

A PROBLEM IN DYNAMICS

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1. The problem

Three masses, m_1 , m_2 , m_3 are placed in line on a smooth horizontal surface. Initially, m_1 and m_3 are at rest, and m_2 is moving towards m_3 with speed v . Assuming elastic collisions between masses, determine the final velocities of the three masses.

This problem was posed in a first-year Physics course at UNSW, and the students were asked to consider the case in which $m_1, m_3 \gg m_2$. The given solution,

$$v_1 = -v\sqrt{\frac{m_2m_3}{m_1(m_1+m_3)}}, \quad v_2 = v_3 = v\sqrt{\frac{m_1m_2}{m_3(m_1+m_3)}}$$

was unsatisfactory to me, particularly because it was counter-intuitive that we should have $v_2 = v_3$. So it seemed desirable to solve the general problem exactly, and that is what I have done.

2. The solution

Let $(v_1)_n$, $(v_2)_n$, $(v_3)_n$ be the velocities of the masses m_1 , m_2 , m_3 after n collisions, and let

$$\mathbf{v}_n = \begin{pmatrix} (v_1)_n \\ (v_2)_n \\ (v_3)_n \end{pmatrix}.$$

Then

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}.$$

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Upon collision, the velocities of the masses change, and the changes are governed by conservation of momentum and conservation of energy.

From the odd-numbered collisions (those between m_2 and m_3) $(v_1)_{2n+1}$, $(v_2)_{2n+1}$, $(v_3)_{2n+1}$ are determined by

$$\begin{aligned}(v_1)_{2n+1} &= (v_1)_{2n}, \\ m_2(v_2)_{2n+1} + m_3(v_3)_{2n+1} &= m_2(v_2)_{2n} + m_3(v_3)_{2n}, \\ m_2(v_2)_{2n+1}^2 + m_3(v_3)_{2n+1}^2 &= m_2(v_2)_{2n}^2 + m_3(v_3)_{2n}^2.\end{aligned}$$

It follows that

$$\begin{aligned}(v_2)_{2n+1} + (v_2)_{2n} &= (v_3)_{2n+1} + (v_3)_{2n}, \\ \begin{pmatrix} m_2 & m_3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (v_2)_{2n+1} \\ (v_3)_{2n+1} \end{pmatrix} &= \begin{pmatrix} m_2 & m_3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (v_2)_{2n} \\ (v_3)_{2n} \end{pmatrix}\end{aligned}$$

and

$$\begin{pmatrix} (v_2)_{2n+1} \\ (v_3)_{2n+1} \end{pmatrix} = \begin{pmatrix} \frac{m_2 - m_3}{m_2 + m_3} & \frac{2m_3}{m_2 + m_3} \\ \frac{2m_2}{m_2 + m_3} & \frac{m_3 - m_2}{m_2 + m_3} \end{pmatrix} \begin{pmatrix} (v_2)_{2n} \\ (v_3)_{2n} \end{pmatrix}.$$

Thus

$$\mathbf{v}_{2n+1} = R\mathbf{v}_{2n}$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{m_2 - m_3}{m_2 + m_3} & \frac{2m_3}{m_2 + m_3} \\ 0 & \frac{2m_2}{m_2 + m_3} & -\frac{m_2 - m_3}{m_2 + m_3} \end{pmatrix}$$

Similarly, from the even-numbered collisions (those between m_2 and m_1),

$$\mathbf{v}_{2n+2} = L\mathbf{v}_{2n+1}$$

where

$$L = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} & 0 \\ \frac{2m_1}{m_1 + m_2} & -\frac{m_1 - m_2}{m_1 + m_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that if we write

$$m_3/m_2 = r, \quad m_1/m_2 = l$$

then

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{r-1}{r+1} & \frac{2r}{r+1} \\ 0 & \frac{2}{r+1} & \frac{r-1}{r+1} \end{pmatrix}, \quad L = \begin{pmatrix} \frac{l-1}{l+1} & \frac{2}{l+1} & 0 \\ \frac{2l}{l+1} & -\frac{l-1}{l+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It thus becomes clear from the mathematics (if it was not already so from the physics) that the behaviour of the velocities depends not on the masses, but on their ratios.

It follows that

$$\begin{aligned} \mathbf{v}_{2n} &= (LR)^n \mathbf{v}_0, \\ \mathbf{v}_{2n+1} &= R(LR)^n \mathbf{v}_0. \end{aligned}$$

Now,

$$LR = \begin{pmatrix} \frac{l-1}{l+1} & -\frac{2(r-1)}{(l+1)(r+1)} & \frac{4r}{(l+1)(r+1)} \\ \frac{2l}{l+1} & \frac{(l-1)(r-1)}{(l+1)(r+1)} & -\frac{2(l-1)r}{(l+1)(r+1)} \\ 0 & \frac{2}{r+1} & \frac{r-1}{r+1} \end{pmatrix}.$$

The characteristic polynomial of LR is

$$(\lambda - 1) \left(\lambda^2 - 2 \left(\frac{lr - l - r - 1}{(l+1)(r+1)} \right) \lambda + 1 \right).$$

If ϕ is defined by

$$\cos \phi = \sqrt{\frac{lr}{(l+1)(r+1)}}, \quad \sin \phi = \sqrt{\frac{(l+r+1)}{(l+1)(r+1)}},$$

then the characteristic polynomial of LR becomes

$$(\lambda - 1)(\lambda^2 - 2 \cos 2\phi \lambda + 1).$$

Thus the eigenvalues of LR are

$$1, \quad e^{\pm 2\phi i}.$$

If $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are defined by

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 1 \\ -(l+r) \\ 1 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} \sqrt{r(l+r+1)/l} \\ 0 \\ -\sqrt{l(l+r+1)/r} \end{pmatrix}$$

then

$$\begin{aligned} LR\mathbf{p}_1 &= \mathbf{p}_1, \\ LR\mathbf{p}_2 &= \cos 2\phi \mathbf{p}_2 + \sin 2\phi \mathbf{p}_3, \\ LR\mathbf{p}_3 &= -\sin 2\phi \mathbf{p}_2 + \cos 2\phi \mathbf{p}_3. \end{aligned}$$

Thus, if P is the matrix whose columns are $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and M is the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi \\ 0 & \sin 2\phi & \cos 2\phi \end{pmatrix}$$

then

$$LR = PMP^{-1}.$$

It follows that

$$\mathbf{v}_{2n} = (LR)^n \mathbf{v}_0 = PM^n P^{-1} \mathbf{v}_0 = PM^n P^{-1} \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$$

and

$$\mathbf{v}_{2n+1} = R\mathbf{v}_{2n}.$$

It is clear that we need only the central column of P^{-1} , and that is easily verified to be

$$\begin{pmatrix} 1/(l+r+1) \\ -1/(l+r+1) \\ 0 \end{pmatrix}.$$

It follows that

$$P^{-1} \mathbf{v}_0 = P^{-1} \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} v/(l+r+1) \\ -v/(l+r+1) \\ 0 \end{pmatrix}$$

and

$$M^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2n\phi & -\sin 2n\phi \\ 0 & \sin 2n\phi & \cos 2n\phi \end{pmatrix}$$

so

$$M^n P^{-1} \mathbf{v}_0 = \begin{pmatrix} \frac{v}{l+r+1} \\ -\frac{v}{l+r+1} \cos 2n\phi \\ -\frac{v}{l+r+1} \sin 2n\phi \end{pmatrix}$$

and

$$\mathbf{v}_{2n} = PM^n P^{-1} \mathbf{v}_0 = \begin{pmatrix} \frac{v}{l+r+1} - \frac{v}{l+r+1} \left(\cos 2n\phi + \sqrt{\frac{r(l+r+1)}{l}} \sin 2n\phi \right) \\ \frac{v}{l+r+1} + \frac{(l+r)v}{l+r+1} \cos 2n\phi \\ \frac{v}{l+r+1} + \frac{v}{l+r+1} \left(\sqrt{\frac{l(l+r+1)}{r}} \sin 2n\phi - \cos 2n\phi \right) \end{pmatrix}.$$

If we write

$$\bar{v} = \frac{v}{l+r+1} = \frac{m_2 v}{m_1 + m_2 + m_3}$$

then \bar{v} is the velocity of the centre of mass of the system (this velocity is constant), and

(1)

$$\begin{aligned} (v_1)_{2n} &= \bar{v} \left\{ 1 - \left(\cos 2n\phi + \sqrt{\frac{r(l+r+1)}{l}} \sin 2n\phi \right) \right\} \\ (v_2)_{2n} &= \bar{v} \left\{ 1 + (l+r) \cos 2n\phi \right\} \\ (v_3)_{2n} &= \bar{v} \left\{ 1 + \left(\sqrt{\frac{l(l+r+1)}{r}} \sin 2n\phi - \cos 2n\phi \right) \right\} \end{aligned}$$

and from $\mathbf{v}_{2n+1} = R\mathbf{v}_{2n}$,

$$\begin{aligned} (v_1)_{2n+1} &= \bar{v} \left\{ 1 - \left(\cos 2n\phi + \sqrt{\frac{r(l+r+1)}{l}} \sin 2n\phi \right) \right\} \\ (v_2)_{2n+1} &= \bar{v} \left\{ 1 - \left(\frac{lr+r^2+r-l}{r+1} \cos 2n\phi - \frac{2\sqrt{lr(l+r+1)}}{r+1} \sin 2n\phi \right) \right\} \\ (v_3)_{2n+1} &= \bar{v} \left\{ 1 + \left(\frac{2l+r+1}{r+1} \cos 2n\phi + \frac{r-1}{r+1} \sqrt{\frac{l(l+r+1)}{r}} \sin 2n\phi \right) \right\}. \end{aligned}$$

These results can be recast as

$$\begin{aligned} (v_1)_{2n} &= \bar{v} \left\{ 1 - \sqrt{\frac{(l+r)(r+1)}{l}} \sin(2n\phi + \epsilon) \right\} \\ (v_2)_{2n} &= \bar{v} \left\{ 1 + (l+r) \cos 2n\phi \right\} \end{aligned}$$

$$\begin{aligned}
(v_3)_{2n} &= \bar{v} \left\{ 1 + \sqrt{\frac{(l+1)(l+r)}{r}} \sin(2n\phi - \epsilon') \right\} \\
(v_1)_{2n+1} &= \bar{v} \left\{ 1 - \sqrt{\frac{(l+r)(r+1)}{l}} \sin(2n\phi + \epsilon) \right\} \\
(v_2)_{2n+1} &= \bar{v} \left\{ 1 - (l+r) \cos(2n\phi + \epsilon'') \right\} \\
(v_3)_{2n+1} &= \bar{v} \left\{ 1 + \sqrt{\frac{(l+1)(l+r)}{r}} \sin(2n\phi + \epsilon''') \right\}
\end{aligned}$$

where

$$\begin{aligned}
\cos \epsilon &= \sqrt{\frac{r(l+r+1)}{(l+r)(r+1)}}, & \sin \epsilon &= \sqrt{\frac{l}{(l+r)(r+1)}}, \\
\cos \epsilon' &= \sqrt{\frac{l(l+r+1)}{(l+1)(l+r)}}, & \sin \epsilon' &= \sqrt{\frac{r}{(l+1)(l+r)}}, \\
\cos \epsilon'' &= \frac{lr+r^2+r-l}{(l+r)(r+1)}, & \sin \epsilon'' &= \frac{2\sqrt{lr(l+r+1)}}{(l+r)(r+1)}, \\
\cos \epsilon''' &= \frac{r-1}{r+1} \sqrt{\frac{l(l+r+1)}{(l+1)(l+r)}}, & \sin \epsilon''' &= \frac{2l+r+1}{r+1} \sqrt{\frac{r}{(l+1)(l+r)}}.
\end{aligned}$$

and

$$\cos \phi = \sqrt{\frac{lr}{(l+1)(r+1)}}, \quad \sin \phi = \sqrt{\frac{l+r+1}{(l+1)(r+1)}}.$$

Expansion of $\cos 2\epsilon$, $\sin 2\epsilon$, $\cos(\phi + \epsilon)$, $\sin(\phi + \epsilon)$, $\cos(\epsilon + \epsilon')$, $\sin(\epsilon + \epsilon')$ shows that

$$\epsilon'' = 2\epsilon, \quad \epsilon''' = \phi + \epsilon, \quad \phi = \epsilon + \epsilon'.$$

Thus, the results may be recast solely in terms of ϵ , ϵ' as follows.

(2)

If n is even,

$$(v_1)_n = \bar{v} \left\{ 1 - \frac{\sin((n+1)\epsilon + n\epsilon')}{\sin \epsilon} \right\}$$

$$(v_2)_n = \bar{v} \left\{ 1 + \frac{\cos(\epsilon + \epsilon') \cos(n\epsilon + n\epsilon')}{\sin \epsilon \sin \epsilon'} \right\}$$

$$(v_3)_n = \bar{v} \left\{ 1 + \frac{\sin(n\epsilon + (n-1)\epsilon')}{\sin \epsilon'} \right\}$$

while if n is odd,

$$(v_1)_n = \bar{v} \left\{ 1 - \frac{\sin(n\epsilon + (n-1)\epsilon')}{\sin \epsilon} \right\}$$

$$(v_2)_n = \bar{v} \left\{ 1 - \frac{\cos(\epsilon + \epsilon') \cos((n+1)\epsilon + (n-1)\epsilon')}{\sin \epsilon \sin \epsilon'} \right\}$$

$$(v_3)_n = \bar{v} \left\{ 1 + \frac{\sin((n+1)\epsilon + n\epsilon')}{\sin \epsilon'} \right\}$$

where

$$\bar{v} = v \tan \epsilon \tan \epsilon'.$$

Now define “catch-up” velocities δ_n by

$$\delta_{2n+1} = (v_2)_{2n} - (v_3)_{2n}, \quad \delta_{2n+2} = (v_1)_{2n+1} - (v_2)_{2n+1}.$$

From the formulae (1)

$$\begin{aligned} \delta_{2n+1} &= v \left\{ \cos 2n\phi - \sqrt{\frac{l}{r(l+r+1)}} \sin 2n\phi \right\} \\ &= v \sqrt{\frac{(l+r)(r+1)}{r(l+r+1)}} \cos(2n\phi + \epsilon) \\ &= v \frac{\cos((2n+1)\epsilon + 2n\epsilon')}{\cos \epsilon}, \end{aligned}$$

$$\begin{aligned} \delta_{2n+2} &= v \left\{ \frac{r-1}{r+1} \cos 2n\phi - (2l+r+1) \sqrt{\frac{r(l+r+1)}{l}} \sin 2n\phi \right\} \\ &= v \sqrt{\frac{(l+1)(l+r)}{l(l+r+1)}} \cos(2n\phi + \epsilon''') \\ &= v \frac{\cos((2n+2)\epsilon + (2n+1)\epsilon')}{\cos \epsilon'}. \end{aligned}$$

Thus

$$\delta_n = C_n v \cos(n\epsilon + (n-1)\epsilon')$$

where

$$C_n = \begin{cases} \sec \epsilon & \text{if } n \text{ is odd} \\ \sec \epsilon' & \text{if } n \text{ is even.} \end{cases}$$

The number N of collisions is given by

$$\delta_1, \dots, \delta_N > 0, \quad \delta_{N+1} \leq 0,$$

that is,

$$\cos \epsilon, \dots, \cos(N\epsilon + (N-1)\epsilon') > 0, \quad \cos((N+1)\epsilon + N\epsilon') \leq 0,$$

$$N\epsilon + (N-1)\epsilon' < \frac{\pi}{2}, \quad (N+1)\epsilon + N\epsilon' \geq \frac{\pi}{2},$$

$$(N-1)(\epsilon + \epsilon') < \frac{\pi}{2} - \epsilon, \quad N(\epsilon + \epsilon') \geq \frac{\pi}{2} - \epsilon,$$

$$N-1 < (\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon') \leq N,$$

$$\begin{aligned} N &= \lceil (\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon') \rceil \\ &= \left\lceil \frac{\tan^{-1} \sqrt{\frac{r(l+r+1)}{l}}}{\tan^{-1} \sqrt{\frac{(l+r+1)}{lr}}} \right\rceil \end{aligned}$$

and the final velocities are given by (2) with $n = N$.

3. Two numerical examples

If $l = 64$, $r = 16$, then

$$\epsilon = \tan^{-1}\left(\frac{2}{9}\right) \approx 0.218669, \quad \epsilon' = \tan^{-1}\left(\frac{1}{18}\right) \approx 0.055499$$

$$N = \left\lceil (\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon') \right\rceil = 5$$

$$\bar{v} = \frac{1}{81}v$$

$$(v_1)_5 = \frac{1}{81}v \left\{ 1 - \frac{\sin(5\epsilon + 4\epsilon')}{\sin \epsilon} \right\} \approx -0.042718v,$$

$$(v_2)_5 = \frac{1}{81}v \left\{ 1 - \frac{\cos(\epsilon + \epsilon') \cos(6\epsilon + 4\epsilon')}{\sin \epsilon \sin \epsilon'} \right\} \approx -0.023981v,$$

$$(v_3)_5 = \frac{1}{81}v \left\{ 1 + \frac{\sin(6\epsilon + 5\epsilon')}{\sin \epsilon'} \right\} \approx 0.234872v.$$

The exact values for the final velocities, calculated from

$$\mathbf{v}_5 = RLRLR\mathbf{v}_0$$

are

$$(v_1)_5 = -\frac{177344}{4151485}v, \quad (v_2)_5 = -\frac{99555}{4151485}v, \quad (v_3)_5 = \frac{975066}{4151485}v.$$

If $l = 16$, $r = 64$, then

$$\epsilon = \tan^{-1}\left(\frac{1}{18}\right) \approx 0.055499, \quad \epsilon' = \tan^{-1}\left(\frac{2}{9}\right) \approx 0.218669$$

$$N = \lceil (\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon') \rceil = 6$$

$$\bar{v} = \frac{1}{81}v$$

$$(v_1)_6 = \frac{1}{81}v \left\{ 1 - \frac{\sin(7\epsilon + 6\epsilon')}{\sin \epsilon} \right\} \approx -0.208350v,$$

$$(v_2)_6 = \frac{1}{81}v \left\{ 1 + \frac{\cos(\epsilon + \epsilon') \cos(6\epsilon + 6\epsilon')}{\sin \epsilon \sin \epsilon'} \right\} \approx -0.060879v,$$

$$(v_3)_6 = \frac{1}{81}v \left\{ 1 + \frac{\sin(6\epsilon + 5\epsilon')}{\sin \epsilon'} \right\} \approx 0.068664v.$$

The exact values are

$$(v_1)_6 = -\frac{281112126}{1349232625}v, \quad (v_2)_6 = -\frac{82140335}{1349232625}v, \quad (v_3)_6 = \frac{92643234}{1349232625}v.$$

4. Asymptotic analysis

We can show that if $l, r \geq 1$, that is, $m_1, m_3 \geq m_2$, then

$$(v_1)_N = -v \sqrt{\frac{m_2 m_3}{m_1(m_1 + m_3)}} \left\{ 1 - \frac{13}{8} \theta_1 \left(\sqrt{\frac{m_2}{m_1}} + \sqrt{\frac{m_2}{m_3}} \right) \right\},$$

$$(v_3)_N = v \sqrt{\frac{m_1 m_2}{m_3(m_1 + m_3)}} \left\{ 1 + \frac{3}{2} \theta_2 \left(\sqrt{\frac{m_2}{m_1}} + \sqrt{\frac{m_2}{m_3}} \right) \right\}$$

where $-\frac{4}{13} < \theta_1 < 1$, $-\frac{5}{12} < \theta_2 < 1$.

We have

$$(v_1)_N = -T + \bar{v}$$

where

$$T = \bar{v} \frac{\sin((N+1)\epsilon + N\epsilon')}{\sin \epsilon} \quad \text{or} \quad \bar{v} \frac{\sin(N\epsilon + (N-1)\epsilon')}{\sin \epsilon}.$$

Now,

$$N\epsilon + (N-1)\epsilon' < \frac{\pi}{2} \leq (N+1)\epsilon + N\epsilon'.$$

So

$$\frac{\pi}{2} \leq (N+1)\epsilon + N\epsilon' < \frac{\pi}{2} + (\epsilon + \epsilon') \quad \text{and} \quad \frac{\pi}{2} - (\epsilon + \epsilon') \leq N\epsilon + (N-1)\epsilon' < \frac{\pi}{2}.$$

It follows that

$$1 \geq \sin((N+1)\epsilon + N\epsilon') > \cos(\epsilon + \epsilon') \quad \text{and} \quad \cos(\epsilon + \epsilon') \leq \sin(N\epsilon + (N-1)\epsilon') < 1.$$

Thus

$$\bar{v} \frac{\cos(\epsilon + \epsilon')}{\sin \epsilon} \leq T \leq \frac{\bar{v}}{\sin \epsilon}.$$

Now,

$$\begin{aligned} \frac{\bar{v}}{\sin \epsilon} &= \bar{v} \sqrt{\frac{lr + r^2 + r + l}{l}} \\ &= \bar{v} \sqrt{\frac{r(l+r+1)}{l}} \sqrt{1 + \frac{l}{r(l+r+1)}} \\ &= v \sqrt{\frac{r}{l(l+r+1)}} \sqrt{1 + \frac{l}{r(l+r+1)}} \end{aligned}$$

$$\begin{aligned}
&= v \sqrt{\frac{r}{l(l+r)}} \sqrt{1 + \frac{l}{r(l+r+1)}} / \sqrt{1 + \frac{1}{l+r}} \\
&= v \sqrt{\frac{r}{l(l+r)}} \left(1 + \frac{1}{2}\theta_3\left(\frac{1}{r}\right)\right) \left(1 - \frac{1}{2}\theta_4\frac{1}{l+r}\right), \quad 0 < \theta_3 < 1, 0 < \theta_4 < 1 \\
&= v \sqrt{\frac{r}{l(l+r)}} \left(1 + \frac{1}{2}\theta_5\left(\frac{1}{l} + \frac{1}{r}\right)\right) \left(1 - \frac{1}{8}\theta_6\left(\frac{1}{l} + \frac{1}{r}\right)\right), \quad 0 < \theta_5 < 1, 0 < \theta_6 < 1 \\
&= v \sqrt{\frac{r}{l(l+r)}} \left(1 + \frac{1}{2}\theta_7\left(\frac{1}{l} + \frac{1}{r}\right)\right), \quad -\frac{1}{4} < \theta_7 < 1.
\end{aligned}$$

Also,

$$\begin{aligned}
\cos \phi &= \frac{1}{\sqrt{1 + \frac{1}{l}}} \frac{1}{\sqrt{1 + \frac{1}{r}}} \\
&= \left(1 - \frac{1}{2}\theta_8\left(\frac{1}{l}\right)\right) \left(1 - \frac{1}{2}\theta_9\left(\frac{1}{r}\right)\right), \quad 0 < \theta_8 < 1, 0 < \theta_9 < 1 \\
&= \left(1 - \frac{1}{2}\theta_{10}\left(\frac{1}{l} + \frac{1}{r}\right)\right), \quad 0 < \theta_{10} < 1.
\end{aligned}$$

So

$$\begin{aligned}
\frac{\bar{v} \cos \phi}{\sin \epsilon} &= v \sqrt{\frac{r}{l(l+r)}} \left(1 + \frac{1}{2}\theta_7\left(\frac{1}{l} + \frac{1}{r}\right)\right) \left(1 - \frac{1}{2}\theta_{10}\left(\frac{1}{l} + \frac{1}{r}\right)\right) \\
&= v \sqrt{\frac{r}{l(l+r)}} \left(1 - \frac{5}{8}\theta_{11}\left(\frac{1}{l} + \frac{1}{r}\right)\right), \quad -\frac{4}{5} < \theta_{11} < 1.
\end{aligned}$$

It follows that

$$T = v \sqrt{\frac{r}{l(l+r)}} \left(1 - \frac{5}{8}\theta_{12}\left(\frac{1}{l} + \frac{1}{r}\right)\right), \quad -\frac{4}{5} < \theta_{12} < 1.$$

Also,

$$\begin{aligned}
\bar{v} &= \frac{v}{l+r+1} = v \sqrt{\frac{r}{l(l+r+1)}} \sqrt{\frac{l}{r(l+r+1)}} \\
&= v \sqrt{\frac{r}{l(l+r)}} \sqrt{\frac{l+r}{l+r+1}} \theta_{13}\left(\frac{1}{\sqrt{r}}\right), \quad 0 < \theta_{13} < 1 \\
&= v \sqrt{\frac{r}{l(l+r)}} \theta_{14}\left(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}}\right), \quad 0 < \theta_{14} < 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
(v_1)_N &= -T + \bar{v} = -v \sqrt{\frac{r}{l(l+r)}} \left(1 - \theta_{14} \left(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}} \right) - \frac{5}{8} \theta_{12} \left(\frac{1}{l} + \frac{1}{r} \right) \right) \\
&= -v \sqrt{\frac{r}{l(l+r)}} \left(1 - \theta_{14} \left(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}} \right) - \frac{5}{8} \theta_{15} \left(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}} \right) \right), & -\frac{4}{5} < \theta_{15} < 1 \\
&= -v \sqrt{\frac{r}{l(l+r)}} \left(1 - \frac{13}{8} \theta_{16} \left(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}} \right) \right), & -\frac{4}{13} < \theta_{16} < 1,
\end{aligned}$$

as claimed. The proof of the result for $(v_3)_N$ is similar. ■