STIRLING WITHOUT WALLIS

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Abstract. It is fairly easy to show that
\[ n! \sim Cn^{n+\frac{1}{2}}e^{-n} \text{ as } n \to \infty, \]
and it is then standard procedure to use Wallis’s product to show that
\[ C = \sqrt{2\pi}. \]
The purpose of this note is to show that there is an alternative route to determining \( C \).

1. Introduction

It is fairly easy to show that
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and it is then standard procedure to use Wallis’s product to show that
\[ C = \sqrt{2\pi}. \]
The purpose of this note is to show that there is an alternative route to determining \( C \), and consequently a non-standard way to derive Wallis’s product.

2. The usual procedure, from Wallis to Stirling

If we let
\[ u_n = n! / n^{n+\frac{1}{2}}e^{-n}, \]
then
\[ \frac{u_n}{u_{n-1}} \approx \exp \left\{ -\frac{1}{12n^2} \right\}, \]
from which it follows that
\[ u_n \to C \text{ as } n \to \infty, \]
where \( C \) is a non-zero constant, and so
\[ n! \sim Cn^{n+\frac{1}{2}}e^{-n} \text{ as } n \to \infty. \]

Now, Wallis’s product, which follows from the fact that
\[ \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta \]
is a decreasing function of $n$, together with the facts that
\[
\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \text{ if } n \text{ is odd},
\]
\[
= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{\pi}{2} \text{ if } n \text{ is even}
\]
says that
\[
\frac{\pi}{4} = \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{5^2}\right) \left(1 - \frac{1}{7^2}\right) \cdots ,
\]
or, equivalently,
\[
\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{4^2}\right) \left(1 - \frac{1}{6^2}\right) \cdots ,
\]
or, yet again,
\[
\pi = \lim_{n \to \infty} \frac{2^{4n} n!^4}{n(2n)!^2}.
\]
Taken together with
\[
n! \sim C n^{n+\frac{1}{2}} e^{-n},
\]
this gives
\[
\frac{C^2}{2} = \pi,
\]
so
\[
C = \sqrt{2\pi}
\]
and
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,
\]
which is Stirling’s formula.

For all this, see, for example, [1], Vol. II, pp. 616–618.

3. STIRLING WITHOUT WALLIS

In this section we show that
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
\]
without using Wallis’s product.

We start with the series for $e^n$,
\[
e^n = 1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!} + \cdots
\]
\[
= \sum_{k=0}^{\infty} \frac{n^k}{k!}.
\]
The (equally) largest term in this expansion occurs when \( k = n \), and is \( H = \frac{n^n}{n!} \). Nearby terms are given by

\[
\frac{n^{n+k}}{(n+k)!} = H \cdot \frac{n^{n+k}}{(n+k)!} / \frac{n^n}{n!}
\]

\[
= H \cdot \frac{n^{n+k}}{(n+1)\cdots n} \cdot \frac{n}{n+k}
\]

\[
= H \cdot \exp \left\{ -\log \left( 1 + \frac{1}{n} \right) - \cdots - \log \left( 1 + \frac{k}{n} \right) \right\}
\]

\[
\approx H \cdot \exp \left\{ -\frac{1}{n} - \cdots - \frac{k}{n} \right\}
\]

\[
\approx H \cdot \exp \left\{ -\frac{k^2 + k}{2n} \right\},
\]

for terms to the right of \( n \), or by

\[
\frac{n^{n-k}}{(n-k)!} = H \cdot \frac{n^{n-k}}{(n-k)!} / \frac{n^n}{n!}
\]

\[
= H \cdot \frac{n^{-1} \cdots n-k+1}{n}
\]

\[
= H \cdot \exp \left\{ \log \left( 1 - \frac{1}{n} \right) + \cdots + \log \left( 1 - \frac{k-1}{n} \right) \right\}
\]

\[
\approx H \cdot \exp \left\{ -\frac{1}{n} - \cdots - \frac{k-1}{n} \right\}
\]

\[
\approx H \cdot \exp \left\{ -\frac{k^2 - k}{2n} \right\}
\]

for terms to the left of \( n \).

So the distribution function is close to

\[
f(x) = H \exp \left\{ -\frac{(x-n)^2 + (x-n)}{2n} \right\}
\]

\[
= H \exp \left\{ -\frac{(x-n+\frac{1}{2})^2 - \frac{1}{4}}{2n} \right\}
\]

\[
= H \exp \left\{ -\frac{(x-(n-\frac{1}{2}))^2}{2n} + \frac{1}{8n} \right\}.
\]

Thus the terms are distributed roughly normally about the mean \( (n - \frac{1}{2}) \) with standard deviation \( \sigma \) given by

\[
\sigma^2 = n,
\]

or

\[
\sigma = \sqrt{n}.
\]
Figure 1. The case $n = 1000$, showing the points $(k, \frac{n^k}{k!})$ for $900 \leq k \leq 1100$, together with the normal $y = \frac{n^n}{n!} \exp \left\{ -\frac{(x - (n - \frac{1}{2}))}{2n} + \frac{1}{8n} \right\}$

It follows that

$$e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} \approx H e^{\frac{1}{2n}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{x^2}{2n} \right\} \, dx \approx H \sigma \sqrt{2\pi} = \sqrt{2\pi n} \frac{n^n}{n!},$$

and so

$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

Of course, this argument can be tightened (with a fair bit of trouble) to give

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

From this, we easily find that

$$\lim_{n \to \infty} \frac{2^n n!^4}{n(2n)!^2} = \pi,$$

which is Wallis’s product!

References


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