}{p}\right) \\
&= \epsilon a\left(\frac{n}{p}\right),
\end{aligned}
$$

where

$$\epsilon = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12}. \end{cases}$$

This completes the proof of Theorem 1 in the case $r = 1$.

Proof of Theorem 1 in the case $r = 4$

We have

$$(q)_\infty^4 = (q)_\infty (q)_\infty^3 = f_1(q) f_2(q) = \sum_{\alpha \equiv 1 \pmod 6,\ \beta \equiv 1 \pmod 4} (-1)^{(\alpha-1)/6} \beta q^{(\alpha^2+3\beta^2-4)/24}.$$

Thus

$$a(n) = \sum_{\substack{\alpha \equiv 1 \pmod 6,\ \beta \equiv 1 \pmod 4 \\ (\alpha^2+3\beta^2-4)/24 = n}} (-1)^{(\alpha-1)/6} \beta = \sum_{\substack{\alpha \equiv 1 \pmod 6,\ \beta \equiv 1 \pmod 4 \\ \alpha^2+3\beta^2 = 24n+4}} (-1)^{(\alpha-1)/6} \beta.$$

Therefore

$$a(pn + 4\Delta) = a\left(pn + \frac{p^2-1}{6}\right) = \sum_{\substack{\alpha \equiv 1 \pmod 6,\ \beta \equiv 1 \pmod 4 \\ \alpha^2+3\beta^2 = 24pn+4p^2}} (-1)^{(\alpha-1)/6} \beta.$$

The condition $\alpha^2 + 3\beta^2 = 24pn + 4p^2$ implies $\alpha^2 + 3\beta^2 \equiv 0 \pmod p$. Since $\left(\frac{-3}{p}\right) = -1$ [7, Theorem 96], this implies $\alpha,\ \beta \equiv 0 \pmod p$. If $p \equiv 5 \pmod{12}$, let $\alpha = -p\alpha',\ \beta = p\beta'$, while if $p \equiv 11 \pmod{12}$, let $\alpha = -p\alpha'$, $\beta = -p\beta'$. Then $\alpha',\ \beta'$ are integers and $\alpha' \equiv 1 \pmod 6$, $\beta' \equiv 1 \pmod 4$. If $p \equiv 5 \pmod{12}$ then, modulo 2,

$$\frac{\alpha-1}{6} = \frac{-p\alpha'-1}{6} = \frac{\alpha'-1}{6} - \alpha'\frac{p+1}{6} \equiv \frac{\alpha'-1}{6} + \alpha' \equiv \frac{\alpha'-1}{6} + 1$$

and

$$(-1)^{(\alpha-1)/6}\beta = -(-1)^{(\alpha'-1)/6} p\beta',$$

while if $p \equiv 11 \pmod{12}$,

$$\frac{\alpha-1}{6} = \frac{-p\alpha'-1}{6} = \frac{\alpha'-1}{6} - \alpha'\frac{p+1}{6} \equiv \frac{\alpha'-1}{6}$$

and once again,

$$(-1)^{(\alpha-1)/6}\beta = -(-1)^{(\alpha'-1)/6}p\beta'.$$

Furthermore, the condition $\alpha^2+3\beta^2 = 24pn+4p^2$ becomes ${\alpha'}^2+3{\beta'}^2 = 24n/p+4$. Therefore

$$\begin{aligned} a\left(pn+\frac{p^2-1}{6}\right) &= \sum_{\substack{\alpha\equiv 1 \pmod 6,\ \beta\equiv 1 \pmod 4\\ \alpha^2+3\beta^2=24pn+4p^2}} (-1)^{(\alpha-1)/6}\beta \\ &= -p \sum_{\substack{\alpha'\equiv 1 \pmod 6,\ \beta'\equiv 1 \pmod 4\\ {\alpha'}^2+3{\beta'}^2=24n/p+4}} (-1)^{(\alpha'-1)/6}\beta' \\ &= -p\ a\left(\frac{n}{p}\right). \end{aligned}$$

This completes the proof of the case $r = 4$ of Theorem 1.

Proof of Theorem 1 in the case $r = 8$

We have

$$(q)_\infty^8 = f_7(q) = \frac{1}{2}\sum_{\alpha,\beta\equiv 1 \pmod 3} (2\alpha-\beta)(2\beta-\alpha)(\alpha+\beta)q^{(\alpha^2-\alpha\beta+\beta^2-1)/3}.$$

Therefore

$$a(n) = \frac{1}{2}\sum_{\substack{\alpha,\beta\equiv 1 \pmod 3\\ \alpha^2-\alpha\beta+\beta^2=3n+1}} (2\alpha-\beta)(2\beta-\alpha)(\alpha+\beta)$$

and so

$$a(pn+8\Delta) = a\left(pn+\frac{p^2-1}{3}\right) = \frac{1}{2}\sum_{\substack{\alpha,\beta\equiv 1 \pmod 3\\ \alpha^2-\alpha\beta+\beta^2=3pn+p^2}} (2\alpha-\beta)(2\beta-\alpha)(\alpha+\beta).$$

Consider first the case $p = 2$. The right hand side of $\alpha^2-\alpha\beta+\beta^2 = 3pn+p^2$ is even, and this implies α and β are both even. Let $\alpha = -2\alpha'$, $\beta = -2\beta'$. Then α' and β' are integers, $\alpha',\ \beta' \equiv 1 \pmod 3$ and ${\alpha'}^2-\alpha'\beta'+{\beta'}^2 = 3n/2+1$. Therefore

$$\begin{aligned} a(2n+1) &= \frac{1}{2}\sum_{\substack{\alpha',\beta'\equiv 1 \pmod 3\\ {\alpha'}^2-\alpha'\beta'+{\beta'}^2=3n/2+1}} -8(2\alpha'-\beta')(2\beta'-\alpha')(\alpha'+\beta') \\ &= -8a\left(\frac{n}{2}\right). \end{aligned}$$

Now consider the case $p \equiv 2 \pmod 3$, $p > 2$, p prime. The equation $\alpha^2-\alpha\beta+\beta^2 = 3pn+p^2$ is equivalent to $(\alpha+\beta)^2+3(\alpha-\beta)^2 = 12pn+4p^2$ and so

$(\alpha+\beta)^2+3(\alpha-\beta)^2 \equiv 0 \pmod p$. Since $\left(\frac{-3}{p}\right) = -1$, [7, Theorem 96] this implies $\alpha+\beta \equiv 0 \pmod p$ and $\alpha-\beta \equiv 0 \pmod p$. Consequently $\alpha,\ \beta \equiv 0 \pmod p$. Let $\alpha = -p\alpha'$, $\beta = -p\beta'$. Then $\alpha',\ \beta'$ are integers, $\alpha',\ \beta' \equiv 1 \pmod 3$ and ${\alpha'}^2 - \alpha'\beta' + {\beta'}^2 = 3n/p+1$. Therefore

$$\begin{aligned} a\left(pn+\frac{p^2-1}{3}\right) &= \frac{1}{2} \sum_{\substack{\alpha',\beta' \equiv 1 \pmod 3 \\ {\alpha'}^2-\alpha'\beta'+{\beta'}^2=3n/p+1}} -p^3(2\alpha'-\beta')(2\beta'-\alpha')(\alpha'+\beta') \\ &= -p^3 a\left(\frac{n}{p}\right). \end{aligned}$$

Combining the two cases, we see that if $p \equiv 2 \pmod 3$ is prime, then

$$a\left(pn+\frac{p^2-1}{3}\right) = -p^3 a\left(\frac{n}{p}\right).$$

This completes the proof of Theorem 1 in the case $r = 8$.

Proof of Theorem 2 in the case $(r,s) = (1,5)$
We have

$$(q)_\infty (q^2)_\infty^5 = (q)_\infty^3 \frac{(q^2)_\infty^5}{(q)_\infty^2} = f_2(q) f_6(q) = \sum_{\substack{\alpha \equiv 1 \pmod 4 \\ \beta \equiv 1 \pmod 3}} \alpha\beta(-1)^{\beta-1} q^{(3\alpha^2+8\beta^2-11)/24}.$$

Thus

$$a(n) = \sum_{\substack{\alpha \equiv 1 \pmod 4,\ \beta \equiv 1 \pmod 3 \\ 3\alpha^2+8\beta^2=24n+11}} \alpha\beta(-1)^{\beta-1}.$$

Therefore

$$a(pn+11\Delta) = a\left(pn+\frac{11p^2-11}{24}\right) = \sum_{\substack{\alpha \equiv 1 \pmod 4,\ \beta \equiv 1 \pmod 3 \\ 3\alpha^2+8\beta^2=24pn+11p^2}} \alpha\beta(-1)^{\beta-1}.$$

The condition $3\alpha^2+8\beta^2 = 24pn+11p^2$ implies $3\alpha^2+8\beta^2 \equiv 0 \pmod p$, therefore $6\alpha^2+(4\beta)^2 \equiv 0 \pmod p$. By [7, Theorems 85, 93 and 96], we have

$$\left(\frac{-6}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{-3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1,5,7 \text{ or } 11 \pmod{24} \\ -1 & \text{if } p \equiv 13,17,19 \text{ or } 23 \pmod{24}. \end{cases}$$

Therefore $p \equiv 13, 17, 19$ or $23 \pmod{24}$ implies $\alpha \equiv 0 \pmod p$ and $4\beta \equiv 0 \pmod p$, and consequently $\beta \equiv 0 \pmod p$. Let

$$\alpha = \begin{cases} p\alpha' & \text{if } p \equiv 1 \pmod 4 \\ -p\alpha' & \text{if } p \equiv -1 \pmod 4 \end{cases}, \beta = \begin{cases} p\beta' & \text{if } p \equiv 1 \pmod 6 \\ -p\beta' & \text{if } p \equiv -1 \pmod 6 \end{cases}.$$

Then $\alpha' \equiv 1 \pmod 4$, $\beta' \equiv 1 \pmod 6$ and $3{\alpha'}^2 + 8{\beta'}^2 = 24n/p + 11$. Furthermore $\beta \equiv \beta' \pmod 2$, so $\alpha\beta(-1)^{\beta-1} = \epsilon p^2 \alpha'\beta'(-1)^{\beta'-1}$, where

$$\epsilon = \begin{cases} 1 & \text{if } p \equiv 13 \text{ or } 23 \pmod{24} \\ -1 & \text{if } p \equiv 17 \text{ or } 19 \pmod{24} \end{cases}.$$

Therefore

$$a\left(pn + \frac{11p^2 - 11}{24}\right) = \epsilon p^2 \sum_{\substack{\alpha' \equiv 1 \pmod 4,\ \beta' \equiv 1 \pmod 3 \\ 3{\alpha'}^2 + 8{\beta'}^2 = 24n/p + 11}} \alpha'\beta'(-1)^{\beta'-1} = \epsilon p^2 a\left(\frac{n}{p}\right).$$

This completes the proof of the case $(r, s) = (1, 5)$ of Theorem 2.

Proof of Theorem 1 in the case $r = 2$

We have

$$(q)_\infty^2 = f_1(q)^2 = \sum_{\alpha,\ \beta \equiv 1 \pmod 6} (-1)^{(\alpha+\beta-2)/6} q^{(\alpha^2+\beta^2-2)/24},$$

thus

(7) $$a(n) = \sum_{\substack{\alpha,\ \beta \equiv 1 \pmod 6 \\ \alpha^2+\beta^2 = 24n+2}} (-1)^{(\alpha+\beta-2)/6}$$

and therefore

$$a(pn + 2\Delta) = a\left(pn + \frac{p^2 - 1}{12}\right) = \sum_{\substack{\alpha,\ \beta \equiv 1 \pmod 6 \\ \alpha^2+\beta^2 = 24pn+2p^2}} (-1)^{(\alpha+\beta-2)/6}.$$

The condition $\alpha^2 + \beta^2 = 24pn + 2p^2$ implies $\alpha^2 + \beta^2 \equiv 0 \pmod p$.

Case 1

If $p \equiv 7$ or $11 \pmod{12}$, then $\left(\frac{-1}{p}\right) = -1$ [7, Theorem 82]. Consequently, $\alpha,\ \beta \equiv 0 \pmod p$. If $p \equiv 7 \pmod{12}$, let $\alpha = p\alpha'$, $\beta = p\beta'$, while if $p \equiv 11 \pmod{12}$, let $\alpha = -p\alpha'$, $\beta = -p\beta'$. Then α' and β' are integers, $\alpha',\ \beta' \equiv 1 \pmod 6$ and $(-1)^{(\alpha+\beta-2)/6} = (-1)^{(\alpha'+\beta'-2)/6}$. Furthermore, the condition $\alpha^2 + \beta^2 = 24pn + 2p^2$ becomes ${\alpha'}^2 + {\beta'}^2 = 24n/p + 2$. Therefore

$$\begin{aligned} a\left(pn + \frac{p^2-1}{12}\right) &= \sum_{\substack{\alpha,\ \beta \equiv 1 \pmod 6 \\ \alpha^2+\beta^2 = 24pn+2p^2}} (-1)^{(\alpha+\beta-2)/6} \\ &= \sum_{\substack{\alpha',\ \beta' \equiv 1 \pmod 6 \\ {\alpha'}^2+{\beta'}^2 = 24n/p+2}} (-1)^{(\alpha'+\beta'-2)/6} \\ &= a\left(\frac{n}{p}\right). \end{aligned}$$

Case 2
If $p \equiv 5 \pmod{12}$, then $\left(\frac{-1}{p}\right) = +1$, and the condition $\alpha^2+\beta^2 \equiv 0 \pmod{p}$ does **not** imply $\alpha,\ \beta \equiv 0 \pmod{p}$. Therefore a different method is required. Case 2 is a corollary of the following Lemma.

Lemma Suppose $p \equiv 5 \pmod{12}$ is prime, let $\Delta_k = (p^{2k}-1)/24$, and let $(q)_\infty^2 = \sum_{n\geq 0} a(n)q^n$. Then the coefficients $a(n)$ satisfy
(i) $a(p^{2k}n + 2\Delta_k) = (-1)^k a(n)$ if $12n+1 \not\equiv 0 \pmod{p}$,
(ii) $a(p^{2k+1}n + 2\Delta_{k+1}) = 0$ if $n \not\equiv 0 \pmod{p}$.

Theorem (Case 2) With $p \equiv 5 \pmod{12}$, $\Delta = \Delta_1 = \frac{p^2-1}{24}$ and $(q)_\infty^2 = \sum_{n\geq 0} a(n)q^n$,

$$a(pn+2\Delta) = -a\left(\frac{n}{p}\right).$$

Proof of Theorem
If $p \not| n$ then by part (ii) of the Lemma, $a(pn+2\Delta) = 0 = -a\left(\frac{n}{p}\right)$.
If $p|n$, we can write $n = pn'$. We are then trying to show

$$a(p^2n' + 2\Delta) = -a(n').$$

We can write $12n'+1 = p^r m$ with $m \not\equiv 0 \pmod{p}$ for some $r \geq 0$.
Suppose $r = 2k$. Then $m \equiv 1 \pmod{12}$ and

$$\begin{aligned}
a(p^2n'+2\Delta) &= a\left(p^2\left(\frac{p^{2k}m-1}{12}\right)+2\Delta\right)\\
&= a\left(p^2\left(\frac{p^{2k}m-1}{12}\right)+\frac{p^2-1}{12}\right)\\
&= a\left(p^{2k+2}\left(\frac{m-1}{12}\right)+\frac{p^{2k+2}-1}{12}\right)\\
&= a\left(p^{2k+2}\left(\frac{m-1}{12}\right)+2\Delta_{k+1}\right)\\
&= (-1)^{k+1}a\left(\frac{m-1}{12}\right)
\end{aligned}$$

while

$$a(n') = a\left(\frac{p^{2k}m-1}{12}\right)$$

$$
\begin{aligned}
&= a\left(p^{2k}\left(\frac{m-1}{12}\right)+\frac{p^{2k}-1}{12}\right)\\
&= a\left(p^{2k}\left(\frac{m-1}{12}\right)+2\Delta_k\right)\\
&= (-1)^k a\left(\frac{m-1}{12}\right).
\end{aligned}
$$

On the other hand, if $r = 2k+1$ then $m \equiv p \pmod{12}$ and

$$
\begin{aligned}
a(p^2n'+2\Delta) &= a\left(p^2\left(\frac{p^{2k+1}m-1}{12}\right)+\frac{p^2-1}{12}\right)\\
&= a\left(p^{2k+3}\left(\frac{m-p}{12}\right)+\frac{p^{2k+4}-1}{12}\right)\\
&= a\left(p^{2k+3}\left(\frac{m-p}{12}\right)+2\Delta_{k+2}\right)\\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
a(n') &= a\left(\frac{p^{2k+1}m-1}{12}\right)\\
&= a\left(p^{2k+1}\left(\frac{m-p}{12}\right)+\frac{p^{2k+2}-1}{12}\right)\\
&= a\left(p^{2k+1}\left(\frac{m-p}{12}\right)+2\Delta_{k+1}\right)\\
&= 0.
\end{aligned}
$$

This completes the proof of the Theorem.

Proof of Lemma

We have

$$a(n) = \sum_{\substack{\alpha_1\equiv 1,\ \alpha_2\equiv 1 \pmod 6\\ (\alpha_1^2+\alpha_2^2-2)/24=n}} (-1)^{(\alpha_1+\alpha_2-2)/6}.$$

Now let the Gaussian integer α be defined by $\alpha = \alpha_1 + i\alpha_2$ and let $\mathrm{N}(\alpha) = \alpha_1^2+\alpha_2^2 = \alpha\overline{\alpha}$, $\mathrm{tr}(\alpha) = \alpha_1+\alpha_2$ and $\varepsilon(\alpha) = (-1)^{(\mathrm{tr}(\alpha)-2)/6}$.

Then

$$a(n) = \sum_{\substack{\alpha\equiv 1+i \pmod 6\\ \mathrm{N}(\alpha)=24n+2}} \varepsilon(\alpha).$$

Also,

$$a(p^{2k}n+2\Delta_k) = \sum_{\substack{\alpha\equiv 1+i \pmod 6\\ \mathrm{N}(\alpha)=p^{2k}(24n+2)}} \varepsilon(\alpha)$$

since

$$24(p^{2k}n + 2\Delta_k) + 2 = 24(p^{2k}n + \frac{p^{2k}-1}{12}) + 2 = p^{2k}(24n+2).$$

So now we must consider the equation

$$\mathrm{N}(\alpha) = p^{2k}(24n+2),$$

or,

$$\alpha\overline{\alpha} = p^{2k}(24n+2).$$

Now it can be shown that $p \equiv 5 \pmod{12}$ can be uniquely factored

$$p = \pi\overline{\pi}$$

with $\pi \equiv 1 + 2i \pmod 6$.
Thus we have to consider

$$\alpha\overline{\alpha} = p^{2k}(24n+2) = \pi^{2k}\overline{\pi}^{2k}(24n+2)$$

with $12n+1 \not\equiv 0 \pmod p$. There are just three possibilities, since factorisation is (essentially) unique in $\mathbf{Z}[i]$. They are: both π and $\overline{\pi}|\alpha$, $\overline{\pi} \not| \alpha$ and $\pi \not| \alpha$.
In the first case, $p|\alpha$, so write $\alpha = -p\alpha'$. Then $\alpha' \equiv 1 + i \pmod 6$, $\mathrm{N}(\alpha') = p^{2k-2}(24n+2)$ and $\varepsilon(\alpha) = \varepsilon(\alpha')$.
In the second, we have $\pi^{2k}|\alpha$, so write

$$\alpha = -i\pi^2\alpha_1', \quad \alpha_1' = -i\pi^2\alpha_2', \ \cdots, \alpha_{k-1}' = -i\pi^2\alpha'.$$

Then $\alpha_1', \ \alpha_2', \ \cdots, \alpha_{k-1}', \ \alpha' \equiv 1 + i \pmod 6$, $\mathrm{N}(\alpha') = 24n+2$, $\overline{\pi} \not| \alpha'$ and

$$\varepsilon(\alpha) = -\varepsilon(\alpha_1') = \ \cdots \ = (-1)^k\varepsilon(\alpha').$$

Note that since $12n + 1 \not\equiv 0 \pmod p$, the restriction $\overline{\pi} \not| \alpha'$ is superfluous.
Similarly in the third case, $\overline{\pi}^{2k}|\alpha$, so write

$$\alpha = i\overline{\pi}^2\alpha_1', \quad \alpha_1' = i\overline{\pi}^2\alpha_2', \ \cdots, \alpha_{k-1}' = i\overline{\pi}^2\alpha'.$$

Then $\alpha_1', \ \alpha_2', \ \cdots, \alpha_{k-1}', \ \alpha' \equiv 1 + i \pmod 6$, $\mathrm{N}(\alpha') = 24n+2$, $\pi \not| \alpha'$, again superfluous, and

$$\varepsilon(\alpha) = -\varepsilon(\alpha_1') = \ \cdots \ = (-1)^k\varepsilon(\alpha').$$

Thus we have

$$\begin{aligned}
a(p^{2k}n + 2\Delta_k) &= \sum_{p|\alpha}\varepsilon(\alpha) + \sum_{\overline{\pi}\not|\alpha}\varepsilon(\alpha) + \sum_{\pi\not|\alpha}\varepsilon(\alpha) \\
&= \sum_{\substack{\alpha'\equiv 1+i \pmod 6 \\ \mathrm{N}(\alpha')=p^{2k-2}(24n+2)}} \varepsilon(\alpha') + 2(-1)^k \sum_{\substack{\alpha'\equiv 1+i \pmod 6 \\ \mathrm{N}(\alpha')=24n+2}} \varepsilon(\alpha') \\
&= a(p^{2(k-1)}n + 2\Delta_{k-1}) + 2(-1)^k a(n).
\end{aligned}$$

It now follows by induction on k that $a(p^{2k}n + 2\Delta_k) = (-1)^k a(n)$.

A similar argument shows that if $n \not\equiv 0 \pmod p$,

$$a(p^{2k+1}n + 2\Delta_{k+1}) = a(p^{2(k-1)+1}n + 2\Delta_k) + (-1)^k a(pn + 2\Delta),$$

and hence by induction on k,

$$a(p^{2k+1}n + 2\Delta_{k+1}) = \left(\frac{1 + (-1)^k}{2}\right) a(pn + 2\Delta).$$

Now,

$$a(pn + 2\Delta) = \sum_{\substack{\alpha \equiv 1+i \pmod 6 \\ \mathrm{N}(\alpha) = p(24n+2p)}} \varepsilon(\alpha)$$

since $24(pn + 2\Delta) + 2 = 24(pn + \frac{p^2-1}{12}) + 2 = 24pn + 2p^2 = p(24n + 2p)$. So we need to consider

$$\mathrm{N}(\alpha) = p(24n + 2p),$$

or,

$$\alpha\overline{\alpha} = p(24n + 2p).$$

If $n \not\equiv 0 \pmod p$ then $\pi|\alpha$ or $\overline{\pi}|\alpha$, but not both. In the first case, write $\alpha = \pi\alpha'$, in the second $\alpha = -i\overline{\pi}\alpha'$. In either case, $\alpha' \equiv 3 + i \pmod 6$ and $\mathrm{N}(\alpha') = 24n + 2p$.
So

$$\begin{aligned} a(pn + 2\Delta) &= \sum_{\pi|\alpha} \varepsilon(\alpha) + \sum_{\overline{\pi}|\alpha} \varepsilon(\alpha) \\ &= \sum_{\substack{\alpha' \equiv 3+i \pmod 6 \\ \mathrm{N}(\alpha') = 24n+2p}} \varepsilon(\pi\alpha') + \sum_{\substack{\alpha' \equiv 3+i \pmod 6 \\ \mathrm{N}(\alpha') = 24n+2p}} \varepsilon(-i\overline{\pi}\alpha') \\ &= 0. \end{aligned}$$

(To see this, consider $\alpha' = a + ib$ in both sums.) This completes the proof of the lemma.

Much the same argument as the one given here, with the ring $\mathbf{Z}[\sqrt{-2}]$ (also a unique factorisation domain) in place of $\mathbf{Z}[i]$, may be used to prove the case $r = s = 1,\ p \equiv 3 \pmod 8$ of theorem 2.

4 Remaining Proofs

The remaining cases of Theorems 1 – 4 can be proved in a similar fashion to the previous section, using $f_1, \cdots, f_{13}$ and various combinations of $f_1, \cdots, f_6$ as follows.

For Theorem 1, use

$$\begin{aligned}
(q)_\infty &= f_1(q)\\
(q)_\infty^2 &= f_1(q)^2\\
(q)_\infty^3 &= f_2(q)\\
(q)_\infty^4 &= f_1(q)f_2(q)\\
(q)_\infty^6 &= f_2(q)^2\\
(q)_\infty^8 &= f_7(q)\\
(q)_\infty^{10} &= f_8(q)\\
(q)_\infty^{14} &= f_{12}(q)
\end{aligned}$$

For Theorem 2, use

$$\begin{aligned}
(q)_\infty(q^2)_\infty &= f_1(q)f_1(q^2)\\
(q)_\infty^2(q^2)_\infty^2 &= f_2(q^2)f_3(q)\\
(q)_\infty^3(q^2)_\infty^3 &= f_2(q)f_2(q^2)\\
(q)_\infty^5(q^2)_\infty^5 &= f_9(q)
\end{aligned}$$

$$\begin{aligned}
(q)_\infty^4(q^2)_\infty^2 &= f_6(q^{1/2})f_6(-q^{1/2}) & (q)_\infty^2(q^2)_\infty^4 &= f_5(q^2)f_6(-q)\\
(q)_\infty^3(q^2)_\infty &= f_1(q^2)f_2(q) & (q)_\infty(q^2)_\infty^3 &= f_1(q)f_2(q^2)\\
(q)_\infty^5(q^2)_\infty &= f_2(q^2)f_5(q) & (q)_\infty(q^2)_\infty^5 &= f_2(q)f_6(q)\\
(q)_\infty^2/(q^2)_\infty &= f_3(q) & (q^2)_\infty^2/(q)_\infty &= f_4(q)\\
(q)_\infty^3/(q^2)_\infty &= f_1(q)f_3(q) & (q^2)_\infty^3/(q)_\infty &= f_1(q^2)f_4(q)\\
(q)_\infty^5/(q^2)_\infty &= f_1(q^2)f_5(q) & (q^2)_\infty^5/(q)_\infty &= f_1(q)f_6(q)\\
(q)_\infty^4/(q^2)_\infty^2 &= f_3(q)^2 & (q^2)_\infty^4/(q)_\infty^2 &= f_4(q)^2\\
(q)_\infty^5/(q^2)_\infty^2 &= f_5(q) & (q^2)_\infty^5/(q)_\infty^2 &= f_6(q)\\
(q)_\infty^6/(q^2)_\infty^2 &= f_1(q)f_5(q) & (q^2)_\infty^6/(q)_\infty^2 &= f_1(q^2)f_6(q)\\
(q)_\infty^8/(q^2)_\infty^2 &= f_2(q)f_5(q) & (q^2)_\infty^8/(q)_\infty^2 &= f_2(q^2)f_6(q)\\
(q)_\infty^7/(q^2)_\infty^3 &= f_3(q)f_5(q) & (q^2)_\infty^7/(q)_\infty^3 &= f_4(q)f_6(q)\\
(q)_\infty^9/(q^2)_\infty^3 &= f_2(q^{1/2})f_2(-q^{1/2}) & (q^2)_\infty^9/(q)_\infty^3 &= f_2(q^4)f_2(-q)\\
(q)_\infty^{10}/(q^2)_\infty^4 &= f_5(q)^2 & (q^2)_\infty^{10}/(q)_\infty^4 &= f_6(q)^2\\
(q)_\infty^{14}/(q^2)_\infty^4 &= f_{10}(q) & (q^2)_\infty^{14}/(q)_\infty^4 &= f_{11}(q)\\
(q)_\infty^{11}/(q^2)_\infty^5 &= f_5(q^{1/2})f_5(-q^{1/2}) & (q^2)_\infty^{11}/(q)_\infty^5 &= f_6(q^2)f_5(-q).
\end{aligned}$$

For Theorem 3, use

$$\begin{aligned}
(q)_\infty(q^3)_\infty &= f_1(q)f_1(q^3)\\
(q)_\infty^3(q^3)_\infty^3 &= f_2(q)f_2(q^3)\\
(q)_\infty^7(q^3)_\infty^7 &= f_{13}(q)
\end{aligned}$$

$$(q)_\infty^3(q^3)_\infty = f_1(q^3)f_2(q) \qquad (q)_\infty(q^3)_\infty^3 = f_1(q)f_2(q^3)$$

For Theorem 4, use

$$\begin{aligned}
(q)_\infty (q^4)_\infty &= f_1(q) f_1(q^4) \\
(q)_\infty^2 (q^4)_\infty^2 &= f_1(q^2) f_6(-q) \\
(q)_\infty^3 (q^4)_\infty^3 &= f_2(q) f_2(q^4)
\end{aligned}$$

$$\begin{aligned}
(q)_\infty^3 (q^4)_\infty &= f_1(q^4) f_2(q) & \quad (q)_\infty (q^4)_\infty^3 &= f_1(q) f_2(q^4) \\
(q)_\infty^5 / (q^4)_\infty &= f_3(q^2) f_5(q) & \quad (q^4)_\infty^5 / (q)_\infty &= f_4(q) f_6(q^2)
\end{aligned}$$

5 Conjectures

The following conjectures were suggested by computer search.

Conjecture 1. *Theorem 2 holds for the following additional values of* r, s, p *and* ϵ*:*

(r,s)	p	ϵ
$(9,9)$	$p \equiv 7 \pmod 8$	1
$(7,3),\ (3,7)$	$p \equiv 7$ or $11 \pmod{12}$	$\begin{cases} 1 & \text{if } p \equiv 7 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$
$(7,1),\ (1,7)$	$p = 31$	-1
$(3,-1),\ (-1,3)$	$p \equiv 7$ or $11 \pmod{24}$	$\begin{cases} 1 & \text{if } p \equiv 7 \pmod{24} \\ -1 & \text{if } p \equiv 11 \pmod{24} \end{cases}$
$(7,-1),\ (-1,7)$	$p \equiv 7$ or $11 \pmod{12}$	$\begin{cases} 1 & \text{if } p \equiv 7 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$
$(13,-5),\ (-5,13)$	$p = 31$	-1
$(15,-5),\ (-5,15)$	$p \equiv 19$ or $23 \pmod{24}$	$\begin{cases} 1 & \text{if } p \equiv 23 \pmod{24} \\ -1 & \text{if } p \equiv 19 \pmod{24} \end{cases}$
$(16,-6),\ (-6,16)$	$p \equiv 11 \pmod{12}$	1
$(17,-7),\ (-7,17)$	$p \equiv 7 \pmod 8$	1
$(18,-8),\ (-8,18)$	$p \equiv 11 \pmod{12}$	1
$(19,-9),\ (-9,19)$	$p \equiv 19$ or $23 \pmod{24}$	$\begin{cases} 1 & \text{if } p \equiv 23 \pmod{24} \\ -1 & \text{if } p \equiv 19 \pmod{24} \end{cases}$

Conjecture 2. *Theorem 3 holds for the following additional values of* r, s, p *and* ϵ*:*

(r,s)	p	ϵ
$(2,2)$	$p \equiv 5 \pmod 6$	-1
$(5,3),\ (3,5)$	$p \equiv 5 \pmod 6$	$\begin{cases} 1 & \text{if } p \equiv 5 \pmod{12} \\ -1 & \text{if } p \equiv 11 \pmod{12} \end{cases}$
$(3,-1),\ (-1,3)$	$p \equiv 5 \pmod 6$	1
$(5,-1),\ (-1,5)$	$p \equiv 5 \pmod 6$	$\begin{cases} 1 & \text{if } p \equiv 5 \pmod{12} \\ -1 & \text{if } p \equiv 11 \pmod{12} \end{cases}$
$(9,-1),\ (-1,9)$	$p \equiv 11 \pmod{12}$	-1
$(10,-2),\ (-2,10)$	$p \equiv 5 \pmod 6$	-1
$(11,-3),\ (-3,11)$	$p \equiv 11 \pmod{12}$	-1

Conjecture 3. *Theorem 4 holds for the following additional values of* r, s, p *and* ϵ*:*

(r,s)	p	ϵ
$(5,5)$	$p \equiv 19 \text{ or } 23 \pmod{24}$	1
$(7,-1),\ (-1,7)$	$p \equiv 7 \pmod 8$	1

Remark. Since this paper was written, Robin Chapman has proved the cases $(r,s) = (3,-1)$ and $(r,s) = (-1,3)$ of Conjecture 1.

This investigation was partially supported by a grant to the second author from the ARC.

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