The sum of consecutive squares a square

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My aim is to find infinitely many solutions to the diophantine equation

\[(m + 1)^2 + \cdots + n^2 = \text{square}.\]

Note that I am not interested in finding all solutions of this equation, so I can be as choosey as I like.

First we note that the left–hand–side is

\[
\frac{1}{6}n(n+1)(2n+1) - \frac{1}{6}m(m+1)(2m+1) = \frac{1}{6}(n-m)(2m^2+2mn+2n^2+3m+3n+1).
\]

In order to make this a square, I shall choose the factor \((n-m)\) to be a square, \(n-m = \nu^2\). So the right–hand–side becomes

\[
\frac{1}{6}\nu^2(2m^2 + 2mn + 2n^2 + 3m + 3n + 1) = \frac{1}{12}\nu^2(4m^2 + 4mn + 4n^2 + 6m + 6n + 2) = \frac{1}{12}\nu^2\left(3(m + n + 1)^2 + (n - m)^2 - 1\right).
\]

I shall now choose \(\nu\) to be not divisible by 2 or 3, so the right–hand–side should be written

\[
\nu^2\left(\frac{3(m + n + 1)^2 + (n - m)^2 - 1}{12}\right).
\]

Thus we require

\[3(m + n + 1)^2 + \nu^4 - 1 = 12p^2\]

for some integer \(p\).

Now \(\nu^4 - 1\) is even, so \(m + n + 1\) is even. Write \(m + n + 1 = 2q\). Then

\[12q^2 + \nu^4 - 1 = 12p^2.\]

Indeed, \(\nu\) is odd, so \(\nu^2 \equiv 1 \pmod{8}\), \(\nu^4 \equiv 1 \pmod{16}\), while \(\nu \equiv \pm 1 \pmod{3}\), \(\nu^2 \equiv 1 \pmod{3}\), \(\nu^4 \equiv 1 \pmod{3}\) so \(\nu^4 - 1\) is divisible by 48. It follows that

\[p^2 - q^2 = \frac{\nu^4 - 1}{12}.
\]
is divisible by 4, \( p \equiv q \pmod{2} \), and we can write

\[
\frac{p + q}{2}, \frac{p - q}{2} = \frac{\nu^4 - 1}{48},
\]

where \( \frac{p + q}{2}, \frac{p - q}{2} \) and \( \frac{\nu^4 - 1}{48} \) are all integers.

I choose

\[
\frac{p + q}{2} = \frac{\nu^4 - 1}{48}, \quad \frac{p - q}{2} = 1,
\]

so

\[
p = \frac{\nu^4 + 47}{48}, \quad q = \frac{\nu^4 - 49}{48}.
\]

So now we have

\[
m + n + 1 = 2q = \frac{\nu^4 - 49}{24}, \quad n - m = \nu^2;
\]

from which it follows that

\[
m + 1 = \frac{\nu^4 - 24\nu^2 - 25}{48}, \quad n = \frac{\nu^4 + 24\nu^2 - 73}{48}
\]

and

\[
(m + 1)^2 + \cdots + n^2 = \nu^2 \left( \frac{12p^2}{12} \right) = \nu^2 p^2 = \left( \frac{\nu(\nu^4 + 47)}{48} \right)^2.
\]

So finally, for each \( \nu \) not divisible by 2 or 3, \( k \) consecutive squares summing to a square are given by

\[
a^2 + \cdots + (a + k - 1)^2 = b^2
\]

where

\[
a = \frac{\nu^4 - 24\nu^2 - 25}{48}, \quad k = \nu^2, \quad b = \frac{\nu^5 + 47\nu}{48}.
\]