Equilateral convex pentagons which tile the plane

It is shown that an equilateral convex pentagon tiles the plane if and only if it has two angles adding to $180^\circ$ or it is the unique equilateral convex pentagon with angles $A$, $B$, $C$, $D$, $E$ satisfying $A + 2B = 360^\circ$, $C + 2E = 360^\circ$, $A + C + 2D = 360^\circ$, ($A \approx 70.88^\circ$, $B \approx 144.56^\circ$, $C \approx 89.26^\circ$, $D \approx 99.93^\circ$, $E \approx 135.37^\circ$).

Although the area of mathematical tilings has been of interest for a long time there is still much to be discovered. We do not even know which convex polygons tile the plane. Furthermore, for those polygons which do tile, new tilings are being found. It is known that all triangles and quadrilaterals tile the plane and those convex hexagons which do tile the plane have been classified. It is also known that no convex $n$-gon with $n \geq 7$ tiles.

In this paper we consider the problem of finding all equilateral convex pentagons which tile the plane. The upshot of our study is the following:

THEOREM. An equilateral convex pentagon tiles the plane if and only if it has two angles adding to $180^\circ$, or it is the unique equilateral convex pentagon $X$ with angles $A$, $B$, $C$, $D$, $E$ satisfying $A + 2B = 360^\circ$, $C + 2E = 360^\circ$, $A + C + 2D = 360^\circ$, ($A \approx 70.88^\circ$, $B \approx 144.56^\circ$, $C \approx 89.26^\circ$, $D \approx 99.93^\circ$, $E \approx 135.37^\circ$).

Thus the list of equilateral convex pentagons which tile, to be found in Schattschneider’s paper [2], is complete. We also note that, with this theorem, the only convex polygons whose ability to tile is still in question are the nonequilateral convex pentagons.

It should be remarked that in obtaining this result we make no assumptions regarding periodicity of any tiling. (Yet it is a fact that every equilateral convex pentagon which tiles does so in a periodic manner.)

Our method of proof is interesting if only for the fact that it works only for the problem at hand—it could not, for instance, handle the problems of finding all convex pentagons or all equilateral convex hexagons which tile. In various places in the proof computer calculations are used.

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With no further ado, let us begin.

1. INITIAL REDUCTION

We can suppose that any tiling of an equilateral convex pentagon is edge-to-edge. For if not, it has one or more “fault-lines”. It is easy to see that such fault-lines are necessarily parallel to one another, and that there are at most a countable infinity of them. So the tiling can be slipped along the fault-lines to become edge-to-edge (see Fig. 1).

Thus we need concern ourselves only with the ways in which the angles match up at the vertices of the tiling, in other words, relations between the angles $A, B, C, D, E$ of the pentagon of the form

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^\circ,$$

where $m_A, \cdots, m_E$ are nonnegative integers. First we show that the set of possible relations is finite. In an equilateral convex pentagon, each angle is greater than $\cos^{-1}\left(\frac{7}{8}\right)$ (see Fig. 2), since if any angle were less than or equal to this, the polygon would fail to be a convex pentagon. So we have

$$A, B, C, D, E > \cos^{-1}\left(\frac{7}{8}\right) > 28^\circ$$

and since $m_A A + m_B B + \cdots + m_E E = 360^\circ$, we have

$$m_A + m_B + \cdots + m_E \leq 12.$$ 

Further, since $A, B, C, D, E$ are all less than $180^\circ$,

$$m_A + \cdots + m_E \geq 3.$$ 

Thus there are a finite (if large) set of relations that might be satisfied by some equilateral convex pentagon. We proceed to show how the above list may be whittled down.
There is a lot of duplication; if, for example, a pentagon satisfies \( A + 2B = 360^\circ \), it also satisfies, after suitable relettering, any one of ten different relations (assuming, as we shall from now on, that the angles of the pentagon are \( A, B, C, D, E \) in that order around the pentagon.) We remove such trivial duplication by adopting, without loss of generality, the following conventions. We shall suppose, in everything that follows, that \( B \) is the largest angle of the pentagon, that is, \( B \geq A, C, D, E \), and, further, that of the two angles adjacent to \( B \), namely \( A \) and \( C \), \( A \) is the smaller, that is, \( A \leq C \).

With the above conventions, we have

\[
\text{LEMMA 1. } A \leq C \leq D \leq E \leq B.
\]

Proof. Suppose \( B \) is fixed, and \( A \) decreases from its value when equal to \( C \) (see Fig. 3(a)) when \( D \) is equal to \( E \). As \( A \) decreases, \( E \) increases, \( D \) decreases and \( C \) increases, until \( E \) becomes equal to \( B \) and \( C \) becomes equal to \( D \) (see Fig. 3(b)). Thus we have

\[
A \leq C \leq \lim C = \lim D \leq D \leq E \leq B,
\]

or,

\[
A \leq C \leq D \leq E \leq B.
\]

Lemma 1 will prove useful later in reducing the list of relations.

2. THE GEOMETRY OF AN EQUILATERAL CONVEX PENTAGON

In order to proceed we need to make a careful study of the geometry of the equilateral convex pentagon. Indeed, we prove
LEMMA 2.

\[ 108^\circ \leq B < 180^\circ, \]
\[ 180^\circ - \frac{1}{2} B - \sin^{-1}(\sin(\frac{1}{2} B) - \frac{1}{2}) \geq A \geq 180^\circ - B + 2 \sin^{-1}(1/4 \sin(\frac{1}{2} B)), \]
\[ D = \cos^{-1}(\cos A + \cos B - \cos (A + B) - \frac{1}{2}), \]
\[ C = 270^\circ - B - \frac{1}{2} D + \theta, \]
\[ E = 270^\circ - A - \frac{1}{2} D - \theta \]

where
\[ \theta = \tan^{-1}((\sin A - \sin B)/(1 - \cos A - \cos B)). \]

Proof. The greatest value of \( A \) occurs when \( A = C \) (see Fig. 4(a)), and then
\[ \sin(\frac{1}{2} B) - \sin(180^\circ - (A + \frac{1}{2} B)) = \frac{1}{2} \]
or
\[ A = 180^\circ - \frac{1}{2} B - \sin^{-1}(\sin(\frac{1}{2} B) - \frac{1}{2}). \]
The smallest value of \( A \) occurs when \( B = E \) (see Fig. 4(b)), and then
\[ \sin(\frac{1}{2} A) + \sin(B + \frac{1}{2} A - 180^\circ) = \frac{1}{2} \]
or
\[ 2 \sin(\frac{1}{2} A + \frac{1}{2} B - 90^\circ) \cos(90^\circ - \frac{1}{2} B) = \frac{1}{2} \]
or
\[ 2 \sin(\frac{1}{2} A + \frac{1}{2} B - 90^\circ) \sin(\frac{1}{2} B) = \frac{1}{2}, \]
\[ A = 180^\circ - B + 2 \sin^{-1}(1/4 \sin(\frac{1}{2} B)). \]
In order to find $D$ in terms of $A$ and $B$, we calculate the length of the diagonal $CE$ in two different ways. From Fig. 5(a) we see that
\[ CE^2 = (2 \sin \frac{1}{2} D)^2 \]
while from Fig. 5(b) we have
\[ CE^2 = (1 - \cos A - \cos B)^2 + (\sin A - \sin B)^2. \]
Equating these expressions yields
\[ \cos D = \cos A + \cos B - \cos (A + B) - \frac{1}{2}. \]
Now from Fig. 6 it is easy to see that
\[ C = (180^\circ - B) + (90^\circ - \frac{1}{2} D) + \theta = 270^\circ - B - \frac{1}{2} D + \theta \]
and
\[ E = (180^\circ - A) + (90^\circ - \frac{1}{2} D) - \theta = 270^\circ - A - \frac{1}{2} D - \theta, \]
where
\[ \tan \theta = (\sin A - \sin B)/(1 - \cos A - \cos B). \]
In particular it follows from Lemma 2 that in an equilateral convex pentagon the angles $C$, $D$ and $E$ are uniquely determined by the angles $A$ and $B$, so we can identify the equilateral convex pentagon with a point in the $AB$ plane. The region $P$ in the $AB$ plane which results from this identification is shown in Fig. 7. We indicate certain polygons on the boundary of the region. These are: $R$, the regular pentagon ($A = B = C = D = E = 108^\circ$); $H$, the house pentagon ($A = 60^\circ$, $B = E = 150^\circ$, $C = D = 90^\circ$); $T$, the isosceles triangle of Fig. 2 ($A = \cos^{-1}(\frac{7}{8})$, $B = E = 180^\circ$, $C = D = \cos^{-1}(\frac{1}{3})$); and $Q$, the quadrilateral which is half a regular hexagon ($A = C = 60^\circ$, $B = 180^\circ$, $D = E = 150^\circ$). Note
that the part of the boundary of the region lying along the line $B = 180^\circ$ joining $T$ and $Q$ consists of (points representing) quadrilaterals, not convex pentagons.

3. FURTHER REDUCTION

Now $\cos^{-1}\left(\frac{7}{8}\right) \approx 28.96^\circ < A < 108^\circ$ and $108^\circ \leq B < 180^\circ$ in $P$. Also $60^\circ < C \leq 108^\circ$, $\cos^{-1}\left(\frac{1}{4}\right) \approx 75.52^\circ < D < 120^\circ$, and $108^\circ \leq E < 180^\circ$ for all points $(A, B) \in P$. (Note that these constraints are determined by values at $Q$, $R$ and $T$). These constraints allow just 220 solutions to the equation

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^\circ.$$

In order to determine which of our 220 relations are “good” in the sense that they are actually satisfied by some equilateral convex pentagon, we proceed as follows. For each set $(m_A, \ldots, m_E)$ we consider the function of $A, B$ defined over the region $P$ by $m_A A + \cdots + m_E E$, and which we will denote simply by $m_A m_B \cdots m_E$. (Thus, for example, 20010 denotes $2A + D$.) The relation $m_A A + \cdots + m_E E = 360^\circ$ is good if and only if the function $m_A m_B \cdots m_E$ intersects the level set $360^\circ$ for some $(A, B) \in P$. However, rather than test each of the 220 functions in this way, we cut down the required work as follows: Define a partial order on the set of functions by writing

$$m_A \cdots m_E < m'_A \cdots m'_E$$

if $m_A A + \cdots + m_E E \leq m'_A A + \cdots + m'_E E$ simply by virtue of the fact that $A \leq C \leq D \leq E \leq B$. If $m_A m_B m_C m_D m_E < m'_A m'_B m'_C m'_D m'_E$ and $m'_A A + \cdots + m'_E E < 360^\circ$ then $m_A A + \cdots + m_E E < 360^\circ$. Hence we may discard such an $m_A \cdots m_E$. Similarly if $m_A m_B m_C m_D m_E > m'_A m'_B m'_C m'_D m'_E$ and $m'_A A + \cdots + m'_E E > 360^\circ$ then we may discard $m_A m_B m_C m_D m_E$.

**LEMMA 3.**

$$B + C + D < 360^\circ$$
$$2D + E < 360^\circ$$
\[2A + 2E > 360^\circ\]
\[A + B + 2C > 360^\circ\]
\[A + C + D + E > 360^\circ\]
\[2A + 2C + E > 360^\circ\]
\[4C + D > 360^\circ\]
\[4A + C + E > 360^\circ\]
\[2A + 4C > 360^\circ\]
\[7A + E > 360^\circ\]
\[5A + 3C > 360^\circ\]
\[8A + 2C > 360^\circ\]
\[10A + C > 360^\circ\]

\textit{Proof.} Figure 8 shows regions of the \(AB\) plane and demonstrates that the thirteen associated equalities do not meet \(P\).
The following 107 relations may be eliminated by use of the partial order <.

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This leaves 100 relations. Each of these is satisfied by pentagons lying on a
curve crossing the region \( P \) (see Fig. 9).

**Lemma 4.** There are precisely 100 relations satisfied by some equilateral convex pentagon, namely:

1. \( A + 2E = 360^\circ \)
2. \( A + B + E = 360^\circ \)
3. \( A + 2B = 360^\circ \)
4. \( C + 2E = 360^\circ \)
5. \( B + C + E = 360^\circ \)
6. \( 2B + C = 360^\circ \)
7. \( B + 2D = 360^\circ \)
8. \( D + 2E = 360^\circ \)
9. \( B + D + E = 360^\circ \)
10. \( 2B + D = 360^\circ \)
11. \( 3E = 360^\circ \)
12. \( B + 2E = 360^\circ \)
13. \( 2B + E = 360^\circ \)
14. \( 3B = 360^\circ \)
15. \( 4A = 360^\circ \)
16. \( 3A + C = 360^\circ \)
17. \( 3A + D = 360^\circ \)
18. \( 3A + E = 360^\circ \)
19. \( 3A + B = 360^\circ \)
20. \( 2A + 2C = 360^\circ \)
21. \( 2A + C + D = 360^\circ \)
22. \( 2A + C + E = 360^\circ \)
23. \( 2A + B + C = 360^\circ \)
24. \( 2A + 2D = 360^\circ \)
25. \( 2A + D + E = 360^\circ \)
26. \( 2A + B + D = 360^\circ \)
27. \( A + 3C = 360^\circ \)
28. \( A + 2C + D = 360^\circ \)
29. \( A + 2C + E = 360^\circ \)
30. \( A + C + 2D = 360^\circ \)
31. \( A + 3D = 360^\circ \)
32. \( 4C = 360^\circ \)
33. \( 3C + D = 360^\circ \)
34. \( 3C + E = 360^\circ \)
35. \( 2C + 2D = 360^\circ \)
36. \( C + 3D = 360^\circ \)
37. \( 4D = 360^\circ \)
38. \( 5A = 360^\circ \)
39. \( 4A + C = 360^\circ \)
40. \( 4A + D = 360^\circ \)
41. \( 4A + E = 360^\circ \)
42. \( 4A + B = 360^\circ \)
43. \( 3A + 2C = 360^\circ \)
44. \( 3A + C + D = 360^\circ \)
45. \( 3A + C + E = 360^\circ \)
46. \( 3A + B + C = 360^\circ \)
47. \( 3A + 2D = 360^\circ \)
48. \( 3A + D + E = 360^\circ \)
49. \( 3A + B + D = 360^\circ \)
50. \( 2A + 3C = 360^\circ \)
51. \( 2A + 2C + D = 360^\circ \)
52. \( 2A + C + 2D = 360^\circ \)
53. \( 2A + 3D = 360^\circ \)
54. \( A + 3C = 360^\circ \)
55. \( A + 3C + D = 360^\circ \)
56. \( A + 2C + 2D = 360^\circ \)
57. \( A + C + 3D = 360^\circ \)
58. \( A + 4D = 360^\circ \)
59. \( 5C = 360^\circ \)
60. \( 6A = 360^\circ \)
61. \( 5A + C = 360^\circ \)
62. \( 5A + D = 360^\circ \)
63. \( 5A + E = 360^\circ \)
64. \( 5A + B = 360^\circ \)
65. \( 4A + 2C = 360^\circ \)
66. \( 4A + C + D = 360^\circ \)
67. \( 4A + 2D = 360^\circ \)
68. \( 3A + 3C = 360^\circ \)
69. \( 3A + 2C + D = 360^\circ \)
70. \( 3A + C + 2D = 360^\circ \)
71. \( 3A + 3D = 360^\circ \)
72. \( 7A = 360^\circ \)
73. \( 6A + C = 360^\circ \)
74. \( 6A + D = 360^\circ \)
75. \( 6A + E = 360^\circ \)
76. \( 6A + B = 360^\circ \)
77. \( 5A + 2C = 360^\circ \)
78. \( 5A + C + D = 360^\circ \)
79. \( 5A + 2D = 360^\circ \)
80. \( 4A + 3C = 360^\circ \)
81. \( 4A + 2C + D = 360^\circ \)
82. \( 4A + C + 2D = 360^\circ \)
83. \( 4A + 3D = 360^\circ \)
84. \( 8A = 360^\circ \)
85. \( 7A + C = 360^\circ \)
86. \( 7A + D = 360^\circ \)
87. \( 6A + 2C = 360^\circ \)
88. \( 6A + C + D = 360^\circ \)
89. \( 6A + 2D = 360^\circ \)
90. \( 9A = 360^\circ \)
91. \( 8A + C = 360^\circ \)
92. \( 8A + D = 360^\circ \)
93. \( 7A + 2C = 360^\circ \)
94. \( 7A + C + D = 360^\circ \)
95. \( 7A + 2D = 360^\circ \)
96. \( 10A = 360^\circ \)
97. \( 9A + C = 360^\circ \)
98. \( 9A + D = 360^\circ \)
99. \( 11A = 360^\circ \)
100. \( 12A = 360^\circ \)
4. PROOF OF MAIN THEOREM

We next observe that if an equilateral convex pentagon tiles the plane, it simultaneously satisfies at least two of the 100 relations. For, every angle of the tiling pentagon is involved in some such relation, yet no one relation involves all five angles. So we must consider intersections of the 100 relations (see Fig. 9). Six of the relations coincide.

**Lemma 5.** The following relations are equivalent:

(A) \( \angle A + \angle B + \angle E = 360^\circ \)

(B) \( 2\angle A + \angle C = 360^\circ \)

(C) \( 2\angle A + \angle D + \angle E = 360^\circ \)

(D) \( 2\angle C + 2\angle D = 360^\circ \)

(E) \( 3\angle A + \angle C + \angle D = 360^\circ \)

(F) \( 6\angle A = 360^\circ \)

**Proof.** If \( A = 60^\circ \) then the pentagon is an equilateral triangle joined along one edge to a rhombus and it is clear that all 6 relations hold. Clearly (A) \( \Leftrightarrow \) (D) since \( \angle A + \angle B + \angle C + \angle D + \angle E = 540^\circ \).

Now

\[ A > 60^\circ \Rightarrow \angle C + \angle D > 180^\circ \Rightarrow 3\angle A + \angle C + \angle D > 360^\circ \]

and

\[ A < 60^\circ \Rightarrow \angle C + \angle D < 180^\circ \Rightarrow 3\angle A + \angle C + \angle D < 360^\circ . \]

Hence (A) \( \Leftrightarrow \) (D) \( \Leftrightarrow \) (E) \( \Leftrightarrow \) (F) and (F) \( \Rightarrow \) (B), (F) \( \Rightarrow \) (C).

We now show that (B) \( \Rightarrow \) (F): Assume that \( 2\angle A + \angle B + \angle C = 360^\circ \). Let \( A = (0,0) \) and \( B = (1,0) \). Then

\[ D = (\cos A, \sin A) \]

\[ C = (1 - \cos B, \sin B) \]

\[ E = (1 - \cos B - \cos (B + C - 180^\circ), \sin B + \sin (B + C - 180^\circ)) . \]
Now $(DE)^2 = 1$. Hence

$$(\cos A - 1 + \cos B - \cos (B + C))^2 + (\sin A - \sin B + \sin (B + C))^2 = 1.$$ 

Also $B + C = 360^\circ - 2A$, so

$$1 + \cos^2 A + \cos^2 B + \cos^2 2A - 2\cos A - 2\cos B + 2\cos 2A \\
+ 2\cos A \cos B - 2\cos A \cos 2A - 2\cos B \cos 2A + \sin^2 A \\
+ \sin^2 B + \sin^2 2A - 2\sin A \sin B - 2\sin A \sin 2A \\
+ 2\sin B \sin 2A = 1$$

or

$$4 - 2\cos A - 2\cos B + 2\cos 2A + 2\cos (A + B) \\
- 2\cos (2A - A) - 2\cos (B + 2A) = 1$$

or

$$4 - 4\cos A + 2\cos 2A = 1 + 2\cos B + 2\cos (2A + B) - 2\cos (A + B) \\
= 1 + 4\cos (A + B) \cos A - 2\cos (A + B).$$

Hence

$$3 - 4\cos A + 2\cos 2A = 2\cos (A + B)(2\cos A - 1)$$

or

$$1 - 4\cos A + 4\cos^2 A = 2\cos (A + B)(2\cos A - 1).$$

Hence $\cos A = \frac{1}{2}$ or $\cos A - \cos (A + B) = \frac{1}{2}$, which is not possible in $P$ except at $R$, and here $2A + B + C \neq 360^\circ$. So $\cos A = \frac{1}{2}$ and $A = 60^\circ$. To show $(C) \Rightarrow (F)$ we argue as above with $E$ replacing $B$.

Any equilateral convex pentagon which satisfies these 6 relations is said to be of Type 1.

The two relations

5. $B + C + E = 360^\circ$

24. $2A + 2D = 360^\circ$
coincide, and any equilateral convex pentagon which satisfies these is said to be of Type 2(a).

The two relations

\[ 9. \quad B + D + E = 360^\circ \]
\[ 20. \quad 2A + 2C = 360^\circ \]

coincide, and any equilateral convex pentagon which satisfies these is said to be of Type 2(b). Together, Types 2(a) and 2(b) constitute Type 2.

Type 1 consists of all equilateral convex pentagons with two adjacent angles adding to \(180^\circ\), while Type 2 consists of all equilateral convex pentagons with two nonadjacent angles adding to \(180^\circ\). All these pentagons tile the plane (see Figs. 10, 11).

To find all other equilateral convex pentagons which tile the plane, we must consider all those which simultaneously satisfy at least two of the remaining 90 relations.

We can shorten this task as follows: Any tiling pentagon satisfies at least one of the 14 relations involving \(B\), namely:

\[ 3. \quad A + 2B = 360^\circ \]
\[ 6. \quad 2B + C = 360^\circ \]
\[ 7. \quad B + 2D = 360^\circ \]
\[ 10. \quad 2B + D = 360^\circ \]
\[ 12. \quad B + 2E = 360^\circ \]
\[ 13. \quad 2B + E = 360^\circ \]
\[ 14. \quad 3B = 360^\circ \]
\[ 19. \quad 3A + B = 360^\circ \]
\[ 26. \quad 2A + B + D = 360^\circ \]
\[ 42. \quad 4A + B = 360^\circ \]
\[ 46. \quad 3A + B + C = 360^\circ \]
\[ 49. \quad 3A + B + D = 360^\circ \]
64. $5A + B = 360^\circ$
76. $6A + B = 360^\circ$

Further, the relations involving $B$ that it satisfies cannot all belong to the second half of this list, since in each of these $m_A > m_B$, and so, in tiling there would be more angles $A$ than $B$, an impossibility. Thus the pentagon satisfies at least two relations, including at least one of the seven relations in the first half of the above list. There are 54 such pentagons.

Of these 54, only three satisfy sets of relations involving all 5 angles:

- $3, 4, 30 \quad A + 2B = 360^\circ, \ C + 2E = 360^\circ, \ A + C + 2D = 360^\circ$
- $3, 8, 28 \quad A + 2B = 360^\circ, \ D + 2E = 360^\circ, \ A + 2C + D = 360^\circ$
- $7, 45, 67 \quad B + 2D = 360^\circ, \ 3A + C + E = 360^\circ, \ 4A + 2D = 360^\circ$

Of these, the third does not tile since $m_D > m_B$ in the only relation involving $B$. The second does not tile; for suppose it does, and consider any tiling of it near a vertex where $A + 2B = 360^\circ$. We see from Fig. 12 that $E$ and $A$ or $E$ and $C$ are forced together, but these combinations do not occur in any of the relations satisfied by that pentagon, yielding a contradiction.

This leaves only the first, which does tile the plane (see Fig. 13), and the theorem is proved.