On the parity of $p(n)$ II

1. INTRODUCTION

Not a great deal is known about the parity of $p(n)$. Since Kolberg [3], we have known that $p(n)$ is infinitely often even, infinitely often odd. Recently, together with M. V. Subbarao [2], I proved that for every $r$, $p(16n + r)$ is infinitely often even, infinitely often odd. The main results of this note are that

for every $r$, $p(12n + r)$ is infinitely often even, infinitely often odd

and

for every $r$, $p(40n + r)$ is infinitely often even, infinitely often odd.

In order to establish these results, we obtain congruences modulo 2 for the generating functions of $p(3n + r)$, $r = 0, 1, 2$ and of $p(5n + r)$, $r = 0, 1, 2, 3, 4$. These congruences appear in the important recent paper of Frank Garvan and Dennis Stanton [1], but our derivation of them is rather more straightforward, relying only on the triple product identity. We use these congruences to obtain recurrences modulo 2 for $p(12n + r)$ and for $p(40n + r)$ from which we deduce our results via standard “Kolberg-type” arguments.

Garvan and Stanton obtain congruences for the generating functions of $p(7n + r)$, $r = 0, 2, 6$. We derive these, and show how they, together with an identity of Ramanujan, yield the result

$p(56n + r)$, $r \equiv 0, 2, 5 \text{ or } 6 \pmod{7}$, is infinitely often even, infinitely often odd.

I would like to put on record my thanks to D. W. Trenerry for his help with the computations and to F. G. Garvan for helpful discussions.
2. \( p(12n + r) \) IS INFINITELY OFTEN EVEN, INFINITELY OFTEN ODD

We have, modulo 2,

\[
\sum p(n)q^n = \frac{1}{(q;q)\infty} \equiv (q, q^2)\infty
\]

\[
= (q; q^6)\infty (q^3; q^6)\infty (q^6; q^6)\infty
\]

\[
\equiv \frac{(q; q^6)\infty (q^5; q^6)\infty}{(q^3; q^4)\infty}
\]

\[
\equiv \frac{(q; q^6)\infty (q^5; q^6)\infty (q^6; q^6)\infty}{(q^3; q^4)\infty}
\]

\[
\equiv \frac{1}{(q^3; q^3)\infty} \sum q^{3a^2-2a}
\]

\[
= \frac{1}{(q^3; q^3)\infty}\left\{ \sum q^{3(3a)^2-2(3a)} + \sum q^{3(3a+1)^2-2(3a+1)} + \sum q^{3(3a-1)^2-2(3a-1)} \right\}
\]

\[
= \frac{1}{(q^3; q^3)\infty}\left\{ \sum q^{27a^2-6a} + q \sum q^{27a^2-24a} + q^5 \sum q^{27a^2-24a} \right\}.
\]

So

\[
\sum p(3n)q^n \equiv \frac{1}{(q^3; q^3)\infty} \sum q^{9a^2-2a},
\]

\[
\sum p(3n + 1)q^n \equiv \frac{1}{(q^3; q^3)\infty} \sum q^{9a^2-4a},
\]

\[
\sum p(3n + 2)q^n \equiv \frac{1}{(q^3; q^3)\infty} \sum q^{9a^2-8a}.
\]

We now multiply by \((q; q)^4\). Since

\[
(q; q)\infty \equiv \sum q^{(3a^2-a)/2}
\]

and

\[
(q; q)^4 \equiv (q^4; q^4)\infty,
\]
we have
\[ \sum q^{2(3a^2-a)} \sum p(3n)q^n \equiv \sum q^{(3a^2-2a)/2+(9b^2-2b)} = \sum c_0(n)q^n, \]
\[ \sum q^{2(3a^2-a)} \sum p(3n+1)q^n \equiv \sum q^{(3a^2-a)/2+(9b^2-4b)} = \sum c_1(n)q^n, \]
\[ \sum q^{2(3a^2-a)} \sum p(3n+2)q^n \equiv \sum q^{(3a^2-a)/2+(9b^2-8b)+1} = \sum c_2(n)q^n. \]

If we now write \( p_r(n) = p(12n + r) \), we have, modulo 2, \((*)\)
\[ p_r(n) + p_r(n-1) + p_r(n-2) + p_r(n-5) + p_r(n-7) + \cdots \]
\[ = \begin{cases} 
  c_0(4n) & \text{if } r = 0 \\
  c_1(4n) & \text{if } r = 1 \\
  c_2(4n) & \text{if } r = 2 \\
  c_0(4n+1) & \text{if } r = 3 \\
  c_1(4n+1) & \text{if } r = 4 \\
  c_2(4n+1) & \text{if } r = 5 \\
  c_0(4n+2) & \text{if } r = 6 \\
  c_1(4n+2) & \text{if } r = 7 \\
  c_2(4n+2) & \text{if } r = 8 \\
  c_0(4n+3) & \text{if } r = 9 \\
  c_1(4n+3) & \text{if } r = 10 \\
  c_2(4n+3) & \text{if } r = 11.
\end{cases} \]

Now, \((3a^2-a)/2+(9b^2-2b) \not\equiv 4, 17, 30, 43, 56, 69, 95, 108, 121, 134, 147, 160 \pmod{169}\), so \(c_0(n) = 0\) for \(n \equiv 4, 17, 30, 43, 56, 69, 95, 108, 121, 134, 147, 160 \pmod{169}\). Similarly
\[ c_1(n) = 0 \text{ for } n \equiv 8, 21, 34, 47, 60, 73, 86, 99, 112, 125, 151, 164 \pmod{169}, \]
\[ c_2(n) = 0 \text{ for } n \equiv 12, 38, 51, 64, 77, 90, 103, 116, 129, 142, 155, 168 \pmod{169}. \]

It follows that if \(n \equiv m_r \pmod{169}\), where \(m_r\) is given by the table
\[
\begin{array}{c|cccccccccccc}
  r & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
  m_r & 1 & 2 & 3 & 4 & 5 & 19 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{array}
\]
then (*) becomes

\[(**)
\]

\[p_r(n) + p_r(n - 1) + p_r(n - 2) + p_r(n - 5) + p_r(n - 7) + \cdots \equiv 0 \pmod{2}.
\]

Next, let \(k_r\) be given by the table

<table>
<thead>
<tr>
<th>(r)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_r)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

Then \(p_r(k_r)\) is odd.

Finally let \(l_r\) be given by the table

<table>
<thead>
<tr>
<th>(r)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_r)</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>3</td>
<td>2</td>
<td>-2</td>
<td>2</td>
<td>-2</td>
<td>-2</td>
<td>2</td>
</tr>
</tbody>
</table>

Then \((3l_r^2 + l_r)/2 + k_r \equiv m_r \pmod{169}\).

Suppose \(p_r(n)\) is odd (alternatively even) for \(n \geq n_0\). We can suppose \(n_0 \equiv l_r \pmod{169}\), and that \(2n_0 + 1 > k_r\).

Let \(N = (3n_0^2 + n_0)/2 + k_r\).

Then \(N \equiv (3l_r^2 + l_r)/2 + k_r \equiv m_r \pmod{169}\), and (**) becomes

\[(***)
\]

\[p_r(N) + p_r(N - 1) + p_r(N - 2) + p_r(N - 5) + p_r(N - 7) + \cdots
\]
\[+ \cdots + p_r(n_0 + k_r) + p_r(k_r) \equiv 0 \pmod{2}.
\]

(The condition \(2n_0 + 1 > k_r\) ensures that \(p_r(k_r)\) is the last term on the left.)

But the left-hand-side of (***) is odd: there is an odd number, \(2n_0 + 1\), of terms of which the last is odd while the others are all odd (alternatively even). So we have a contradiction, and our result is proved.