Climbing stairs

I was recently given the following problem. If a person climbs a staircase two or three steps at a time, how many ways can (s)he climb a staircase of \( n \) stairs?

This problem is fairly standard, yet its solution can be written very neatly, indeed surprisingly so. In finding the solution I learnt a few things about cubics and algebraic numbers, and I would like to present them to you.

Let \( a_n \) be the number of ways of climbing an \( n \)-staircase. Then \( a_0 = 1 \), \( a_1 = 0 \), \( a_2 = 1 \) and \( a_n = a_{n-2} + a_{n-3} \) for \( n \geq 3 \). The sequence \( (a_n) \) begins

\[
1, 0, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, \ldots
\]

Let \( F(q) = \sum_{n \geq 0} a_n q^n \). Then \((1 - q^2 - q^3)F(q) = 1\), and

\[
F(q) = \frac{1}{1 - q^2 - q^3}.
\]

We now write

\[
F(q) = \frac{1}{(1 - \alpha q)(1 - \beta q)(1 - \gamma q)}; \quad \alpha + \beta + \gamma = 0, \quad \alpha \beta + \beta \gamma + \gamma \alpha = -1, \quad \alpha \beta \gamma = 1.
\]

It follows that \( \alpha, \beta, \gamma \) are the roots of

\[
z^3 = z + 1.
\]

It is easy to check that this cubic has only one real root, \( \alpha > 1 \), and two complex conjugate roots, \( \beta \) and \( \gamma \).

The real root \( \alpha \) of the cubic can be found as follows:

Let \( z = u + v \).

\[
z^3 = (u + v)^3 = 3uv(u + v) + u^3 + v^3
\]

\[
= 3uvz + u^3 + v^3
\]

\[
= z + 1
\]

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If $uv = \frac{1}{3}$, $u^3 + v^3 = 1$, $u^3v^3 = \frac{1}{27}$.

$u^3$ and $v^3$ are roots of $x^2 - x + \frac{1}{27} = 0$, $x = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{23}{27}}$.

$u = 3^{\frac{1}{3}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{23}{27}}}$, $v = 3^{\frac{1}{3}} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{23}{27}}}$.

and finally

$\alpha = 3^{\frac{1}{3}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{23}{27}}} \pm 3^{\frac{1}{3}} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{23}{27}}}$.

Note that $|\beta|^2 = |\gamma|^2 = \beta \gamma = \frac{1}{\alpha} < 1$.

But we can say much more:

$\beta + \gamma = -\alpha$, $\beta \gamma = \frac{1}{\alpha} = \alpha^2 - 1$ since $\alpha^3 = \alpha + 1$.

It follows that

$$(\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta \gamma$$

$$= \alpha^2 - 4(\alpha^2 - 1)$$

$$= 4 - 3\alpha^2$$

(which is negative) and so

$\beta - \gamma = i\sqrt{3\alpha^2 - 4}$

since w.l.o.g, $\text{im } \beta > 0$.

Hence

$\beta = -\frac{1}{2} \alpha + \frac{i}{2} \sqrt{3\alpha^2 - 4}$,

$\gamma = -\frac{1}{2} \alpha - \frac{i}{2} \sqrt{3\alpha^2 - 4}$. 
Now, it is easy to check (though a non-trivial exercise to discover) that

\[
\sqrt{3\alpha^2 - 4} = \frac{1}{\sqrt{23}} (4 + 9\alpha - 6\alpha^2),
\]

so

\[
\beta = -\frac{1}{2} \alpha + i \frac{1}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2),
\]

\[
\gamma = -\frac{1}{2} \alpha - i \frac{1}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2).
\]

Returning to \( F(q) \), we can write

\[
F(q) = \frac{A}{1 - \alpha q} + \frac{B}{1 - \beta q} + \frac{C}{1 - \gamma q},
\]

from which it follows that

\[
a_n = A\alpha^n + B\beta^n + C\gamma^n
\]

\[
= A\alpha^n + B\left(-\frac{1}{2} \alpha + \frac{i}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2)\right)^n + C\left(-\frac{1}{2} \alpha - \frac{i}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2)\right)^n.
\]

Putting \( n = 0, 1, 2 \), we have

\[
A + B + C = 1,
\]

\[
A\alpha + B\left(-\frac{1}{2} \alpha + \frac{i}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2)\right) + C\left(-\frac{1}{2} \alpha - \frac{i}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2)\right) = 0,
\]

\[
A\alpha^2 + B\left(-\frac{1}{2} \alpha + \frac{i}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2)\right)^2 + C\left(-\frac{1}{2} \alpha - \frac{i}{2 \sqrt{23}} (4 + 9\alpha - 6\alpha^2)\right)^2 = 1,
\]

Solving these, we find (try it!)

\[
A = \frac{\alpha^2}{3\alpha^2 - 1} = \alpha^2 \frac{1}{23} (4 + 9\alpha - 6\alpha^2) = \frac{1}{23} (9 + 3\alpha - 2\alpha^2),
\]

\[
B = \frac{1}{46} (14 - 3\alpha + 2\alpha^2) + \frac{\alpha}{2\sqrt{23}} i,
\]
\[ C = \frac{1}{46} (14 - 3\alpha + 2\alpha^2) - \frac{\alpha}{2\sqrt{23}}i. \]

Thus,
\[
a_n = \frac{1}{23} (9 + 3\alpha - 2\alpha^2)\alpha^n \\
+ \left( \frac{1}{46} (14 - 3\alpha + 2\alpha^2) + \frac{\alpha}{2\sqrt{23}}i \right) \left( -\frac{1}{2} \alpha + \frac{i}{2} \frac{1}{\sqrt{23}} (4 + 9\alpha - 6\alpha^2) \right)^n \\
+ \left( \frac{1}{46} (14 - 3\alpha + 2\alpha^2) - \frac{\alpha}{2\sqrt{23}}i \right) \left( -\frac{1}{2} \alpha - \frac{i}{2} \frac{1}{\sqrt{23}} (4 + 9\alpha - 6\alpha^2) \right)^n.
\]

We have
\[
a_n = A\alpha^n + B\beta^n + B\overline{\beta}^n,
\]
so
\[
|a_n - A\alpha^n| \leq 2|B||\beta|^n.
\]

Now,
\[
|B|^2 = \frac{1}{23} (2 - \alpha + \alpha^2), \\
\alpha|B|^2 = \frac{1}{23} (1 + 3\alpha - \alpha^2), \\
\alpha^2|B|^2 = \frac{1}{23} (-1 + 3\alpha^2), \\
|B|^2 = \frac{1}{23} \frac{3\alpha^2 - 1}{\alpha^2} = \frac{1}{23\alpha^2}. \\
|B| = \frac{1}{\sqrt{23\alpha}}, \\
|\beta| = \frac{1}{\alpha^2},
\]
and
\[
|B||\beta|^n = \frac{1}{\sqrt{23\alpha}} \frac{1}{(\alpha^2)^n}.
\]
Thus we have the remarkable result,

$$\text{If } q_n = \frac{1}{23} (9 + 3\alpha - 2\alpha^2)\alpha^n, \text{ then } |a_n - q_n| \leq \frac{2}{\sqrt{23q_n}}.$$ 

Finally,

$$\text{For } n > 1, \ a_n \text{ is the integer closest to } \frac{1}{23} (9 + 3\alpha - 2\alpha^2)\alpha^n.$$