A new proof of an old formula

The number of sub-cubes in an $n \times n \times n$ stack of $1 \times 1 \times 1$ cubes is $1^3 + 2^3 + \cdots + n^3$; the number of rectangles on an $n \times n$ checkerboard is \( \binom{n+1}{2}^2 = (1 + 2 + \cdots + n)^2 \).

Position an $n \times n \times n$ stack of cubes in the first octant in $\mathbb{R}^3$ so that one corner is at the origin, the opposite corner at $(n, n, n)$; position an $n \times n$ checkerboard in the first quadrant in $\mathbb{R}^2$ with one corner at the origin, the opposite corner at $(n, n)$.

Let the sub-cube whose corner closest to the origin is $(x, y, z)$ and whose corner furthest from the origin is $(x + k, y + k, z + k)$ correspond, if $x \geq y$, to the rectangle whose corner closest to the origin is $(y, z)$ and whose corner furthest from the origin is $(x + k, z + k)$, or, if $x < y$, to the rectangle whose corner closest to the origin is $(z, x)$ and whose corner furthest from the origin is $(z + k, y + k)$.

This is a one-to-one correspondence between the sub-cubes and the rectangles. It follows that

\[ 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2. \]