

# Jet Spaces and Nonrigid Carnot Groups

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# Abstract

This thesis studies the phenomenon of rigidity of Carnot groups and is based in part on the author's publications. One of the main results is the expansion of the class of known nonrigid Carnot groups. This expansion is achieved by realising the generic jet spaces, denoted  $J^k(\mathbb{R}^m, \mathbb{R}^n)$ , as Carnot groups and using prolongation to show that they are nonrigid. This result encapsulates all the previously known examples of nonrigid Carnot groups.

Using the analytic definition of quasiconformality, we apply Bäcklund's theorem on  $J^k(\mathbb{R}, \mathbb{R})$ , where  $k \geq 2$ , to obtain a Liouville type theorem for 1-quasiconformal maps, and an explicit characterisation of the quasiconformal automorphism group in the group of diffeomorphisms.

Carnot groups are in some sense a generalisation of the Heisenberg group, for which the theory is well developed. The jet space model gives a new approach to theory of the Heisenberg group, and the thesis presents some established results in terms of the jet space model. The complexified Heisenberg group is also considered and understood in the context of internal and external symmetry.

Every Carnot group has a growth vector which records the strata dimensions of the group. The thesis finishes with examples which demonstrate that the growth vector is not a rigidity invariant.



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# Contents

<b>Abstract</b>	<b>iii</b>
<b>Acknowledgments</b>	<b>v</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Carnot Groups</b>	<b>5</b>
2.1 Introduction . . . . .	5
2.2 Contact maps . . . . .	8
2.3 Cauchy characteristics . . . . .	11
2.4 Pansu differentiability . . . . .	12
2.5 Quasiconformal maps . . . . .	21
2.6 Second kind coordinates . . . . .	22
<b>3 Jet Spaces</b>	<b>25</b>
3.1 Introduction . . . . .	25
3.2 Contact structure . . . . .	27
3.3 Contact transformations . . . . .	29
3.4 Prolongation . . . . .	29
3.5 Cauchy characteristics . . . . .	33
<b>4 Jet Spaces as Carnot Groups</b>	<b>37</b>
4.1 Introduction . . . . .	37
4.2 Example: $J^1(\mathbb{R}^m, \mathbb{R}^n)$ . . . . .	37

4.3	Example: $J^k(\mathbb{R}, \mathbb{R})$ . . . . .	39
4.4	$J^k(\mathbb{R}^m, \mathbb{R})$ and $J^k(\mathbb{R}^m, \mathbb{R}^n)$ . . . . .	42
4.5	Subgroups . . . . .	45
4.6	Example: (2,1,2,1) . . . . .	47
<b>5</b>	<b>Quasiconformal Maps on <math>J^k(\mathbb{R}, \mathbb{R})</math></b>	<b>49</b>
5.1	Contact preliminaries . . . . .	49
5.2	Quasiconformal preliminaries . . . . .	51
5.3	1-quasiconformal maps . . . . .	54
5.4	Contact automorphisms . . . . .	59
5.5	Quasiconformal automorphisms . . . . .	61
<b>6</b>	<b>The Heisenberg Group</b>	<b>73</b>
6.1	Preliminaries . . . . .	73
6.2	1-Quasiconformal maps . . . . .	75
6.3	Quasiconformal point automorphisms . . . . .	78
6.4	Equivariant lifts . . . . .	80
<b>7</b>	<b>Rigidity and Strata Dimension</b>	<b>83</b>
7.1	Introduction . . . . .	83
7.2	Grassmanian prolongation . . . . .	84
7.3	Nonrigid (3,2,1) . . . . .	84
7.4	Rigid (3,2,1) . . . . .	86

# Chapter 1

## Introduction

A Carnot group  $G$  is a connected, simply connected, stratified nilpotent Lie group, equipped with a left-invariant subriemannian metric, defined on a left-invariant subbundle of the tangent bundle. The subbundle is called the horizontal bundle and the metric is called the Carnot–Carathéodory metric. Diffeomorphisms which preserve the horizontal bundle are called contact maps, and  $G$  is said to be rigid when the space of contact maps is finite-dimensional. Quasiconformal maps are defined with respect to the Carnot–Carathéodory metric, which in a weak sense, implies they must also be contact maps. Carnot groups are naturally equipped with dilations. Together with left translations, dilations provide trivial examples of contact maps. In the rigid case, these tend to be the only examples.

Quasiconformal mappings on Carnot groups were first considered by Mostow [18]. In the proof of his celebrated rigidity theorem, Carnot groups arise as the boundaries of noncompact rank one symmetric spaces with negative sectional curvature, i.e., the hyperbolic spaces  $H_K^n$ , where  $K = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  and the 16-dimensional Cayley hyperbolic plane  $H_O^{16}$ . A homeomorphism  $h : M \rightarrow N$  between negatively curved locally symmetric spaces of rank one lifts to a homeomorphism  $\tilde{h} : H_{K_1}^{n_1} \rightarrow H_{K_2}^{n_2}$ , equivariant with respect to the action of the fundamental groups of  $M$  and  $N$ , which implies that  $\tilde{h}$  induces a quasiconformal map of the spheres at infinity. The rigidity theorem follows by showing that

the induced quasiconformal map is conformal, implying  $\tilde{h}$  is equivalent to an isometry,  $K_1 = K_2$  and  $n_1 = n_2$ .

For  $H_{\mathbb{R}}^n$ , the boundary is the one point compactification of  $\mathbb{R}^{n-1}$ , hence the quasiconformal theory is relative to the euclidean metric, and it is well known that euclidean spaces are not rigid. For  $H_{\mathbb{C}}^n$  the boundary is the one point compactification of the Heisenberg group  $H_n$ . In this setting Korányi and Reimann [15, 16] showed that the spaces of contact and quasiconformal maps are also infinite-dimensional, and furthermore, Reimann and Ricci [24] proved similar results for the complexified Heisenberg group  $\mathbb{C} \otimes H_{\mathbb{R}}^3$ .

Pansu [22] established the rigidity of the remaining cases  $H_K^n$  where  $K = \mathbb{H}$ , and  $H_{\mathcal{O}}^{16}$ . These cases are particular examples of Heisenberg-type groups, and Reimann [23] established the rigidity of all Heisenberg-type groups with center of dimension at least 3. Pansu's crucial observation in [22] was that contact and quasiconformal mappings are differentiable in a certain sense, and that the derivative must be an isomorphism of the group. Nilpotent Lie groups usually have few automorphisms and so Carnot groups are usually expected to be rigid.

The work of [18, 15, 22, 16, 14] gives rise to a general theory of quasiconformal mapping on Carnot groups. However rigidity could possibly make such a general theory redundant except in a handful of known nonrigid cases, thus motivating the search for nonrigid groups. The possibility of a general theory being redundant was pointed out by Heinonen in [13], and more recently Tyson [30] asks the question: Are there Carnot groups of step 3 or higher which are nonrigid? The answer is yes, the simplest examples being the model filiform groups treated in [32] and first observed in [9].

The Heisenberg and model filiform groups are related by the fact that they are the generic jet spaces  $J^1(\mathbb{R}^m, \mathbb{R}^n)$  and  $J^k(\mathbb{R}, \mathbb{R})$ . This suggests that all generic jet spaces might be nonrigid Carnot groups. This is in fact the case and published in [33].

Jet spaces usually arise as examples of a more general situation where in-

stead of a Carnot group, we have a manifold and a distribution given by a frame of vector fields which generate the tangent space at each point by Lie brackets. Again, a transformation of the manifold is a contact transformation if it preserves the distribution and the question of rigidity applies. Such structures are called subriemannian or Carnot–Carathéodory manifolds and they arise in various branches of mathematics, e.g., differential equations, calculus of variations, and control theory. In particular, jet spaces are fundamental to the geometric study of partial differential equations.

A contact map on the jet space  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  can always be extended or prolonged to a contact map of the jet space  $J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$ . The prolongation process gives rise to two types of contact map on  $J^k(\mathbb{R}^m, \mathbb{R}^n)$ , known as point transformations and Lie tangent transformations. Point transformations arise as the  $k$ -fold prolongation of a diffeomorphism of  $J^0(\mathbb{R}^m, \mathbb{R}^n) \equiv \mathbb{R}^m \times \mathbb{R}^n$  and a Lie tangent transformation arises as the  $(k - 1)$ -fold prolongation of a contact map on  $J^1(\mathbb{R}^m, \mathbb{R}^n)$ . Consequently the jet spaces are nonrigid, however there is a classical rigidity theorem of Bäcklund [3] which shows that every contact transformation is either a point or Lie tangent transformation. The theorem of Bäcklund has a long history with varying statements and proofs in various contexts, see for example [3, 2, 27, 20, 21, 28]. A development of quasiconformal mapping for jet spaces requires a good understanding of Bäcklund’s theorem, so a proof based on Cauchy characteristics is given.

In this thesis we show that the jet spaces are Carnot groups, thus providing a large family of nonrigid Carnot groups supporting a nontrivial quasiconformal mapping theory and giving a positive answer to [30]. This result appears in [33]. The difficulty in determining the multiplication arises from the complexity of the Baker–Campbell–Hausdorff formula. The theory of external and internal symmetry sheds light on the rigidity of Carnot subgroups of the jet spaces, in particular we treat the case of the complexified Heisenberg group, which was previously treated in [24]. The jet spaces fall short of being a typical model of Carnot groups, particularly in the role that Cauchy characteristics play. In

particular, the example due to Cartan, in [34], of a Carnot group with strata dimensions  $(2, 1, 2, 1)$  gives clear contrast.

The folklore rule of thumb is that noncommutativity should reflect rigidity in the sense that a high degree of noncommutativity should imply more rigidity. The problem here is that we don't know what the measure of noncommutativity should be. An obvious consideration is that a measure of noncommutativity should be the step and the dimensions of the strata. However, examples show that such data tells us almost nothing. We consider the Carnot group associated with the Hilbert Cartan equation (see [34]), and the jet space  $J^4(\mathbb{R}, \mathbb{R})$ . Both groups have strata dimensions  $(2, 1, 1, 1, 1)$  but opposite rigidity. Another example is constructed by using Grassmanian prolongation to produce a non-rigid group with strata dimensions  $(3, 2, 1)$ . The rigid example is the group of  $4 \times 4$  unipotent real upper triangular matrices.

# Chapter 2

## Carnot Groups

### 2.1 Introduction

A nilpotent Lie algebra  $\mathfrak{g}$  is said to admit an  $n$ -step *stratification* if

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n,$$

such that  $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$ , where  $j = 1, \dots, n-1$ , and  $\mathfrak{g}_n$  is contained in the center  $Z(\mathfrak{g})$ . A connected, simply connected nilpotent Lie group  $G$ , with stratified Lie algebra  $\mathfrak{g}$ , equipped with an inner product  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ , such that  $\mathfrak{g}_i \perp \mathfrak{g}_j$  when  $i \neq j$ , is called a Carnot group.

A left invariant vector field  $X \in \Gamma(TG)$  has the form  $\mathfrak{g} \mapsto (\tau_g)_*(V)$ , where  $V \in \mathfrak{g} \equiv T_e G$ , and it follows that the left invariant vector fields inherit the stratification of  $\mathfrak{g}$ . In particular, if  $L_i$  is the subbundle of  $TG$  defined by  $L_i(g) = (\tau_g)_*(\mathfrak{g}_i)$ , then  $L_{i+1}(g) = [L_1, L_i](g)$  where  $i = 1, \dots, n-1$ . The inner product of  $\mathfrak{g}$  induces an inner product on  $T_g G$  by setting

$$\langle V, W \rangle_g = \langle (\tau_{g^{-1}})_*(V), (\tau_{g^{-1}})_*(W) \rangle_{\mathfrak{g}},$$

and it follows that  $L_i(g) \perp L_j(g)$  when  $i \neq j$ . The horizontal tangent space at  $g \in G$  is the subspace  $L_1(g) \subseteq T_g G$ , and a curve  $\gamma : I \rightarrow G$ , is said to be horizontal if  $\dot{\gamma}(t) \in L_1(\gamma(t))$  for all  $t \in I$ . If  $H(g_1, g_2)$  denotes the set of

horizontal curves joining  $g_1$  to  $g_2$ , then the *Carnot–Carathéodory* distance is

$$d(g_1, g_2) = \inf_{\gamma \in \mathcal{H}(g_1, g_2)} \int \|\dot{\gamma}(t)\| dt,$$

where  $\|\dot{\gamma}(t)\| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}}$ . The theorem of Chow, see [8], implies that  $G$  is path connected via horizontal curves, and that  $d$  is a metric. By definition  $d$  is left-invariant, i.e.,  $d(\tau_g(g_1), \tau_g(g_2)) = d(g_1, g_2)$ .

For simply connected nilpotent Lie groups, the exponential map  $\exp : \mathfrak{g} \rightarrow G$ , is a diffeomorphism. Moreover, the exponential map is an isomorphism  $(\mathfrak{g}, \star) \rightarrow G$  when we define  $X \star Y = \exp^{-1}(\exp(X) \exp(Y))$ . The dilation  $\delta_t \in \text{Aut}(\mathfrak{g})$  is defined by  $\delta_t(X) = \sum_{j=1}^n t^j X_j$ , where  $X = \sum_{j=1}^n X_j$  and  $X_j \in \mathfrak{g}_j$ . Dilation of  $g \in G$  is defined by  $g \mapsto \exp \circ \delta_t \circ \exp^{-1}(g)$ , and where no confusion arises, we denote dilation on  $G$  by  $\delta_t(g)$ . By definition, the Carnot–Carathéodory distance is homogeneous with respect to dilation, that is  $d(\delta_t(g_1), \delta_t(g_2)) = t d(g_1, g_2)$ .

If  $G_1$  and  $G_2$  are Carnot groups, isomorphic as Lie groups, with isomorphism  $h : G_1 \rightarrow G_2$ , then there exists a Lie algebra isomorphism  $\tilde{h} : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ , such that

$$\exp_1 \circ \tilde{h} = h \circ \exp_2, \quad (2.1.1)$$

However, Lie algebra isomorphisms  $\tilde{h} : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  do not always respect the stratifications, for example  $Y \mapsto \sum_k \frac{1}{k!} (\text{ad } X)^k(Y)$  always defines a Lie algebra automorphism of  $\mathfrak{g}$ , but does not preserve the stratification of  $\mathfrak{g}$ . By definition, dilations preserve stratification and a Lie algebra isomorphism  $\tilde{h} : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  respects stratification if and only if it intertwines dilations. Furthermore,  $h$  is an isometry if and only if  $\tilde{h}$  is an isometry.

The Baker–Campbell–Hausdorff formula, see [26], is the explicit expression

$$X \star Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{0 < p_i + q_i \\ 1 \leq i \leq n}} \frac{1}{C_{p,q}} T(X^{p_1}, Y^{q_1}, \dots, X^{p_n}, Y^{q_n}),$$

where

$$C_{p,q} = p_1! q_1! \dots p_n! q_n! \sum_{i=1}^n p_i + q_i$$

and

$$\begin{aligned}
& T(X^{p_1}, Y^{q_1}, \dots, X^{p_n}, Y^{q_n}) \\
&= \begin{cases} (\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \dots (\operatorname{ad} X)^{p_n} (\operatorname{ad} Y)^{q_n-1} Y & \text{if } q_n \geq 1 \\ (\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \dots (\operatorname{ad} X)^{p_n-1} X & \text{if } q_n = 0. \end{cases} \quad (2.1.2)
\end{aligned}$$

The expansion to order 4 takes the form

$$\begin{aligned}
X \star Y &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) \\
&\quad + \frac{1}{48}([Y, [X, [Y, X]]] - [X, [Y, [X, Y]]]) + \dots \quad .
\end{aligned}$$

By construction, the pair  $(\mathfrak{g}, \star)$  is a Lie group with Lie algebra  $\mathfrak{g}$  such that  $\operatorname{Aut}(\mathfrak{g}, \star) = \operatorname{Aut}(\mathfrak{g})$ . Furthermore, any Carnot group  $G$  with Lie algebra  $\mathfrak{g}$  is group isomorphic to  $(\mathfrak{g}, \star)$  via a stratification preserving isomorphism, and when an inner product  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$  is given, the isomorphism becomes an isometry when we define

$$\langle V, W \rangle_X = \langle (\tau_{X^{-1}})_*(V), (\tau_{X^{-1}})_*(W) \rangle_{\mathfrak{g}} .$$

It follows from these observations that the theory of Carnot groups can be developed in the context of the model  $(\mathfrak{g}, \star)$ .

Choosing an orthonormal basis, say  $\mathcal{B}$ , identifies  $\mathfrak{g}$  with  $\mathbb{R}^{\dim \mathfrak{g}}$  and  $X \star Y$  becomes polynomial in the coordinates  $X$  and  $Y$  of degree  $\leq n-1$ . The triple  $(\mathfrak{g}, \star, \mathcal{B})$  is said to be a *normal model of the first kind*.

Let

$$\{ e_{i,\alpha} \mid i = 1, \dots, n, \alpha = 1, \dots, d_i = \dim \mathfrak{g}_i \}$$

denote a basis of  $\mathfrak{g}$  such that

$$\mathfrak{g}_i = \operatorname{span}\{ e_{i,\alpha} \mid \alpha = 1, \dots, d_i = \dim \mathfrak{g}_i \},$$

and let

$$\{ \lambda_{i,\beta} \mid i = 1, \dots, n, \beta = 1, \dots, d_i = \dim \mathfrak{g}_i \} \subset \mathfrak{g}^*$$

denote the corresponding dual basis such that

$$\lambda_{i,\beta}(e_{j,\alpha}) = \begin{cases} 1 & \text{if } i = j \text{ and } \beta = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

The vector fields  $X_{i,\alpha}$ , defined by  $X_{i,\alpha}(X) = (\tau_X)_*(e_{i,\alpha})$ , form a basis for the left-invariant vector fields of  $(\mathfrak{g}, \star)$ , and the corresponding dual forms on  $\mathfrak{g}$  are

$$\theta_{i,\alpha}|_X = (\tau_X^{-1})^* \lambda_{i,\alpha}.$$

If  $f : \Omega_1 \rightarrow \Omega_2$  is a diffeomorphism between open sets  $\Omega_1, \Omega_2 \subseteq \mathfrak{g}$ , and  $V \in T_X \mathfrak{g}$ , where  $X \in \Omega_1$ , then

$$f_*(V_X) = \sum_i \sum_\alpha \sum_j \sum_\beta \lambda_{i,\alpha}(f_*(e_{j,\beta})) \lambda_{j,\beta}(V_X) e_{i,\alpha} \quad (2.1.3)$$

$$\begin{aligned} &= \sum_i \sum_\alpha \sum_j \sum_\beta (\theta_{i,\alpha})_{f(X)} (f_*(X_{j,\beta}(X))) \theta_{j,\beta}(V_X) X_{i,\alpha}(f(X)) \\ &= \sum_i \sum_\alpha \sum_j \sum_\beta \theta_{i,\alpha}(f_* X_{j,\beta}) \theta_{j,\beta}(V_X) X_{i,\alpha}(f(X)). \end{aligned} \quad (2.1.4)$$

We use the notation  $Jf$  and  $Df$  to denote the matrices with block form

$$Jf_{i,j} = (\lambda_{i,\alpha}(f_* e_{j,\beta}))_{\alpha,\beta} \quad \text{and} \quad Df_{i,j} = (\theta_{i,\alpha}(f_* X_{j,\beta}))_{\alpha,\beta}.$$

Note that the substitutions

$$(X_{i,\alpha})(X) = \tau_{X*}(e_{i,\alpha})$$

and

$$(\theta_{i,\alpha})_X = (\tau_X^{-1})^* \lambda_{i,\alpha}$$

show that

$$Df(X) = J(\tau_{f(X)}^{-1} \circ f \circ \tau_X)(0). \quad (2.1.5)$$

## 2.2 Contact maps

A local  $C^1$  diffeomorphism  $f : G \rightarrow G$  that preserves horizontal curves is called a contact map. If  $f$  is a contact map, then  $f_*(L_1(g)) = L_1(f(g))$ , moreover the

contact maps of  $G$  correspond with contact maps of  $(\mathfrak{g}, \star)$  via the exponential map. The trivial examples are left translations and dilations.

Let  $f$  be a contact map of  $\Omega \subseteq (\mathfrak{g}, \star)$ , then  $\theta_{i,1}(f_*X_{1,\beta}) = 0$  when  $i > 1$  and  $1 \leq \beta \leq d_1$ . However more is true, in fact  $\theta_{i,\alpha}(f_*X_{j,\beta}) = 0$  when  $1 \leq j < i$ , and  $Df(X)$  is block upper triangular. We can give two proofs, the first is in some sense classical and assumes that  $f$  is  $C^2$  for the validity of the formula

$$(f_*[V, W])_{f(p)} = [f_*V, f_*W]_{f(p)},$$

where  $V$  and  $W$  are vector fields. The second proof uses the Pansu derivative and only requires that  $f$  is  $C^1$ , and demonstrates some of the significance of the Pansu derivative.

The first proof proceeds by observing that, because  $f_*(L_1(X)) = L_1(f(X))$ ,

$$f_*(L_1(X) \oplus \cdots \oplus L_j(X)) \subseteq L_1(f(X)) \oplus \cdots \oplus L_j(f(X)), \quad (2.2.1)$$

since for any pair of smooth vector fields  $V$  and  $W$  we have

$$(f_*[V, W])_{f(X)} = [f_*V, f_*W]_{f(X)}. \quad (2.2.2)$$

For example, if  $V, W \in \Gamma(L_1)$ , then  $[V, W] \in \Gamma(L_1 \oplus L_2)$ , and it follows that  $f_*V, f_*W \in \Gamma(L_1)$  implying that  $[f_*V, f_*W] \in \Gamma(L_1 \oplus L_2)$ . From (2.2.2),

$$f_*[V, W] \in \Gamma(L_1 \oplus L_2). \quad (2.2.3)$$

If  $D_{ij} = (\theta_{i,\alpha}(f_*X_{j,\beta}))$ , then  $Df$  has the following block form:

$$\begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ 0 & D_{22} & \cdots & D_{2n} \\ 0 & D_{32} & \cdots & D_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & D_{n2} & \cdots & D_{nn} \end{pmatrix}.$$

Since

$$(f_*[V, W])_{f(X)} = Df(X)\theta([V, W]_X),$$

(2.2.3) implies that for all  $V, W \in \Gamma(L_1)$ ,

$$D_{i2}\theta_2([V, W]_X) = 0 \quad i = 3, \dots, n.$$

In particular, if  $V = X_{1,a}$  and  $W = X_{1,b}$  then

$$[X_{1,a}, X_{1,b}]_X = \sum_{\gamma=1}^{n_2} c_{a,b}^\gamma X_{2,\gamma},$$

where  $n_2 = \dim \mathfrak{g}_2$ , and

$$D_{i2}\theta_2([X_{1,a}, X_{1,b}]_X) = \begin{pmatrix} \theta_{i,1}(f_*X_{2,1}) & \dots & \theta_{i,1}(f_*X_{2,n_2}) \\ \vdots & & \vdots \\ \theta_{i,n_i}(f_*X_{2,1}) & \dots & \theta_{i,n_i}(f_*X_{2,n_2}) \end{pmatrix} \begin{pmatrix} c_{a,b}^1 \\ \vdots \\ c_{a,b}^{n_2} \end{pmatrix}$$

where  $n_i = \dim \mathfrak{g}_i$ . It follows that each row in  $D_{i2}$  is orthogonal to all the vectors  $(c_{a,b}^\gamma)_\gamma$ . By the bracket generating property,

$$\dim \text{span} \left\{ (c_{a,b}^\gamma)_\gamma \mid a, b = 1, \dots, n_1 \right\} = n_2$$

forcing  $D_{i2} = 0$  for  $i = 3, \dots, n$ , and

$$f_*(L_1(X) \oplus L_2(X)) \subseteq L_1(f(X)) \oplus L_2(f(X)).$$

Next assume that

$$f_*(L_1(X) \oplus \dots \oplus L_\ell(X)) \subseteq L_1(f(X)) \oplus \dots \oplus L_\ell(f(X)), \quad \ell = 1, \dots, k-1,$$

that is,  $D_{ij} = 0$  when  $1 \leq j < i < k$ . Let  $V \in \Gamma(L_1)$  and

$$W \in \Gamma(L_1 \oplus \dots \oplus L_{k-1}).$$

Then

$$[V, W] \in \Gamma(L_1 \oplus \dots \oplus L_k)$$

and

$$f_*[V, W] = [f_*V, f_*W] \in \Gamma(L_1 \oplus \dots \oplus L_k). \quad (2.2.4)$$

Since

$$(f_*[V, W])_{f(X)} = Df(X)\theta([V, W]_X),$$

(2.2.4) implies that for all  $V \in \Gamma(L_1)$  and  $W \in \Gamma(L_1 \oplus \cdots \oplus L_{k-1})$ ,

$$D_{ik}\theta_k([V, W]_X) = 0, \quad i = k + 1, \dots, n.$$

Again, the bracket generating property implies  $D_{ik} = 0$  for  $i = k + 1, \dots, n$  and

$$f_*(L_1(X) \oplus \cdots \oplus L_k(X)) \subseteq L_1(f(X)) \oplus \cdots \oplus L_k(f(X)).$$

Hence induction gives (2.2.1), and it follows that, if  $f$  is a contact diffeomorphism, then

$$\theta_{i,\alpha}(f_*X_{j,\beta}) = 0, \quad 1 \leq j < i. \quad (2.2.5)$$

## 2.3 Cauchy characteristics

A vector field  $V \in \Gamma(L_1 \oplus \cdots \oplus L_j)$  is called a Cauchy characteristic of order  $j$  if

$$[V, W] \in \Gamma(L_1 \oplus \cdots \oplus L_j) \quad \text{for all } W \in \Gamma(L_1 \oplus \cdots \oplus L_j).$$

We denote the set of Cauchy characteristics of order  $j$  by  $\mathcal{C}_j$ . It follows from (2.2.1) that a contact map must also preserve each  $\mathcal{C}_j$ , since

$$\begin{aligned} [f_*\mathcal{C}_j, \Gamma(L_1 \oplus \cdots \oplus L_j)] &= [f_*\mathcal{C}_j, f_*\Gamma(L_1 \oplus \cdots \oplus L_j)] \\ &= f_*[\mathcal{C}_j, \Gamma(L_1 \oplus \cdots \oplus L_j)] \\ &\subseteq f_*\Gamma(L_1 \oplus \cdots \oplus L_j) \\ &\subseteq \Gamma(L_1 \oplus \cdots \oplus L_j). \end{aligned} \quad (2.3.1)$$

Note that the case where  $j = n$  is trivial in the sense that every vector field is a characteristic of order  $n$  and the case where  $j = 1$  is trivial in the sense that there are no nonzero characteristics of order 1.

## 2.4 Pansu differentiability

Let  $f$  be a map of some open set  $\Omega \subseteq (\mathfrak{g}, \star)$  into  $(\mathfrak{g}, \star)$ , and let

$$\psi_X = \tau_{f(X)}^{-1} \circ f \circ \tau_X.$$

Then  $f$  is said to be Pansu differentiable on  $\Omega$  if for every  $X \in \Omega$ , the limit

$$\lim_{t \rightarrow 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z) \tag{2.4.1}$$

converges locally uniformly with respect to  $Z \in (\mathfrak{g}, \star)$ , and the map

$$\phi_X(Z) = \lim_{t \rightarrow 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z),$$

called the Pansu derivative of  $f$  at  $X$ , is an element of  $\text{Aut}(\mathfrak{g}, \star)$ . The topology of the convergence is the metric topology induced by the Euclidean norm, the Carnot–Carathéodory distance, or the gauge metric of Nagel–Stein–Wainger [19].

**Theorem 2.4.1** *Let  $\Omega \subseteq \mathfrak{g}$  be an open set and let  $f : \Omega \rightarrow \mathfrak{g}$  be  $C^1$ . Then  $f$  is a contact map if and only if it is Pansu differentiable at every  $X \in \Omega$ .*

Proof. Suppose  $f$  is Pansu differentiable at  $X \in \Omega$ . Since

$$\frac{1}{t^i} \lambda_{i,\alpha} \circ \psi_X(t^j e_{j,\beta}) = \lambda_{i,\alpha} \circ \delta_{1/t} \circ \psi_X \circ \delta_t(e_{j,\beta}),$$

it follows that

$$\lim_{t \rightarrow 0} \frac{1}{t^i} \lambda_{i,\alpha} \circ \psi_X(t^j e_{j,\beta}) = \lambda_{i,\alpha} \circ \phi_X(e_{j,\beta}).$$

In particular, if  $f$  is  $C^1$  in a neighborhood of  $X$ , then  $Df(X) = J\psi_X(0)$  is block upper triangular, and  $\phi_X$  is given by the diagonal part of  $Df(X)$ . Moreover  $f$  is a contact map since  $Df(X)_{i,1} = 0$  when  $i > 1$ , i.e.,  $f$  preserves horizontal curves.

In the rest of this section we first outline the proof that a  $C^1$  contact map  $f : \Omega \rightarrow (\mathfrak{g}, \star)$  is Pansu differentiable at every  $X \in \Omega$ , and then provide the details. The proof uses Lemma 1.40 of [11], which states that there exists a

constant  $C > 0$ , an integer  $m$ , and a map  $g : \{1, \dots, m\} \rightarrow \{1, \dots, d_1\}$ , such that every  $W \in (\mathfrak{g}, \star)$  has the form

$$W = w_1 e_{1,g(1)} \star \dots \star w_m e_{1,g(m)}, \quad (2.4.2)$$

where  $|w_i| \leq C|W|^{1/n}$ .

The proof proceeds first by observing that for every  $C^1$  horizontal curve  $\gamma$  such that  $\gamma(0) = 0$ , we have

$$\lim_{t \rightarrow 0} \delta_{1/t} \circ \gamma(t) = \gamma'_1(0).$$

Next we observe that since  $f$  preserves horizontal curves, the curve  $\gamma(t) = \psi_X(tZ)$  is horizontal when  $Z \in \mathfrak{g}_1$ , and  $\gamma(0) = 0$ . It follows that

$$\lim_{t \rightarrow 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z) = \gamma'_1(0),$$

and the smoothness of  $f$  implies that the convergence is uniform when  $Z \in D_R = \{Z \in \mathfrak{g}_1 \mid |Z| \leq R\}$ . Next we use Pansu's decomposition [22], i.e., if  $Y, Z \in \mathfrak{g}$  then

$$\begin{aligned} \delta_{1/t} \circ \psi_X \circ \delta_t(Y \star Z) &= \delta_{1/t} \left( f(X)^{-1} \star f(X \star \delta_t(Y) \star \delta_t(Z)) \right) \\ &= \delta_{1/t} \left( f(X)^{-1} \star f(X \star \delta_t(Y)) \star f(X \star \delta_t(Y))^{-1} \star f(X \star \delta_t(Y) \star \delta_t(Z)) \right) \\ &= \left( \delta_{1/t} \circ \tau_{f(X)}^{-1} \circ f \circ \tau_X \circ \delta_t(Y) \right) \star \left( \delta_{1/t} \circ \tau_{f(X \star \delta_t(Y))}^{-1} \circ f \circ \tau_{X \star \delta_t(Y)} \circ \delta_t(Z) \right). \end{aligned}$$

Now assume that the limit

$$\lim_{t \rightarrow 0} \delta_{1/t} \circ \tau_{f(X)}^{-1} \circ f \circ \tau_X \circ \delta_t(Y)$$

converges uniformly when  $Y$  is an element of the  $j - 1$  fold product  $D_R^{j-1} = D_R \star \dots \star D_R$ . Assume further that  $Z \in D_R$ , and that

$$\lim_{t \rightarrow 0} \delta_{1/t} \circ \tau_{f(X \star \delta_t(Y))}^{-1} \circ f \circ \tau_{X \star \delta_t(Y)} \circ \delta_t(Z) = \lim_{t \rightarrow 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Z),$$

and the convergence is uniform when  $Y \in D_R^{j-1}$  and  $Z \in D_R$ . Then Pansu's decomposition shows that

$$\lim_{t \rightarrow 0} \delta_{1/t} \circ \psi_X \circ \delta_t(Y \star Z) = \phi_X(Y) \star \phi_X(Z)$$

and the limit converges uniformly when  $Y \in D_R^{j-1}$  and  $Z \in D_R$ . By induction on  $j$  and (2.4.2), it then follows that

$$\lim_{t \rightarrow 0} \delta_{1/t} \circ \psi_X \circ \delta_t(W) = w_1 \phi_X(e_{1,g(1)}) \star \cdots \star w_m \phi_X(e_{1,g(m)})$$

uniformly when  $|W| \leq R$ , and consequently  $\phi_X \in \text{Aut}(\mathfrak{g}, \star)$ .

Now we come to the details.

**Lemma 2.4.2** *If  $\gamma$  is a  $C^1$  horizontal curve such that  $\gamma(0) = 0$ , then*

$$\lim_{s \rightarrow 0} \delta_{1/s} \circ \gamma(s) = \gamma'_1(0).$$

Proof. By definition,  $\gamma$  is horizontal if and only if

$$\gamma'(s) = (\tau_{\gamma(s)})_* \Big|_0(v(s)) \quad (2.4.3)$$

for some  $v(s) \in \mathfrak{g}_1$ . If  $V \in \mathfrak{g} = T_0\mathfrak{g}$ , then (2.1.2) shows that there are constants  $C_k$  such that

$$(\tau_X)_* \Big|_0(V) = \sum_{k=0}^{n-1} C_k (\text{ad } X)^k(V). \quad (2.4.4)$$

For example,  $C_0 = 1$ ,  $C_1 = 1/2$ ,  $C_2 = 1/12$  and  $C_3 = 0$ . Together, (2.4.3) and (2.4.4) show that

$$\begin{aligned} \gamma'(s) &= \sum_{k=0}^{n-1} C_k (\text{ad } \gamma(s))^k(v(s)) \\ &= v(s) + \sum_{k=1}^{n-1} C_k (\text{ad } \gamma(s))^k(v(s)) \end{aligned}$$

which implies  $v(s) = \gamma'_1(s)$ . It follows that

$$\gamma'(s) = \sum_{k=0}^{n-1} C_k (\text{ad } \gamma(s))^k(\gamma'_1(s)).$$

Since  $(\text{ad } \gamma(s))^k(\gamma'_1(s))$  is a weighted sum of terms

$$[\gamma_{\ell_k}(s), [\dots [\gamma_{\ell_1}(s), \gamma'_1(s)] \dots]],$$

and

$$\lambda_j \left( [\gamma_{\ell_k}(s), [\dots [\gamma_{\ell_1}(s), \gamma'_1(s)] \dots]] \right) = 0$$

unless  $\ell_k + \dots + \ell_1 = j - 1$ , it follows that

$$\begin{aligned} & \frac{1}{s^{j-1}} \lambda_j \left( (\text{ad } \gamma(s))^k (\gamma'_1(s)) \right) \\ &= \begin{cases} \lambda_j \left( (\text{ad } \delta_{1/s} \circ \gamma(s))^k (\gamma'_1(s)) \right) & \text{if } 1 \leq k \leq j - 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.4.5)$$

In particular,

$$\gamma'_j(s) = \sum_{k=1}^{j-1} C_j \lambda_j \left( \left( \text{ad } (\gamma_1(s) + \dots + \gamma_{j-1}(s)) \right)^k (\gamma'_1(s)) \right). \quad (2.4.6)$$

When  $j = 2$ , (2.4.6) implies that

$$\gamma'_2(s) = C_1 [\gamma_1(s), \gamma'_1(s)],$$

and it follows that

$$\lim_{t \rightarrow 0} \frac{\gamma'_2(t)}{t} = \lim_{t \rightarrow 0} C_1 \left[ \frac{\gamma_1(t)}{t}, \gamma'_1(t) \right] = 0.$$

Furthermore,

$$\begin{aligned} \left| \frac{\gamma_2(s)}{s^2} \right| &= \left| \frac{C_1}{s^2} \int_0^s [\gamma_1(t), \gamma'_1(t)] dt \right| \\ &\leq \frac{|C_1|}{s} \int_0^s \left| \left[ \frac{\gamma_1(t)}{t}, \gamma'_1(t) \right] \right| dt \\ &= |C_1| \left| \left[ \frac{\gamma_1(c)}{c}, \gamma'_1(c) \right] \right|, \end{aligned} \quad (2.4.7)$$

where the existence of  $c \in (0, s)$  in the the last line is guaranteed by the Mean Value Theorem. It follows that

$$\lim_{s \rightarrow 0} \frac{\gamma_2(s)}{s^2} = 0.$$

If we assume  $\lim_{t \rightarrow 0} \frac{\gamma_\ell(t)}{t^\ell} = 0$  when  $\ell = 2, \dots, j - 1$ , then (2.4.6) implies

$$\lim_{t \rightarrow 0} \frac{\gamma'_j(t)}{t^{j-1}} = 0.$$

Furthermore, we have

$$\begin{aligned}
\left| \frac{\gamma_j(s)}{s^j} \right| &= \left| \frac{1}{s^j} \int_0^s \gamma_j'(t) dt \right| \\
&\leq \frac{1}{s} \int_0^s \left| \frac{\gamma_j'(t)}{t^{j-1}} \right| dt \\
&= \left| \frac{\gamma_j'(c)}{c^{j-1}} \right|,
\end{aligned} \tag{2.4.8}$$

where the existence of  $c \in (0, s)$  in the the last line is guaranteed by the Mean Value Theorem. It follows that

$$\lim_{s \rightarrow 0} \frac{\gamma_j(s)}{s^j} = 0,$$

and we conclude that

$$\lim_{s \rightarrow 0} \delta_{1/s} \circ \gamma(s) = \gamma_1'(0),$$

as required.  $\square$

For  $Z \in \mathfrak{g}_1$ , let  $\zeta(t) = tZ$ , then

$$(\tau_{\zeta(t)})_* \Big|_0 (\zeta_1'(t)) = \sum_{k=0}^{n-1} C_k (\text{ad } tZ)^k (Z) = Z = \zeta'(t),$$

hence  $\zeta$  is horizontal. It follows that  $\gamma(t) = \psi_x(tZ)$  is a  $C^1$  horizontal curve such that  $\gamma(0) = 0$ , since  $\psi_x$  is a  $C^1$  contact map which fixes 0. Moreover, the previous lemma shows that

$$\lim_{s \rightarrow 0} \delta_{1/s} \circ \psi_x \circ \delta_s(Z) = J\psi_x(0)\lambda(Z) = Df(X)\lambda(Z), \tag{2.4.9}$$

where  $\lambda(Z)$  is the coordinate expression of  $Z$  relative to the basis  $\{e_{i,\alpha}\}$ .

Since  $f \in C^1$ , it follows that

$$(V, X) \mapsto \|J\psi_x(V) - Df(X)\|_\infty$$

and

$$(V, X) \mapsto \|J\psi_x(V)\|_\infty,$$

are both continuous function of  $\mathfrak{g} \times \mathfrak{g}$  into  $\mathbb{R}^+$ , see [25, p. 188]. For a compact subset  $U \subset \mathfrak{g}$ , we define

$$N_1(R, U) = \max\{ \|J\psi_x(V) - Df(X)\|_\infty \mid |V| \leq R, X \in U\}$$

and

$$N_2(R, U) = \max\{ \|J\psi_x(V)\|_\infty \mid |V| \leq R, X \in U\}.$$

**Lemma 2.4.3** *Let  $Z \in D_R$ ,  $X \in U$ , and  $\gamma(t) = \psi_x(tZ)$ . Then*

$$|\gamma'_1(s) - \gamma'_1(0)| \leq P(R)N_1(sR, U)$$

and

$$\left| \frac{\gamma_1(s)}{s} - \gamma'_1(0) \right| \leq P(R)N_1(sR, U)$$

where  $P(R) = R \sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty$ .

Proof. For the first inequality,

$$\begin{aligned} |\gamma'_1(s) - \gamma'_1(0)| &= \left| \sum_{\mu=1}^{d_1} \lambda_{1,\mu} (J\psi_x(sZ)\lambda(Z) - Df(X)\lambda(Z)) e_{1,\mu} \right| \\ &\leq \sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty |J\psi_x(sZ)\lambda(Z) - Df(X)\lambda(Z)| \\ &\leq R \sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty \|J\psi_x(sZ) - Df(X)\|_\infty. \end{aligned}$$

For the second inequality we apply the Mean Value Theorem, that is, there exists  $0 < t_{1,\mu} < s$  such that

$$\begin{aligned} \left| \frac{\gamma_{1,\mu}(s)}{s} - \gamma'_{1,\mu}(0) \right| &= |\gamma'_{1,\mu}(t_{1,\mu}) - \gamma'_{1,\mu}(0)| \\ &= |\lambda_{1,\mu} (J\psi_x(t_{1,\mu}Z)\lambda(Z) - Df(X)\lambda(Z))| \\ &\leq \|\lambda_{1,\mu}\|_\infty R \|J\psi_x(t_{1,\mu}Z) - Df(X)\|_\infty, \end{aligned}$$

and it follows that

$$\left| \frac{\gamma_1(s)}{s} - \gamma'_1(0) \right| \leq R \sum_{\mu=1}^{d_1} \|\lambda_{1,\mu}\|_\infty \|J\psi_x(t_{1,\mu}Z) - Df(X)\|_\infty, \quad (2.4.10)$$

as claimed.  $\square$

**Lemma 2.4.4** *Let  $Z \in D_R$ ,  $X \in U$ , and  $\gamma(t) = \psi_x(tZ)$ . Then for each  $k \geq 2$  there is a positive constant  $Q_k(R, U)$  such that*

$$\left| \frac{\gamma_k(s)}{s^k} \right| \leq Q_k(R, U) N_1(sR, U).$$

Proof. From (2.4.7), there is  $c \in (0, s)$  such that

$$\left| \frac{\gamma_2(s)}{s^2} \right| \leq |C_1| \left| \left[ \frac{\gamma_1(c)}{c}, \gamma'_1(c) \right] \right|. \quad (2.4.11)$$

Since there is a constant  $M > 0$  such that  $||[X, Y]|| \leq M|X| |Y|$ ,

$$\begin{aligned} \left| \left[ \frac{\gamma_1(c)}{c}, \gamma'_1(c) \right] \right| &\leq \left| \left[ \frac{\gamma_1(c)}{c} - \gamma'_1(0), \gamma'_1(c) \right] \right| + |[\gamma'_1(0), \gamma'_1(c) - \gamma'_1(0)]| \\ &\leq M \left( \left| \frac{\gamma_1(c)}{c} - \gamma'_1(0) \right| |\gamma'_1(c)| + |\gamma'_1(0)| |\gamma'_1(c) - \gamma'_1(0)| \right). \end{aligned} \quad (2.4.12)$$

Furthermore, since  $c < s \leq 1$ ,

$$\begin{aligned} |\gamma'_1(c)| &= |\lambda_1(J\psi_x(cZ)Z)| \\ &\leq P(R) \|J\psi_x(cZ)\|_\infty \\ &\leq P(R) N_2(R, U). \end{aligned} \quad (2.4.13)$$

From (2.4.13), (2.4.12), and Lemma 2.4.4,

$$\left| \left[ \frac{\gamma_1(c)}{c}, \gamma'_1(c) \right] \right| \leq 2 M P(R)^2 N_2(R, U) N_1(cR, U),$$

and  $N_1(cR, U) \leq N_1(sR, U)$ , so

$$\left| \frac{\gamma_2(s)}{s^2} \right| \leq 2|C_1| M P(R)^2 N_2(R, U) N_1(sR, U),$$

thus

$$Q_2(R, U) = 2|C_1| M P(R)^2 N_2(R, U).$$

Similarly, by (2.4.8) there is  $c \in (0, s)$  such that

$$\left| \frac{\gamma_j(s)}{s^j} \right| \leq \left| \frac{\gamma'_j(c)}{c^{j-1}} \right|, \quad (2.4.14)$$

and by (2.4.6)

$$\left| \frac{\gamma'_j(c)}{c^{j-1}} \right| \leq \sum_{k=1}^{j-1} |C_j| \| \lambda_j \|_\infty \left| \left( \text{ad } \frac{\gamma_1(c)}{c} + \dots + \frac{\gamma_{j-1}(c)}{c^{j-1}} \right)^k (\gamma'_1(c)) \right|. \quad (2.4.15)$$

We write

$$\begin{aligned} \frac{\gamma_1(c)}{c} + \dots + \frac{\gamma_{j-1}(c)}{c^{j-1}} &= \left( \frac{\gamma_1(c)}{c} - \gamma'_1(0) \right) + \left( \gamma'_1(0) + \frac{\gamma_2(c)}{c} \dots + \frac{\gamma_{j-1}(c)}{c^{j-1}} \right) \\ &= A + B_j, \end{aligned}$$

say, and use the inequality  $|[X, Y]| \leq M|X| |Y|$  to obtain

$$\begin{aligned} \left| (\text{ad } A + B_j)^k (\gamma'_1(c)) \right| &\leq |\gamma'_1(c)| M^k \sum_{\ell=0}^{k-1} \binom{k}{\ell} |A|^{k-\ell} |B_j|^\ell \\ &\quad + |B_j|^{k-1} |[B_j, \gamma'_1(c)]| M^{k-1}. \end{aligned}$$

Furthermore, if we write  $B_j = \gamma'_1(0) + \tilde{B}_j$ , then

$$\begin{aligned} |[B_j, \gamma'_1(c)]| &= |[\gamma'_1(0), \gamma'_1(c)] + [\tilde{B}_j, \gamma'_1(c)]| \\ &\leq |[\gamma'_1(0), \gamma'_1(c)]| + M|\tilde{B}_j| |\gamma'_1(c)| \\ &= \left| \left[ \gamma'_1(0) - \frac{\gamma'_1(c)}{c}, \gamma'_1(c) \right] \right| + M|\tilde{B}_j| |\gamma'_1(c)| \\ &\leq M|A| |\gamma'_1(c)| + M|\tilde{B}_j| |\gamma'_1(c)|, \end{aligned}$$

hence

$$\begin{aligned} \left| (\text{ad } A + B_j)^k (\gamma'_1(c)) \right| &\leq P(R)N_2(R, U)M^k \sum_{\ell=0}^{k-1} \binom{k}{\ell} |A|^{k-\ell} |B_j|^\ell \\ &\quad + P(R)N_2(R, U)M^k |B_j|^{k-1} (|A| + |\tilde{B}_j|) \quad (2.4.16) \end{aligned}$$

From Lemma 2.4.3,

$$|A| \leq P(R)N_1(sR, U),$$

and assuming inductively that

$$\left| \frac{\gamma_\ell(s)}{s^\ell} \right| \leq Q_\ell(R, U)N_1(sR, U)$$

for  $\ell = 2, \dots, j-1$ , we deduce that

$$|B_j| \leq RN_2(R, U) + N_1(R, U) \sum_{\ell=1}^{j-1} Q_\ell(R, U)$$

and

$$|\tilde{B}_j| \leq N_1(sR, U) \sum_{\ell=1}^{j-1} Q_\ell(R, U).$$

These estimates, together with (2.4.16), (2.4.15) and (2.4.14), show that there is a constant  $Q_j(R, U)$  such that

$$\left| \frac{\gamma_j(s)}{s^j} \right| \leq Q_j(R, U) N_1(sR, U),$$

as stated. □

By Lemma (2.4.3) and Lemma (2.4.4), the following result holds.

**Corollary 2.4.5** *If  $Z \in D_R$ , then for each  $X \in \mathfrak{g}$ , the limit*

$$\lim_{s \rightarrow 0} \delta_{1/s} \circ \psi_X \circ \delta_s(Z) = Df(X)\lambda(Z)$$

*converges uniformly with respect to  $Z$ .*

**Lemma 2.4.6** *Let  $X_0, Y \in \mathfrak{g}$  and  $Z \in \mathfrak{g}_1$ . Assume that  $Y, Z \in \overline{B_R(0)}$ . Then*

$$\lim_{s \rightarrow 0} \delta_{1/s} \circ \psi_{X_0 \star \delta_s(Y)} \circ \delta_s(Z) = \lim_{s \rightarrow 0} \delta_{1/s} \circ \psi_{X_0} \circ \delta_s(Z) = Df(X_0)\lambda(Z),$$

*and the convergence is uniform with respect to  $Z$ .*

Proof. Let  $U$  be the compact set given by

$$U = \{X \mid X = X_0 \star W, W \in \overline{B_R(0)}\}.$$

For  $X \in U$  let  $\gamma^{X,Z}(t) = \psi_X(tZ)$ , then

$$\begin{aligned} \left| \delta_{1/s} \circ \gamma^{X_0,Z}(s) - \delta_{1/s} \circ \gamma^{X,Z}(s) \right| &\leq \left| \delta_{1/s} \circ \gamma^{X_0,Z}(s) - (\gamma^{X_0,Z})'(0) \right| \\ &\quad + \left| \delta_{1/s} \circ \gamma^{X,Z}(s) - (\gamma^{X,Z})'(0) \right| \\ &\quad + \left| (\gamma^{X_0,Z})'(0) - (\gamma^{X,Z})'(0) \right|, \end{aligned} \quad (2.4.17)$$

By Lemma 2.4.3 and Lemma 2.4.4,

$$\begin{aligned} \left| \delta_{1/s} \circ \gamma^{X,Z}(s) - (\gamma^{X,Z})'(0) \right| &\leq \left| \frac{\gamma_1^{X,Z}(s)}{s} - (\gamma_1^{X,Z})'(0) \right| + \sum_{k=2}^n \left| \frac{\gamma_k^{X,Z}(s)}{s^k} \right| \\ &\leq \left( P(R) + \sum_{k=2}^n Q_k(R, U) \right) N_1(sR, U), \end{aligned}$$

and (2.4.17) gives

$$\left| \delta_{1/s} \circ \gamma^{X_0, Z}(s) - \delta_{1/s} \circ \gamma^{X, Z}(s) \right| \leq 2H(R, U)N_1(sR, U) + R \|Df(X_0) - Df(X)\|_\infty,$$

where  $H(R, U) = P(R) + \sum_{k=2}^n Q_k(R, U)$ .

Letting  $X = X_0 \star \delta_s(Y)$  in the previous estimate shows that

$$\lim_{s \rightarrow 0} \delta_{1/s} \circ \gamma^{X_0 \star \delta_s(Y), Z}(s) = \gamma^{X_0, Z}'(0) = Df(X_0)\lambda(Z),$$

and the convergence is uniform with respect to  $Y$  and  $Z$ .  $\square$

The proof of Theorem 2.4.1 is now complete.

## 2.5 Quasiconformal maps

The *metric definition* of quasiconformality is as follows. Let  $\Omega_1$  and  $\Omega_2$  be open subsets of  $\mathfrak{g}$ , let  $f : \Omega_1 \rightarrow \Omega_2$  be a homeomorphism and define

$$H_f(X) = \limsup_{r \rightarrow 0} \frac{\sup\{d(f(X), f(Y)) : d(X, Y) = r\}}{\inf\{d(f(X), f(Y)) : d(X, Y) = r\}}.$$

Then  $f$  is said to be  $K$ -quasiconformal if  $H_f$  is bounded and

$$\text{ess sup}_{X \in \Omega_1} |H_f(X)| = \|H_f\|_\infty \leq K.$$

The trivial examples are dilations and left translations which are 1-quasiconformal.

The *analytic definition*, due to Pansu [22], is as follows. A homeomorphism  $f : \Omega_1 \rightarrow \Omega_2$  is  $K$ -quasiconformal if and only if the horizontal distributional derivatives exist and belong to  $L_{\text{loc}}^Q(\Omega_1)$ ,  $f$  is P-differentiable almost everywhere with the P-differential satisfying

$$\|\phi_x\|^Q \leq K |\det Jf(X)|$$

where  $Q = \sum_i i \dim \mathfrak{g}_i$  is the *homogeneous dimension* of  $\mathfrak{g}$ ,

$$\|\phi_x\| = \sup\{\|\phi_x(Y)\|_{\mathfrak{g}} \mid Y \in \mathfrak{g}_1, \|Y\|_{\mathfrak{g}} = 1\}$$

and  $\det Jf(X)$  is the generalised Jacobian determinant of  $f$ . This definition uses the fact that in a first kind model,  $\text{Aut}(\mathfrak{g}, \star) = \text{Aut}(\mathfrak{g})$ . In particular, if  $G$  is any Carnot group, then replacing  $\phi_x$  with the Lie derivative of  $\phi_g$  in the definition above, gives the general definition.

## 2.6 Second kind coordinates

When studying Carnot groups, it suffices to work with a normal model of the first kind, however such models do not always lead to the simplest expressions as they do not exploit the commutativity that is present. In some sense, second kind models do exploit the commutativity that is present.

Let  $\{e_{i,\alpha}\}$  be a basis for  $\mathfrak{g}$  with dual basis  $\lambda_{i,\alpha}$  so that  $X \in \mathfrak{g}$  has the form

$$X = x_{1,1}e_{1,1} + \cdots + x_{n,d_n}e_{n,d_n}$$

where  $x_{i,\alpha} = \lambda_{i,\alpha}(X)$ . The map  $\Phi : \mathfrak{g} \rightarrow G$ , given by

$$\begin{aligned} \Phi(X) &= \exp(x_{1,1}e_{1,1}) \exp(x_{1,2}e_{1,2}) \cdots \exp(x_{n,d_n}e_{n,d_n}) \\ &= \exp(x_{1,1}e_{1,1} \star x_{1,2}e_{1,2} \star \cdots \star x_{n,d_n}e_{n,d_n}), \end{aligned} \quad (2.6.1)$$

is a diffeomorphism, see [31, p. 86], and an isomorphism  $(\mathfrak{g}, \odot) \rightarrow G$  when we define

$$X \odot Y = \Phi^{-1}(\Phi(X)\Phi(Y)).$$

Relative to the chosen basis,  $X \odot Y$  becomes polynomial in the coordinates of  $X$  and  $Y$  of degree  $\leq n - 1$ . The triple  $(\mathfrak{g}, \odot, \{e_{i,\alpha}\})$  is said to be a *normal model of the second kind*.

We have the natural isomorphism  $\rho : (\mathfrak{g}, \odot, \{e_{i,\alpha}\}) \rightarrow (\mathfrak{g}, \star, \{e_{i,\alpha}\})$ , given by

$$\rho = \exp^{-1} \circ \Phi.$$

In particular

$$\rho(X) = x_{1,1}e_{1,1} \star \cdots \star x_{n,d_n}e_{n,d_n}, \quad (2.6.2)$$

which shows that  $\rho$  commutes with dilation and  $J\rho(0) = I$ . It follows that the classes of Pansu differentiable maps, contact maps, and quasiconformal maps on  $(\mathfrak{g}, \odot)$ , are intertwined by  $\rho$  with those of  $(\mathfrak{g}, \star)$ .



# Chapter 3

## Jet Spaces

### 3.1 Introduction

In this section we establish the standard apparatus of jet spaces, see for example [4, 27, 20, 21, 28].

A function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  has  $d(m, k) = \binom{m+k-1}{k}$  distinct  $k$ -th order partial derivatives

$$\partial_I f(p) = \frac{\partial^k f}{\partial x_1^{i_1} \dots \partial x_m^{i_m}}(p)$$

where the  $k$ -index  $I = (i_1, \dots, i_m)$ , satisfies  $|I| = i_1 + \dots + i_m = k$ . We denote the set of  $k$ -indexes by  $I(k)$ , and let

$$\tilde{I}(k) = I(0) \cup \dots \cup I(k).$$

For  $I \in \tilde{I}(k)$  and  $t \in \mathbb{R}^m$ , we define

$$I! = i_1! i_2! \dots i_m! \quad \text{and} \quad t^I = (t^1)^{i_1} (t^2)^{i_2} \dots (t^m)^{i_m},$$

moreover the  $k$ -th order Taylor polynomial of  $f$  at  $p$  is given by

$$T_p^k(f)(t) = \sum_{I \in \tilde{I}(k)} \partial_I f(p) \frac{(t-p)^I}{I!}.$$

If  $D \subseteq \mathbb{R}^m$  is open and  $p \in D$ , then two functions  $f_1, f_2 \in C^k(D, \mathbb{R})$  are defined to be equivalent at  $p$ , denoted  $f_1 \sim_p f_2$ , if and only if  $T_p^k(f_1) = T_p^k(f_2)$ . The

$k$ -jet space over  $D$  is given by

$$J^k(D, \mathbb{R}) = \cup_{p \in D} C^k(D, \mathbb{R}) / \sim_p \quad (3.1.1)$$

where elements are denoted  $j_p^k(\mathbf{f})$ . It comes equipped with the following projections

$$x : J^k(D, \mathbb{R}) \rightarrow D \text{ and } \pi_j^k : J^k(D, \mathbb{R}) \rightarrow J^{k-j}(D, \mathbb{R}), \quad j = 1, \dots, k,$$

where

$$x(j_p^k(\mathbf{f})) = p \quad \text{and} \quad \pi_j^k(j_p^k(\mathbf{f})) = j_p^{k-j}(\mathbf{f}).$$

Global coordinates are given by  $\psi^{(k)} = (x, u^{(k)})$  where

$$u^{(k)} = \{u_I \mid I \in \tilde{I}(k)\}$$

and

$$u_I(j_p^k(\mathbf{f})) = \partial_I \mathbf{f}(p), \quad I \in \tilde{I}(k).$$

It follows that

$$J^k(D, \mathbb{R}) \equiv D \times \mathbb{R}^{d(m,0)} \times \mathbb{R}^{d(m,1)} \times \dots \times \mathbb{R}^{d(m,k)}.$$

If  $\mathbf{f} = (f^1, \dots, f^n)$  is a map  $\mathbf{f} : D \rightarrow \mathbb{R}^n$  then we apply the jet apparatus to the coordinate functions  $f^\ell$ . Thus global coordinates are denoted by  $\psi^{(k)} = (x, u^{(k)})$ , where

$$x(j_p^k(\mathbf{f})) = p \quad \text{and} \quad u_I^\ell(j_p^k(\mathbf{f})) = \partial_I f^\ell(p), \quad I \in \tilde{I}(k), \quad \ell = 1, \dots, n,$$

and

$$u^{(k)} = \{u_I^\ell \mid I \in \tilde{I}(k), \ell = 1, \dots, n\}.$$

It follows that

$$J^k(D, \mathbb{R}^n) \equiv D \times \mathbb{R}^{nd(m,0)} \times \mathbb{R}^{nd(m,1)} \times \dots \times \mathbb{R}^{nd(m,k)}.$$

When making comparisons between jet spaces of different orders, we add the superscript  $(t)$  to coordinate expressions on  $J^t(\mathbb{R}^m, \mathbb{R}^n)$ . In particular we replace  $x$  by  $x^{(t)}$  and we use

$$u^{(t)} = \{u_I^{(t),\ell} \mid I \in \tilde{I}(t), \ell = 1, \dots, n\}.$$

This notation expresses the compatibility of the coordinates with the projections  $\pi_s^t$ , that is:

$$x^{(t)} = x^{(t-s)} \circ \pi_s^t \quad \text{and} \quad u_J^{(t),\ell} = u_J^{(t-s),\ell} \circ \pi_s^t, \quad \text{when} \quad |J| \leq t-s. \quad (3.1.2)$$

We also use the notation

$$\bar{\pi}_s^t = \psi^{(t-s)} \circ \pi_s^t \circ (\psi^{(t)})^{-1}.$$

## 3.2 Contact structure

The  $k$ -jet of a map  $f \in C^k(D, \mathbb{R}^n)$  is the section  $p \mapsto j_p^k(f)$  of the bundle  $x : J^k(D, \mathbb{R}^n) \rightarrow D$ . A contact form  $\theta$  on  $J^k(D, \mathbb{R}^n)$  is a one-form satisfying  $s^*\theta = 0$  for all  $k$ -jets  $s$ . By the chain rule, the contact forms are framed by the set

$$\left\{ \omega_I^\ell = du_I^\ell - \sum_{j=1}^m u_{I+e_j}^\ell dx^j \mid I \in \tilde{I}(k-1), \ell = 1, \dots, n \right\}, \quad (3.2.1)$$

and, see [12], a section  $s$  of  $x : J^k(D, \mathbb{R}^n) \rightarrow D$  is a  $k$ -jet if and only if  $s^*\omega_I^\ell = 0$  for all  $I \in \tilde{I}(k-1)$  and  $\ell = 1, \dots, n$ .

The horizontal tangent bundle  $\mathcal{H}^k$  is defined pointwise by

$$\mathcal{H}_p^k = \left\{ v \in T_p J^k(D, \mathbb{R}^n) \mid \omega_I^\ell(v) = 0, \quad I \in \tilde{I}(k-1), \quad \ell = 1, \dots, n \right\}.$$

In coordinates,

$$\begin{aligned} v &= \sum_{j=1}^m dx^j(v) \frac{\partial}{\partial x^j} + \sum_{\ell=1}^n \sum_{I \in \tilde{I}(k)} du_I^\ell(v) \frac{\partial}{\partial u_I^\ell} \\ &= \sum_{j=1}^m dx^j(v) \frac{\partial}{\partial x^j} + \sum_{\ell=1}^n \sum_{I \in \tilde{I}(k-1)} \left( \sum_{j=1}^m u_{I+e_j}^\ell dx^j(v) \right) \frac{\partial}{\partial u_I^\ell} \\ &\quad + \sum_{\ell=1}^n \sum_{I \in I(k)} du_I^\ell(v) \frac{\partial}{\partial u_I^\ell} \\ &= \sum_{j=1}^m dx^j(v) \left( \frac{\partial}{\partial x^j} + \sum_{\ell=1}^n \sum_{I \in \tilde{I}(k-1)} u_{I+e_j}^\ell \frac{\partial}{\partial u_I^\ell} \right) + \sum_{\ell=1}^n \sum_{I \in I(k)} du_I^\ell(v) \frac{\partial}{\partial u_I^\ell} \\ &= \sum_{j=1}^m dx^j(v) X_j^{(k)} + \sum_{\ell=1}^n \sum_{I \in I(k)} du_I^\ell(v) \frac{\partial}{\partial u_I^\ell}, \end{aligned}$$

where

$$X_j^{(k)} = \frac{\partial}{\partial x^j} + \sum_{\ell=1}^n \sum_{I \in \tilde{I}(k-1)} u_{I+e_j}^\ell \frac{\partial}{\partial u_I^\ell}, \quad j = 1, \dots, m,$$

and it follows that

$$\mathcal{H}^k = \text{span} \left\{ X_j^{(k)} \mid j = 1, \dots, m \right\} \oplus \text{span} \left\{ \frac{\partial}{\partial u_I^\ell} \mid I \in I(k), \ell = 1, \dots, n \right\}.$$

The nontrivial commutators are

$$\left[ \frac{\partial}{\partial u_{I+e_j}^\ell}, X_j^{(k)} \right] = \frac{\partial}{\partial u_I^\ell}, \quad I \in \tilde{I}(k-1), \ell = 1, \dots, n.$$

If  $L_0 = \mathcal{H}^k$  and

$$L_j = \text{span} \left\{ \frac{\partial}{\partial u_I^\ell} \mid I \in I(k-j), \ell = 1, \dots, n \right\},$$

where  $j \geq 1$ , then  $L_j = [L_0, L_{j-1}]$ , where  $j = 1, \dots, k$ . It follows that

$$\mathfrak{X}^k = L_0 \oplus \dots \oplus L_k$$

is a  $(k+1)$ -step stratified nilpotent Lie algebra of vector fields which span  $TJ^k(D, \mathbb{R}^n)$  pointwise.

**Note:** The use of  $0, \dots, k$  to index the strata rather than  $1, \dots, k+1$ , which is the convention for Carnot groups, is chosen to coincide with the order of derivatives. In particular the homogeneous dimension takes the form

$$Q = \sum_{i=1}^k (i+1) \dim L_i.$$

Corresponding to the abstract Lie algebra defined by  $\mathfrak{X}^k$ , there is a Carnot group  $G^{(k)}(m, n)$ , unique up to isomorphism, constructed via the Baker–Campbell–Hausdorff formula. As is shown later, in the case  $D = \mathbb{R}^m$ , we can explicitly determine a multiplication  $\odot$  on  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  such that  $(J^k(\mathbb{R}^m, \mathbb{R}^n), \odot)$  is a Carnot group isomorphic with  $G^{(k)}(m, n)$  and the group induced contact structure agrees with the jet contact structure.

### 3.3 Contact transformations

A diffeomorphism  $f : D_1 \rightarrow D_2$  of domains  $D_1, D_2 \subseteq J^k(\mathbb{R}^m, \mathbb{R}^n)$  is called a contact transformation if  $f_*\mathcal{H}_p^k = \mathcal{H}_{f(p)}^k$ . Equivalently,  $f$  is a contact transformation if it preserves contact forms, i.e., if  $\theta$  is a contact form then  $f^*\theta$  is a contact form.

Let  $v \in \mathcal{H}_p^k$  and

$$\psi^{(k)} \circ f \circ (\psi^{(k)})^{-1}(x, u^{(k)}) = (\xi(x, u^{(k)}), \eta^{(k)}(x, u^{(k)})),$$

where

$$\eta^{(k)}(x, u^{(k)}) = \left\{ \eta_J^\ell(x, u^{(k)}) \mid J \in \tilde{I}(k), \ell = 1, \dots, n \right\}.$$

Then

$$dx^j(f_*v) = d\xi^j(\psi_*^{(k)}v) = \sum_i (X_i^{(k)} \xi^j) dx^i(v) + \sum_q \sum_{I \in I(k)} \frac{\partial \xi^j}{\partial u_I^q} du_I^q(v)$$

and

$$du_J^\ell(f_*v) = d\eta_J^\ell(\psi_*^{(k)}v) = \sum_i (X_i^{(k)} \eta_J^\ell) dx^i(v) + \sum_q \sum_{I \in I(k)} \frac{\partial \eta_J^\ell}{\partial u_I^q} du_I^q(v).$$

If  $f_*v \in \mathcal{H}_{f(p)}^k$  then  $du_J^\ell(f_*v) = \sum_j (u_{J+e_j}^\ell \circ f(p)) dx^j(f_*v)$ , hence a contact diffeomorphism satisfies the contact conditions:

$$X_i^{(k)} \eta_J^\ell = \sum_j \eta_{J+e_j}^\ell (X_i^{(k)} \xi^j), \quad J \in \tilde{I}(k-1), \quad \ell = 1, \dots, n, \quad (3.3.1)$$

$$\frac{\partial \eta_J^\ell}{\partial u_I^q} = \sum_j \eta_{J+e_j}^\ell \frac{\partial \xi^j}{\partial u_I^q}, \quad J \in \tilde{I}(k-1), \quad I \in I(k), \quad \ell = 1, \dots, n. \quad (3.3.2)$$

In the case  $n = 1$  we drop the superscript  $\ell$ .

### 3.4 Prolongation

From a contact transformation  $f$  on  $\Omega \subseteq J^k(\mathbb{R}^m, \mathbb{R}^n)$  we can construct a domain  $\Omega_1 \subset J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$  and a map  $\text{pr}(f) : \Omega_1 \rightarrow \text{pr}(f)(\Omega_1) \subseteq J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$ , called the first prolongation of  $f$ , uniquely determined by the following conditions:

1.  $\text{pr}(f)$  is a contact transformation

2.  $\pi_1^{k+1} \circ \text{pr}(f) = f \circ \pi_1^{k+1}$ .

Let  $\bar{\pi}_1^{k+1} = \psi^{(k)} \circ \pi_1^{k+1} \circ (\psi^{(k+1)})^{-1}$  and

$$\psi^{(k+1)} \circ \text{pr}(f) \circ (\psi^{(k+1)})^{-1} = (\xi^{(k+1)}, \eta^{(k+1)}),$$

then (2) and the compatibility conditions (3.1.2) imply

$$\xi^{(k+1),j} = \xi^{(k),j} \circ \bar{\pi}_1^{k+1}, \quad (3.4.1)$$

and

$$\eta_J^{(k+1),\ell} = \eta_J^{(k),\ell} \circ \bar{\pi}_1^{k+1} \quad (3.4.2)$$

when  $|J| \leq k$ . When  $|J| = k + 1$ , the definition of the coordinate functions  $\eta_J^{(k+1),\ell}$  is given by the contact conditions

$$\omega_I^{(k+1),\ell}(\text{pr}(f)_* X_i^{(k+1)}) = 0, \quad |I| = k, \quad \ell = 1, \dots, n, \quad i = 1, \dots, m. \quad (3.4.3)$$

In coordinates, these conditions give the matrix equation

$$\left[ X_i^{(k+1)}(\eta_I^{(k),\ell} \circ \bar{\pi}_1^{k+1}) \right]_i = \left[ X_i^{(k+1)}(\xi^{(k),j} \circ \bar{\pi}_1^{k+1}) \right]_{ij} \left[ \eta_{I+e_i}^{(k+1),\ell} \right]_i \quad (3.4.4)$$

which serves to define the coordinate functions  $\eta_J^{(k+1),\ell}$ , where  $|J| = k + 1$ , uniquely on  $\Omega_1 = (\psi^{(k+1)})^{-1}(W)$  where

$$W = \left\{ (x^{(k+1)}, u^{(k+1)}) \in \psi^{(k+1)}((\pi_1^{k+1})^{-1}(\Omega)) \mid \det \left[ X_i^{(k+1)}(\xi^{(k),j} \circ \bar{\pi}_1^{k+1}) \right]_{ij} \neq 0 \right\}.$$

It remains to be checked that  $\text{pr}(f)$  is a contact transformation. To this end, note that the compatibility conditions (3.1.2), imply that

$$dx^{(k+1),i} = (\pi_1^{k+1})^* dx^{(k),i} = dx^{(k),i} \circ (\pi_1^{k+1})_*, \quad (3.4.5)$$

and, when  $|J| \leq k$ , that

$$du_J^{(k+1),\ell} = (\pi_1^{k+1})^* du_J^{(k),\ell} = du_J^{(k),\ell} \circ (\pi_1^{k+1})_*. \quad (3.4.6)$$

It follows that

$$\omega_J^{(k+1),\ell} = (\pi_1^{k+1})_* \omega_J^{(k),\ell} = \omega_J^{(k),\ell} \circ (\pi_1^{k+1})_* \quad (3.4.7)$$

when  $|J| \leq k - 1$ . It follows from (3.4.5), (3.4.6) and (3.4.7) that  $(\pi_1^{k+1})_* : \mathcal{H}^{k+1} \rightarrow \mathcal{H}^k$ . In particular

$$(\pi_1^{k+1})_* \frac{\partial}{\partial u_I^{(k+1),\ell}} = \begin{cases} \frac{\partial}{\partial u_I^{(k),\ell}} & |I| \leq k \\ 0 & |I| = k + 1 \end{cases} \quad (3.4.8)$$

and

$$(\pi_1^{k+1})_* X_j^{(k+1)} = X_j^{(k)} + \sum_{\ell} \sum_{|I|=k} du_I^{(k),\ell} \left( (\pi_1^{k+1})_* X_j^{(k+1)} \right) \frac{\partial}{\partial u_I^{(k),\ell}}. \quad (3.4.9)$$

From (2) and (3.4.7), we have

$$\omega_J^{(k+1),\ell} \circ \text{pr}(f)_* = \omega_J^{(k),\ell} \circ f_* \circ (\pi_1^{k+1})_* \quad (3.4.10)$$

when  $|J| \leq k - 1$ , hence (3.4.9) and (3.4.10), together with the fact that  $f$  is a contact transformation, imply

$$\omega_J^{(k+1),\ell} \left( \text{pr}(f)_* X_j^{(k+1)} \right) = 0 \quad (3.4.11)$$

when  $|J| \leq k - 1$ . Furthermore, for  $|I| = k + 1$ , (3.4.8) and (3.4.10), together with the fact that  $f$  is a contact transformation, show that

$$\omega_J^{(k+1),\ell} \left( \text{pr}(f)_* \frac{\partial}{\partial u_I^{(k+1),\ell}} \right) = 0 \quad (3.4.12)$$

when  $|J| \leq k - 1$ .

For  $|J| = k$ , (3.4.5), (3.4.6) and (2) give

$$\begin{aligned} \omega_J^{(k+1),\ell} \circ \text{pr}(f)_* = \\ du_J^{(k),\ell} \circ f_* \circ (\pi_1^{k+1})_* - \sum_j u_{J+e_j}^{(k+1),\ell} \circ \text{pr}(f) dx^{(k),j} \circ f_* \circ (\pi_1^{k+1})_*, \end{aligned}$$

which, by (3.4.8), gives (3.4.12) when  $|J| = k$  and  $|I| = k + 1$ . It follows that  $\text{pr}(f)$  satisfies the contact conditions on  $J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$ .

By definition,  $\text{pr}(f) \circ \text{pr}(f^{-1})$  covers the identity of  $J^k(\mathbb{R}^m, \mathbb{R}^n)$ , and satisfies the contact conditions on  $J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$ . It follows from (3.4.1), (3.4.2) and (3.4.4), that  $\text{pr}(f) \circ \text{pr}(f^{-1})$  agrees with the identity on  $J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$  thus implying  $\text{pr}(f)$  is a diffeomorphism and a contact map.

Iterating the prolongation procedure defines the higher order prolongation  $\text{pr}^t(f)$  with domain  $\Omega_t$ .

Prolongation gives rise to two particular types of contact transformations, known as point and Lie tangent transformations. A point transformation is a prolongation of a diffeomorphism of some  $D \subseteq J^0(\mathbb{R}^m, \mathbb{R}^n) \equiv \mathbb{R}^m \times \mathbb{R}^n$ , and a Lie tangent transformation is a prolongation of a contact transformation on some  $D \subseteq J^1(\mathbb{R}^m, \mathbb{R}^n)$ . It turns out that Lie tangent transformations can form a larger class than point transformations, but there are no other contact transformations beyond Lie tangent transformations. This fact is Bäcklund's theorem.

**Theorem 3.4.1 (Bäcklund)** [3] *If  $n > 1$ , then every contact transformation on  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  is the  $k$ -th order prolongation of a point transformation on  $J^0(\mathbb{R}^m, \mathbb{R}^n)$ . If  $n = 1$ , then every contact transformation on  $J^k(\mathbb{R}^m, \mathbb{R})$  is the  $(k - 1)$ -th order prolongation of a contact transformation on  $J^1(\mathbb{R}^m, \mathbb{R})$ .*

The main step in proving Bäcklund's theorem is to show that the coordinate functions

$$\begin{aligned} \xi^i, \quad i &= 1, \dots, m, \\ \eta^\ell, \quad \ell &= 1, \dots, n, \quad \text{and} \\ \eta_I^\ell, \quad \ell &= 1, \dots, n, \quad \text{where } |I| = 1, \end{aligned}$$

depend on  $u_I^\ell$  only when  $|I| \leq 1$ . We will see that this is a consequence of the fact that contact maps preserve Cauchy characteristics.

### 3.5 Cauchy characteristics

Recall that a vector field  $V \in \Gamma(L_0 \oplus \cdots \oplus L_s)$  is called a Cauchy characteristic vector field of order  $s$  if

$$[V, W] \in \Gamma(L_0 \oplus \cdots \oplus L_s) \quad \text{for all } W \in \Gamma(L_0 \oplus \cdots \oplus L_s).$$

We denote the set of Cauchy characteristics of order  $s$  by  $\mathcal{C}_s$  and, as shown in (2.3.1), contact maps preserve each  $\mathcal{C}_s$ .

For  $s = 1, \dots, k-1$ , let

$$V = \sum_i v_i X_i^{(k)} + \sum_{q=1}^n \sum_{|J| \geq k-s} v_J^q \frac{\partial}{\partial u_J^q} \in \mathcal{C}_s$$

and

$$W = \sum_i w_i X_i^{(k)} + \sum_{q=1}^n \sum_{|J| \geq k-s} w_J^q \frac{\partial}{\partial u_J^q} \in \Gamma(L_0 \oplus \cdots \oplus L_s),$$

then

$$[V, W] = \sum_{i,q} \sum_{|J|=k-s} (v_J^q w_i - v_i w_J^q) \left[ \frac{\partial}{\partial u_J^q}, X_i^{(k)} \right] \quad \text{mod } \Gamma(L_0 \oplus \cdots \oplus L_s).$$

For each  $i$  there is a multi-index  $J$ , where  $|J| = k-s$ , such that  $[\frac{\partial}{\partial u_J^q}, X_i^{(k)}] = \frac{\partial}{\partial u_{J-e_i}^q}$  which forces  $v_i = 0$  since  $W$  is arbitrary. Similarly, for each multi-index  $J$ , where  $|J| = k-s$ , there is an  $i$  such that  $[\frac{\partial}{\partial u_J^q}, X_i^{(k)}] = \frac{\partial}{\partial u_{J-e_i}^q}$  which forces  $v_J^q = 0$  since  $W$  is arbitrary. It follows that  $\mathcal{C}_s$ , where  $s = 1, \dots, k-1$ , consists of the fields

$$V = \sum_q \sum_{k-s+1 \leq |J|} v_J^q \frac{\partial}{\partial u_J^q}.$$

Let  $f$  be a diffeomorphism, then

$$\begin{aligned}
(f_*V)_{f(x,u^{(k)})} &= f_*(V_{(x,u^{(k)})}) \\
&= \sum_q \sum_{k-s+1 \leq |J|} v_J^q f_* \frac{\partial}{\partial u_J^q} \\
&= \sum_i \left( \sum_q \sum_{k-s+1 \leq |J|} v_J^q dx^i (f_* \frac{\partial}{\partial u_J^q}) \right) X_i^{(k)} \\
&\quad + \sum_p \sum_{|I|=k} \left( \sum_q \sum_{k-s+1 \leq |J|} v_J^q du_I^p (f_* \frac{\partial}{\partial u_J^q}) \right) \frac{\partial}{\partial u_I^p} \\
&\quad + \sum_p \sum_{|I| < k} \left( \sum_q \sum_{k-s+1 \leq |J|} v_J^q \omega_I^p (f_* \frac{\partial}{\partial u_J^q}) \right) \frac{\partial}{\partial u_I^p},
\end{aligned}$$

which gives

$$\begin{aligned}
(f_*V)_{f(x,u^{(k)})} &= \sum_i \left( \sum_q \sum_{k-s+1 \leq |J|} v_J^q dx^i (f_* \frac{\partial}{\partial u_J^q}) \right) X_i^{(k)} \\
&\quad + \sum_p \sum_{|I| < k-s+1} \left( \sum_q \sum_{k-s+1 \leq |J|} v_J^q \omega_I^p (f_* \frac{\partial}{\partial u_J^q}) \right) \frac{\partial}{\partial u_I^p} \pmod{\mathcal{C}_s}.
\end{aligned}$$

If  $f$  is a contact map, then  $f_*\mathcal{C}_s \subseteq \mathcal{C}_s$  for all  $s$ , and the previous identity implies that

$$dx^i (f_* \frac{\partial}{\partial u_J^q}) = 0, \quad q = 1, \dots, n, \quad i = 1, \dots, m, \quad 1 < |J| \quad (3.5.1)$$

$$\omega_I^p (f_* \frac{\partial}{\partial u_J^q}) = 0, \quad p, q = 1, \dots, n, \quad |I| < |J|. \quad (3.5.2)$$

Since

$$\omega_I^p (f_* \frac{\partial}{\partial u_J^q}) = du_I^p (f_* \frac{\partial}{\partial u_J^q}) - \sum_{i=1}^m \eta_{I+e_i}^p dx^i (f_* \frac{\partial}{\partial u_J^q}), \quad |I| < k,$$

(3.5.1) and (3.5.2) imply

$$du_I^p (f_* \frac{\partial}{\partial u_J^q}) = 0, \quad p, q = 1, \dots, n, \quad 1 < |J|, \quad |I| < k, \quad |I| < |J|. \quad (3.5.3)$$

From (3.5.1), (3.5.2) and (3.5.3), we see that a contact transformation satisfies the following conditions on coordinate functions

$$\frac{\partial \xi^i}{\partial u_J^q} = 0, \quad q = 1, \dots, n, \quad i = 1, \dots, m, \quad 1 < |J| \quad (3.5.4)$$

$$\frac{\partial \eta_I^p}{\partial u_J^q} = 0, \quad p, q = 1, \dots, n, \quad 1 < |J|, \quad |I| < k, \quad |I| < |J|. \quad (3.5.5)$$

Let  $\tilde{f}$  be a contact map on  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  and let  $a$  and  $b$  be points in the fiber  $(\pi_1^k)^{-1}(p)$  where  $\pi_1^k : J^k(\mathbb{R}^m, \mathbb{R}^n) \rightarrow J^{k-1}(\mathbb{R}^m, \mathbb{R}^n)$ . In coordinates we have

$$\begin{aligned} \tilde{f}(a) &= ( \xi^{(k)}(x^{(k)}(a), u^{(k)}(a)) , \eta^{(k)}(x^{(k)}(a), u^{(k)}(a)) ) \\ \tilde{f}(b) &= ( \xi^{(k)}(x^{(k)}(b), u^{(k)}(b)) , \eta^{(k)}(x^{(k)}(b), u^{(k)}(b)) ) \end{aligned}$$

Since  $\pi_1^k(a) = \pi_1^k(b)$ , the compatibility of the coordinates implies

$$x^{(k)}(a) = x^{(k)}(b) \quad \text{and} \quad u_J^{(k),\ell}(a) = u_J^{(k),\ell}(b), \quad \text{when} \quad |J| \leq k-1.$$

It follows from (3.5.4) and (3.5.5) that

$$\pi_1^k \circ \tilde{f}(a) = \pi_1^k \circ \tilde{f}(b)$$

and the map  $f : J^{k-1}(\mathbb{R}^m, \mathbb{R}^n) \rightarrow J^{k-1}(\mathbb{R}^m, \mathbb{R}^n)$  defined by

$$f(p) = \pi_1^k \circ \tilde{f}(a)$$

is well defined. Since  $\tilde{f}$  is a contact map, and

$$\pi_1^k \circ \tilde{f} = f \circ \pi_1^k,$$

it follows that  $f$  satisfies the contact conditions on  $J^{k-1}(\mathbb{R}^m, \mathbb{R}^n)$ .

Applying the above argument to  $\tilde{f}^{-1}$  gives a map  $g : J^{k-1}(\mathbb{R}^m, \mathbb{R}^n) \rightarrow J^{k-1}(\mathbb{R}^m, \mathbb{R}^n)$ , defined by  $g = \pi_1^k \circ \tilde{f}^{-1}$ , which satisfies

$$\pi_1^k = f \circ g \circ \pi_1^k.$$

It follows that  $f$  and  $g$  are smooth, and  $f \circ g = \text{id}$ , hence  $f$  is a diffeomorphism and therefore a contact map, moreover  $\tilde{f} = \text{pr}(f)$ . By (3.5.4) and (3.5.5), we

can iterate the above “deprolongation” procedure so that  $\tilde{f} = \text{pr}^{k-1}(f_1)$ , where  $f_1$  is a contact map on  $J^1(\mathbb{R}^m, \mathbb{R}^n)$ .

The final step in the proof of Bäcklund’s theorem is to show that when  $n > 1$ , every contact transformation on  $J^1(\mathbb{R}^m, \mathbb{R}^n)$  is a point transformation. A complete proof of this fact can be found in [27, p. 142, Theorem 4.5.12]. For contact flows on  $J^1(\mathbb{R}^m, \mathbb{R}^n)$ , the proof is a little easier. Let

$$f = (\xi, \eta^{(1)}) = (\xi^1, \dots, \xi^m, \eta_{e_1}^1, \dots, \eta_{e_m}^1, \dots, \eta_{e_1}^n, \dots, \eta_{e_m}^n, \eta^1, \dots, \eta^n)$$

be a contact transformation on  $J^1(\mathbb{R}^m, \mathbb{R}^n)$ , then conditions (3.3.1) and (3.3.2) give

$$\begin{aligned} X_i^{(1)}\eta^\ell - \sum_j \eta_{e_j}^\ell (X_i^{(1)}\xi^j) &= 0, \\ \frac{\partial \eta^\ell}{\partial u_{e_\alpha}^q} - \sum_j \eta_{e_j}^\ell \frac{\partial \xi^j}{\partial u_{e_\alpha}^q} &= 0. \end{aligned}$$

If we assume  $f$  is the flow of a vector field

$$\begin{aligned} V &= \sum_j v_j X_j^{(1)} + \sum_\ell \sum_j v_j^\ell \frac{\partial}{\partial u_{e_j}^\ell} + \sum_\ell v^\ell \frac{\partial}{\partial u^\ell} \\ &= \sum_j v_j \frac{\partial}{\partial x^j} + \sum_\ell \sum_j v_j^\ell \frac{\partial}{\partial u_{e_j}^\ell} + \sum_\ell \left( v^\ell + \sum_j v_j u_{e_j}^\ell \right) \frac{\partial}{\partial u^\ell}, \end{aligned}$$

then the contact conditions give

$$\begin{aligned} v_i^\ell &= X_i^{(1)}v^\ell \\ \delta_{q\ell} v_j &= -\frac{\partial v^\ell}{\partial u_{e_j}^q}. \end{aligned} \tag{3.5.6}$$

It follows from (3.5.6) that  $v_j$  and  $v^\ell + \sum_j v_j u_{e_j}^\ell$  are independent of the variables  $u_{e_\alpha}^q$ , which implies that the flow consists of point transformations.

# Chapter 4

## Jet Spaces as Carnot Groups

### 4.1 Introduction

In what follows we establish a multiplication, denoted  $\odot$ , for the jet spaces  $J^k(\mathbb{R}^m, \mathbb{R})$  and  $J^k(\mathbb{R}^m, \mathbb{R}^n)$ . The particular examples  $J^1(\mathbb{R}^m, \mathbb{R}^n)$  and  $J^k(\mathbb{R}, \mathbb{R})$  are simple enough that we can produce  $\odot$  from second kind coordinates using the Baker–Campbell–Hausdorff formula. Owing to the complexity of the Baker–Campbell–Hausdorff formula, this approach is in general difficult. However, the left translation arising from  $\odot$  must be a contact automorphism and thus also a point transformation. The examples demonstrate how to construct the coordinate maps  $\xi$  and  $\eta^\ell$  which through prolongation define  $\odot$ .

### 4.2 Example: $J^1(\mathbb{R}^m, \mathbb{R}^n)$

In this case

$$\mathcal{H}^1 = \text{span} \left\{ X_j^{(1)} \mid j = 1, \dots, m \right\} \oplus \text{span} \left\{ \frac{\partial}{\partial u_{e_j}^\ell} \mid \ell = 1, \dots, n, j = 1, \dots, m \right\}$$

where

$$X_j^{(1)} = \frac{\partial}{\partial x^j} + \sum_{\ell=1}^n u_{e_j}^\ell \frac{\partial}{\partial u_0^\ell}, \quad j = 1, \dots, m$$

and the nontrivial commutators are

$$\left[ \frac{\partial}{\partial u_{e_j}^\ell}, X_j^{(1)} \right] = \frac{\partial}{\partial u_0^\ell}.$$

If  $L_0 = \mathcal{H}^1$  and  $L_1 = \text{span}\{\frac{\partial}{\partial u_0^\ell}\}$  then  $L_1 = [L_0, L_0]$ . It follows that

$$\mathfrak{X}^1 = L_0 \oplus L_1$$

is a 2-step stratified nilpotent Lie algebra of vector fields which span  $TJ^1(\mathbb{R}^m, \mathbb{R}^n)$  pointwise.

Let  $\mathfrak{g}$  denote the abstract Lie algebra over  $\mathbb{R}$  isomorphic with  $\mathfrak{X}^1$ . Denote the basis by

$$\{e_1^{(1)}, \dots, e_m^{(1)}, e_1^1, \dots, e_m^1, \dots, e_1^n, \dots, e_m^n, e^1, \dots, e^n\}$$

where the nontrivial commutation relations are

$$\left[ e_j^\ell, e_j^{(1)} \right] = e^\ell$$

and the isomorphism is given by  $X_j^{(1)} \leftrightarrow e_j^{(1)}$ ,  $\frac{\partial}{\partial u_{e_j}^\ell} \leftrightarrow e_j^\ell$  and  $\frac{\partial}{\partial u^\ell} \leftrightarrow e^\ell$ . The map

$$\sum x_j e_j^{(1)} + \sum u_j^\ell e_j^\ell + \sum u^\ell e^\ell \mapsto m(x, u^{(1)}),$$

where

$$m(x, u^{(1)}) = \begin{pmatrix} 0 & \dots & 0 & u_1^1 & \dots & u_m^1 & u^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & u_1^n & \dots & u_m^n & u^n \\ 0 & \dots & 0 & 0 & \dots & 0 & x^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & x^m \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

is a Lie algebra isomorphism giving a matrix model of  $\mathfrak{g}$ . In coordinates of the

second kind we have

$$\Phi(m(x, u^{(1)})) = \begin{pmatrix} 1 & \dots & 0 & u_1^1 & \dots & u_m^1 & u^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 1 & u_1^n & \dots & u_m^n & u^n \\ 0 & \dots & 0 & 1 & \dots & 0 & x^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & x^m \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and it follows that the second kind coordinate multiplication

$$(x, u^{(1)}) \odot (y, v^{(1)}) = (z, w^{(1)})$$

is defined by  $z = x + y$ ,  $w_{e_j}^\ell = v_{e_j}^\ell + u_{e_j}^\ell$  and

$$w^\ell = u^\ell + v^\ell + \sum_{j=1}^m u_{e_j}^\ell y_j. \quad (4.2.1)$$

### 4.3 Example: $J^k(\mathbb{R}, \mathbb{R})$

In this case the multi-indexes are simply  $j e_1$ , so to simplify notation we use  $\frac{\partial}{\partial u_j}$  instead of  $\frac{\partial}{\partial u_{j e_1}}$ . Similarly, the subscript on  $X_1^{(k)}$  is also redundant so we write  $X^{(k)}$  instead. It follows that

$$\mathcal{H}^k = \text{span} \left\{ X^{(k)}, \frac{\partial}{\partial u_k} \right\} \quad (4.3.1)$$

where

$$X^{(k)} = \frac{\partial}{\partial x} + \sum_{j=0}^{k-1} u_{j+1} \frac{\partial}{\partial u_j} \quad (4.3.2)$$

and the commutation relations are

$$\left[ \frac{\partial}{\partial u_j}, X^{(k)} \right] = \frac{\partial}{\partial u_{j-1}}, \quad j = 1, \dots, k.$$

If  $L_0 = \mathcal{H}^k$  and  $L_j = \text{span}\{\frac{\partial}{\partial u_{k-j}}\}$ , where  $j \geq 1$ , then  $L_j = [L_0, L_{j-1}]$ , where  $j = 1, \dots, k$ , and it follows that

$$\mathfrak{X} = L_0 \oplus \dots \oplus L_k$$

is a  $(k + 1)$ -step stratified nilpotent Lie algebra of vector fields which span  $TJ^k(\mathbb{R}, \mathbb{R})$  pointwise.

Let  $\mathfrak{g}^{(k)}$  denote the abstract Lie algebra over  $\mathbb{R}$  isomorphic with  $\mathfrak{X}^k$ . Denote the basis by  $\{e^{(k)}, e_k, \dots, e_0\}$  where the nontrivial commutators are  $[e_j, e^{(k)}] = e_{j-1}$ , where  $j = 1, \dots, k$ , and the isomorphism is given by the correspondence  $X^{(k)} \leftrightarrow e^{(k)}$  and  $e_j \leftrightarrow \frac{\partial}{\partial u_j}$ . Note that  $\mathfrak{g}^{(1)}$  is the Heisenberg algebra,  $\mathfrak{g}^{(2)}$  is the Engel algebra, and in general  $\mathfrak{g}^{(k)}$  goes by the names model filiform algebra or Goursat algebra.

The map

$$xe^{(k)} + \sum u_j e_j \mapsto \begin{pmatrix} 0 & -x & 0 & \cdots & 0 & u_0 \\ 0 & 0 & -x & \cdots & 0 & u_1 \\ 0 & 0 & 0 & \cdots & 0 & u_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -x & u_{k-1} \\ 0 & 0 & 0 & \cdots & 0 & u_k \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a Lie algebra isomorphism giving a matrix model of  $\mathfrak{g}^{(k)}$ . In coordinates of the second kind, the elements of the corresponding connected, simply connected Lie group  $G^{(k)}$  take the form

$$\exp(xe^{(k)}) \exp(u_k e_k + \cdots + u_0 e_0).$$

Multiplication in second kind coordinates, denoted

$$(x, u_k, \dots, u_0) \odot (y, v_k, \dots, v_0) = (z, w_k, \dots, w_0),$$

can be found by solving

$$\exp(ze^{(k)}) \exp\left(\sum_j w_j e_j\right) \tag{4.3.3}$$

$$= \exp(xe^{(k)}) \exp\left(\sum_j u_j e_j\right) \exp(ye^{(k)}) \exp\left(\sum_j v_j e_j\right) \tag{4.3.4}$$

for  $(z, w_k, \dots, w_0)$ . Using the matrix model we have

$$\exp(xe^{(k)}) \sim \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (-x) & (-x)^2/2! & (-x)^3/3! & \dots & (-x)^k / (k-0)! & 0 \\ 0 & 1 & (-x) & (-x)^2/2! & \dots & (-x)^{k-1}/(k-1)! & 0 \\ 0 & 0 & 1 & (-x) & \dots & (-x)^{k-2}/(k-2)! & 0 \\ 0 & 0 & 0 & 1 & \dots & (-x)^{k-3}/(k-3)! & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (-x) & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and

$$\exp\left(\sum_j u_j e_j\right) \sim \begin{pmatrix} \text{Id} & V(u) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & u_0 \\ 0 & 1 & 0 & 0 & \dots & 0 & u_1 \\ 0 & 0 & 1 & 0 & \dots & 0 & u_2 \\ 0 & 0 & 0 & 1 & \dots & 0 & u_3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & u_k \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Substituting these expressions into (4.3.4) gives

$$\begin{pmatrix} A(z) & A(z)V(w) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A(x)A(y) & A(x)A(y)V(v) + A(x)V(u) \\ 0 & 1 \end{pmatrix}. \quad (4.3.5)$$

From (4.3.5), we have

$$A(z) = A(x)A(y) \quad \text{and} \quad V(w) = V(v) + A(y)^{-1}V(u).$$

It follows that  $z = x + y$ ,  $w_k = v_k + u_k$  and

$$w_s = v_s + u_s + \sum_{j=s+1}^k u_j \frac{y^{j-s}}{(j-s)!}, \quad s = 0, \dots, k-1 \quad (4.3.6)$$

For each  $(x, u^{(k)})$  the previous formula defines a contact transformation in the variable  $(y, v^{(k)})$  and is thus the prolongation of the point transformation

$$(y, v_0) \mapsto \left( x + y, v_0 + \sum_{j=0}^k u_j \frac{y^j}{j!} \right). \quad (4.3.7)$$

#### 4.4 $J^k(\mathbb{R}^m, \mathbb{R})$ and $J^k(\mathbb{R}^m, \mathbb{R}^n)$

We first construct multiplication on the jet spaces  $J^k(\mathbb{R}^m, \mathbb{R})$ . To this end we establish some notation. We write

$$(x, u^{(k)}) \odot (y, v^{(k)}) = (x + y, uv^{(k)}), \quad (4.4.1)$$

where

$$\begin{aligned} (y, v^{(k)}) &= j_y^k(\mathbf{f}), & \mathbf{f}(t) &= \sum_{I \in \bar{I}(k)} v_I \frac{(t-y)^I}{I!}, \\ (x, u^{(k)}) &= j_x^k(\mathbf{g}), & \mathbf{g}(t) &= \sum_{I \in \bar{I}(k)} u_I \frac{(t-x)^I}{I!}, \\ (x+y, uv^{(k)}) &= j_{x+y}^k(\mathbf{h}), & \mathbf{h}(t) &= \sum_{I \in \bar{I}(k)} uv_I \frac{(t-y-x)^I}{I!}. \end{aligned}$$

We write  $I \leq J$  if  $i_\ell \leq j_\ell$  for all  $\ell$  then

$$\begin{aligned} \frac{\partial t^I}{\partial t^J} &= \left( \frac{\partial}{\partial t^1} \right)^{j_1} \cdots \left( \frac{\partial}{\partial t^m} \right)^{j_m} (t^1)^{i_1} (t^2)^{i_2} \cdots (t^m)^{i_m} \\ &= \begin{cases} \frac{I!}{(I-J)!} t^{I-J} & \text{if } J \leq I \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Guided by (4.2.1), (4.3.6) and (4.3.7), we define  $uv_I = v_I + u_I$  when  $|I| = k$  and

$$uv_I = v_I + \sum_{I \leq J} u_J \frac{y^{J-I}}{(J-I)!} = \frac{\partial}{\partial t^I} \mathbf{f}(t) \Big|_{t=y} + \frac{\partial}{\partial t^I} \mathbf{g}(t) \Big|_{t=y+x} \quad (4.4.2)$$

when  $|I| < k$ .

In particular,  $vu_I$  is the  $I$ -th coordinate function  $\eta_I$  of the prolonged point transformation

$$(y, v_0) \mapsto \left( x + y, v_0 + \sum_{0 < J} u_J \frac{y^J}{J!} \right).$$

**Theorem 4.4.1** *The multiplication defined by formulae (4.4.1) and (4.4.2) realises  $J^k(\mathbb{R}^m, \mathbb{R})$  as a Carnot group with the group induced contact structure agreeing with the jet contact structure.*

Proof. To prove associativity we use the notation

$$\left( (z, w^{(k)}) \odot (x, u^{(k)}) \right) \odot (y, v^{(k)}) = (z + x + y, (wu)v^{(k)})$$

and

$$(z, w^{(k)}) \odot \left( (x, u^{(k)}) \odot (y, v^{(k)}) \right) = (z + x + y, w(uv)^{(k)}).$$

By definition,

$$\begin{aligned} (wu)v_I &= v_I + \sum_{I \leq J} wu_J \frac{y^{J-I}}{(J-I)!} \\ &= v_I + \sum_{I \leq J} u_J \frac{y^{J-I}}{(J-I)!} + \sum_{I \leq J} \sum_{J \leq K} w_K \frac{x^{K-J}}{(K-J)!} \frac{y^{J-I}}{(J-I)!} \end{aligned}$$

and

$$\begin{aligned} w(uv)_I &= uv_I + \sum_{I \leq J} w_J \frac{(x+y)^{J-I}}{(J-I)!} \\ &= v_I + \sum_{I \leq J} u_J \frac{y^{J-I}}{(J-I)!} + \sum_{I \leq J} w_J \frac{(x+y)^{J-I}}{(J-I)!}. \end{aligned}$$

Hence associativity will follow if  $(wu)v_I - w(uv)_I = 0$ , where

$$(wu)v_I - w(uv)_I = \sum_{I \leq J} \sum_{J \leq K} w_K \frac{x^{K-J}}{(K-J)!} \frac{y^{J-I}}{(J-I)!} - \sum_{I \leq J} w_J \frac{(x+y)^{J-I}}{(J-I)!}. \quad (4.4.3)$$

Using the multi-index binomial formula

$$(x+y)^{J-I} = \sum_{0 \leq K \leq J-I} \frac{(J-I)!}{(J-I-K)! K!} x^{J-I-K} y^K$$

the sum in (4.4.3) becomes

$$\sum_{I \leq J} \sum_{J \leq K} w_K \frac{x^{K-J}}{(K-J)!} \frac{y^{J-I}}{(J-I)!} - \sum_{I \leq J} \sum_{K \leq J-I} w_J \frac{x^{J-I-K} y^K}{(J-I-K)! K!}. \quad (4.4.4)$$

Exchanging  $J$  and  $K$  in the first sum of (4.4.4) and changing  $K$  to  $K - I$  in the second sum of (4.4.4), we obtain

$$\begin{aligned} (wu)v_I - w(uv)_I &= \sum_{I \leq K} \sum_{K \leq J} w_J \frac{x^{J-K}}{(J-K)!} \frac{y^{K-I}}{(K-I)!} \\ &\quad - \sum_{I \leq J} \sum_{I \leq K \leq J} w_J \frac{x^{J-K} y^{K-I}}{(J-K)!(K-I)!}. \end{aligned}$$

If

$$S_1(I) = \{(J, K) \mid I \leq K \quad \& \quad K \leq J\}$$

and

$$S_2(I) = \{(J, K) \mid I \leq J \quad \& \quad I \leq K \leq J\},$$

then  $S_1(I) \subseteq S_2(I)$  and  $S_2(I) \subseteq S_1(I)$ , hence the right hand side of the previous expression is zero.

From (4.4.2), the point  $(y, v^{(k)})$ , where  $y = -x$  and

$$v_I = \begin{cases} -u_I & |I| = k \\ -\sum_{I \leq J} (-1)^{|J-I|} u_J \frac{x^{J-I}}{(J-I)!} & |I| < k \end{cases}$$

is a right inverse of  $(x, u^{(k)})$ . Since the multiplication is associative, the right inverse is also a left inverse.

The distribution induced by the left translation under  $\odot$  is exactly  $\mathcal{H}^k$ .

Indeed it follows that

$$\frac{\partial}{\partial y^j} uv_J \Big|_{(0,0)} = \begin{cases} u_{J+e_j} & |J| = 0, \dots, k-1 \\ 0 & |J| = k \end{cases} \quad \text{and} \quad \frac{\partial}{\partial v_I} uv_J \Big|_{(0,0)} = \begin{cases} 1 & I = J \\ 0 & \text{otherwise} \end{cases}$$

implying that

$$L_{(x, u^{(k)})_*} \left( \frac{\partial}{\partial y^j} \Big|_{(0,0)} \right) = X_j^{(k)} \Big|_{(x, u^{(k)})} \quad \text{and} \quad L_{(x, u^{(k)})_*} \left( \frac{\partial}{\partial v_I} \Big|_{(0,0)} \right) = \frac{\partial}{\partial v_I} \Big|_{(x, u^{(k)})}.$$

□

Guided by (4.2.1), multiplication on  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  is obtained by applying the multiplication on  $J^k(\mathbb{R}^m, \mathbb{R})$  to the coordinate functions, i.e., we define

$$uv_I^\ell = v_I^\ell + \sum_{I \leq J} u_J^\ell \frac{y^{J-I}}{(J-I)!} = \frac{\partial}{\partial t^I} \mathbf{f}^\ell(t) \Big|_{t=y} + \frac{\partial}{\partial t^I} \mathbf{g}^\ell(t) \Big|_{t=y+x}.$$

## 4.5 Subgroups

Rigidity theory of Carnot subgroups of jet spaces lies within the scope of the theory of internal and external symmetries. If we can view such a subgroup as a submanifold  $S$  of a jet space  $J^k(\mathbb{R}^m, \mathbb{R}^n)$ , then two types of symmetry arise. An external symmetry is a contact transformation on  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  which preserves  $S$ , and an internal symmetry is a transformation of  $S$ , which preserves the contact system on  $S$  obtained via restricting the contact system of  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  to  $S$ . It turns out that internal symmetries can form larger families than external symmetries. An example of this phenomena is given by the complexified Heisenberg group  $J^1(\mathbb{C}, \mathbb{C})$ .

The space  $J^1(\mathbb{C}, \mathbb{C})$  of holomorphic 1-jets is simply the analogue of  $J^1(\mathbb{R}, \mathbb{R})$  with  $\mathbb{C}$  replacing  $\mathbb{R}$  and holomorphic replacing differentiable. The nonrigidity of  $J^1(\mathbb{C}, \mathbb{C})$  was first established in [24], where it is also shown that contact maps must be holomorphic.

If

$$(z, w^{(1)}) = (z, w', w)$$

denotes complex coordinates on  $J^1(\mathbb{C}, \mathbb{C})$ , then the contact structure

$$dw = w'dz$$

induces the second kind coordinate multiplication

$$(z_1, w_1^{(1)}) \odot_{\mathbb{C}} (z_2, w_2^{(1)}) = (z_1 + z_2, w_1' + w_2', w_1 + w_2 + w_1'z_2).$$

By resolving the coordinates on  $J^1(\mathbb{C}, \mathbb{C})$  into real and imaginary parts, we can view  $J^1(\mathbb{C}, \mathbb{C})$  as the subgroup

$$S = \{(x, u^{(1)}) \in J^1(\mathbb{R}^2, \mathbb{R}^2) \mid u_{e_1}^1 - u_{e_2}^2 = 0 \text{ and } u_{e_1}^2 + u_{e_2}^1 = 0\}.$$

Indeed, the map  $\phi : J^1(\mathbb{R}^2, \mathbb{R}^2) \rightarrow J^1(\mathbb{C}, \mathbb{C})$ , given by

$$\phi(x, u^{(1)}) = (x^1 + ix^2, \frac{1}{2}(u_{e_1}^1 + u_{e_2}^2) - \frac{i}{2}(u_{e_2}^1 - u_{e_1}^2), u^1 + iu^2),$$

defines an isomorphism  $S \rightarrow J^1(\mathbb{C}, \mathbb{C})$ . Furthermore,  $\phi$  is a contact transformation when the contact structure on  $S$  is defined via the inclusion map as  $\mathcal{H}_p \cap T_p S$ . In particular we take  $(x^1, x^2, u_{e_1}^1, u_{e_1}^2, u^1, u^2)$  as coordinates on  $S$ , with the inclusion given by

$$\iota : (x^1, x^2, u_{e_1}^1, u_{e_1}^2, u^1, u^2) \mapsto (x^1, x^2, u_{e_1}^1, -u_{e_1}^2, u_{e_1}^1, u^1, u^2).$$

It follows that

$$\begin{aligned} \iota^* dx^1 &= dx^1 & \iota^* du_{e_1}^1 &= du_{e_1}^1 & \iota^* du_{e_1}^2 &= du_{e_1}^2 & \iota^* du^1 &= du^1 \\ \iota^* dx^2 &= dx^2 & \iota^* du_{e_2}^1 &= -du_{e_1}^2 & \iota^* du_{e_2}^2 &= du_{e_1}^1 & \iota^* du^2 &= du^2, \end{aligned}$$

giving

$$\begin{aligned} \iota^* \omega^1 &= du^1 - u_{e_1}^1 dx^1 + u_{e_1}^2 dx^2 \\ \iota^* \omega^2 &= du^2 - u_{e_1}^2 dx^1 - u_{e_1}^1 dx^2 \end{aligned}$$

and

$$\mathcal{H}(S) = \text{span} \left\{ X_1^S, X_2^S, \frac{\partial}{\partial u_{e_1}^1}, \frac{\partial}{\partial u_{e_1}^2} \right\},$$

where

$$X_1^S = \frac{\partial}{\partial x_1} + u_{e_1}^1 \frac{\partial}{\partial u^1} + u_{e_1}^2 \frac{\partial}{\partial u^2}, \quad \text{and} \quad X_2^S = \frac{\partial}{\partial x_2} - u_{e_1}^2 \frac{\partial}{\partial u^1} + u_{e_1}^1 \frac{\partial}{\partial u^2}.$$

A vector field

$$V = v_1 X_1^S + v_2 X_2^S + v_{e_1}^1 \frac{\partial}{\partial u_{e_1}^1} + v_{e_1}^2 \frac{\partial}{\partial u_{e_1}^2} + v^1 \frac{\partial}{\partial u^1} + v^2 \frac{\partial}{\partial u^2}$$

on  $S$  generates a contact flow if and only if  $[V, \mathcal{H}(S)] \subseteq \mathcal{H}(S)$ . It follows that

$$X_1^S v^1 = X_2^S v^2, \quad X_1^S v^2 = -X_2^S v^1, \quad \frac{\partial v^1}{\partial u_{e_1}^1} = \frac{\partial v^2}{\partial u_{e_1}^2} \quad \text{and} \quad \frac{\partial v^2}{\partial u_{e_1}^1} = -\frac{\partial v^1}{\partial u_{e_1}^2}$$

with

$$v_{e_1}^1 = X_1^S v^1, \quad v_{e_1}^2 = X_1^S v^2, \quad v_1 = -\frac{\partial v^1}{\partial u_{e_1}^1} \quad \text{and} \quad v_2 = -\frac{\partial v^2}{\partial u_{e_1}^1}.$$

Putting  $p = v^1 + iv^2$ , it then follows that

$$\bar{Z}p = \frac{1}{2} (X_1^S + iX_2^S) p = 0, \quad \bar{W}'p = \frac{1}{2} \left( \frac{\partial}{\partial u_{e_1}^1} + i \frac{\partial}{\partial u_{e_1}^2} \right) p = 0$$

and

$$\begin{aligned} -\bar{W}p &= [\bar{Z}, \bar{W}']p \\ &= \left[ \frac{1}{2} (X_1^S + iX_2^S), \frac{1}{2} \left( \frac{\partial}{\partial u_{e_1}^1} + i \frac{\partial}{\partial u_{e_1}^2} \right) \right] p \\ &= -\frac{1}{2} \left( \frac{\partial}{\partial u^1} + i \frac{\partial}{\partial u^2} \right) p \\ &= 0. \end{aligned}$$

We conclude that the only restriction on  $p$  is that it be holomorphic in  $z = x^1 + ix^2$ ,  $w' = u_{e_1}^1 + iu_{e_1}^2$  and  $w = u^1 + iu^2$ . If the flow of  $V$  is to extend to a contact flow on  $J^1(\mathbb{R}^2, \mathbb{R}^2)$ , then (3.5.6) shows that  $p$  must be independent of  $w'$ , hence there exist contact vector fields on  $S$  which do not generate flows extending to external symmetries.

More generally, from [1, p. 95], we have the following result which requires the notion of Cauchy–Kovalevskaya form. We say that  $S$  is in Cauchy–Kovalevskaya form, if for some fixed  $j \in \{1, \dots, m\}$ ,  $S$  is defined by

$$u_{ke_j}^\ell = f^\ell(x, \hat{u}^{(k)}), \quad \ell = 1, \dots, n$$

where  $\hat{u}^{(k)}$  is independent of  $u_{ke_j}^\ell$ , where  $\ell = 1, \dots, n$ . If  $k, m \geq 2$  and  $S$  has Cauchy–Kovalevskaya form, then every internal symmetry extends to an external symmetry.

## 4.6 Example: (2,1,2,1)

As we have seen, for  $k \geq 2$ , the Cauchy characteristics of  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  give rise to the jet spaces  $J^{k-t}(\mathbb{R}^m, \mathbb{R}^n)$  as subgroups of  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  such that  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  fibers over  $J^{k-t}(\mathbb{R}^m, \mathbb{R}^n)$ . This scenario is not indicative of Carnot groups in general as the following example, due to Cartan (see [34]), demonstrates.

The 6-dimensional real Lie algebra given by the matrices

$$\begin{pmatrix} 0 & x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} \\ 0 & 0 & x_{1,2} & -x_{2,1} & x_{3,2} \\ 0 & 0 & 0 & x_{1,1} & x_{2,1} \\ 0 & 0 & 0 & 0 & x_{1,2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has nontrivial commutators

$$\begin{aligned} [X_{1,1}, X_{1,2}] &= X_{2,1}, & [X_{1,1}, X_{2,1}] &= -2X_{3,1}, & [X_{1,1}, X_{3,2}] &= X_{4,1}, \\ [X_{1,2}, X_{2,1}] &= 2X_{3,2}, & [X_{1,2}, X_{3,1}] &= -X_{4,1} \end{aligned}$$

and is thus 3-step stratified with strata dimensions  $(2, 1, 2, 1)$ . In second kind coordinates we have

$$\begin{aligned} (x_{1,1}, x_{1,2}, x_{2,1}, x_{3,1}, x_{3,2}, x_{4,1}) \odot (y_{1,1}, y_{1,2}, y_{2,1}, y_{3,1}, y_{3,2}, y_{4,1}) \\ = (w_{1,1}, w_{1,2}, w_{2,1}, w_{3,1}, w_{3,2}, w_{4,1}), \end{aligned}$$

where

$$\begin{aligned} w_{1,1} &= x_{1,1} + y_{1,1}, & w_{1,2} &= x_{1,2} + y_{1,2}, & w_{2,1} &= x_{2,1} + y_{2,1} - y_{1,1}x_{1,2}, \\ w_{3,1} &= x_{3,1} + y_{3,1} + 2y_{1,1}x_{2,1} - y_{1,1}^2x_{1,2}, \\ w_{3,2} &= x_{3,2} + y_{3,2} + 2y_{1,1}y_{1,2}x_{1,2} - 2y_{1,2}x_{2,1} + y_{1,1}x_{1,2}^2, \\ w_{4,1} &= x_{4,1} + y_{4,1} - y_{1,1}x_{3,2} + y_{1,2}x_{3,1} - y_{1,1}^2y_{1,2}x_{1,2} + 2y_{1,1}y_{1,2}x_{2,1} - \frac{y_{1,1}^2x_{1,2}^2}{2}, \end{aligned}$$

giving the following basis of left-invariant fields

$$\begin{aligned} X_{1,1} &= \frac{\partial}{\partial x_{1,1}} - x_{1,2} \frac{\partial}{\partial x_{2,1}} + 2x_{2,1} \frac{\partial}{\partial x_{3,1}} + x_{1,2}^2 \frac{\partial}{\partial x_{3,2}} - x_{3,2} \frac{\partial}{\partial x_{4,1}} \\ X_{1,2} &= \frac{\partial}{\partial x_{1,2}} - 2x_{2,1} \frac{\partial}{\partial x_{3,2}} + x_{3,1} \frac{\partial}{\partial x_{4,1}} \\ X_{2,1} &= \frac{\partial}{\partial x_{2,1}}, & X_{3,1} &= \frac{\partial}{\partial x_{3,1}}, & X_{3,2} &= \frac{\partial}{\partial x_{3,2}} \quad \text{and} \quad X_{4,1} = \frac{\partial}{\partial x_{4,1}}. \end{aligned}$$

The nontrivial Cauchy characteristics are  $\mathcal{C}_2 = \Gamma(L_2)$ , however  $x_{2,1} = 0$  does not give a subgroup. Hence Cauchy characteristics do not give rise to subgroups generally.

# Chapter 5

## Quasiconformal Maps on

$$J^k(\mathbb{R}, \mathbb{R})$$

### 5.1 Contact preliminaries

Since the multi-indexes are simply  $ie_1$ , we simplify notation by using  $u_i$  instead of  $u_{ie_1}$ . Similarly, the subscript on  $X_1^{(k)}$  is also redundant so we write  $X^{(k)}$  instead. Recall from (4.3.1) and (4.3.2) that the basis of left-invariant fields on  $J^k(\mathbb{R}, \mathbb{R})$  is

$$\left\{ X^{(k)}, \frac{\partial}{\partial u_k}, \dots, \frac{\partial}{\partial u_0} \right\},$$

where

$$X^{(k)} = \frac{\partial}{\partial x} + \sum_{j=0}^{k-1} u_{j+1} \frac{\partial}{\partial u_j}$$

and the corresponding basis of dual forms is  $\{dx, du_k, \omega_{k-1}, \dots, \omega_0\}$  where

$$\omega_i = du_i - u_{i+1}dx.$$

Let  $f : \Omega \subseteq J^k(\mathbb{R}, \mathbb{R}) \rightarrow \Omega' \subseteq J^k(\mathbb{R}, \mathbb{R})$  be a contact map. From (2.2.5) or (3.5.2) we have the contact conditions

$$\omega_i(f_*X^{(k)}) = 0, \quad i = 0, \dots, k-1 \quad \text{and} \quad (5.1.1)$$

$$\omega_i\left(f_*\frac{\partial}{\partial u_j}\right) = 0, \quad 0 \leq i < j \leq k, \quad i = 0, \dots, k-1. \quad (5.1.2)$$

To derive the analytic conditions for quasiconformality we compute the invariant differential (2.1.5). We write

$$f(x, u^{(k)}) = (\xi, \eta^{(k)})$$

where

$$\eta^{(k)} = \{\eta_i \mid i = 0, \dots, k\}.$$

By Bäcklund's theorem,

$$\frac{\partial \xi}{\partial u_j} = 0, \quad \frac{\partial \eta_0}{\partial u_j} = 0 \quad \text{and} \quad \frac{\partial \eta_1}{\partial u_j} = 0 \quad \text{when} \quad j > 1, \quad (5.1.3)$$

and

$$\frac{\partial \eta_i}{\partial u_j} = 0 \quad \text{when} \quad i \geq 1 \quad \text{and} \quad j > i. \quad (5.1.4)$$

**Lemma 5.1.1** *If  $k > 1$  and  $f$  is a contact diffeomorphism of  $\Omega \subseteq J^k(\mathbb{R}, \mathbb{R})$ , then*

$$\omega_i \left( f_* \frac{\partial}{\partial u_i} \right) = \frac{\partial \eta_k}{\partial u_k} (X^{(k)} \xi)^{k-i}, \quad i = 0, \dots, k-1.$$

*Proof.* Using Cartan's formula:  $\omega([X, Y]) = X\omega(Y) - Y\omega(X) - 2d\omega(X, Y)$ , and the contact conditions, we have

$$\begin{aligned} \omega_i \left( f_* \frac{\partial}{\partial u_i} \right) &= \omega_i \left( f_* \left[ \frac{\partial}{\partial u_{i+1}}, X^{(k)} \right] \right) \\ &= 2du_{i+1} \wedge dx \left( f_* \frac{\partial}{\partial u_{i+1}}, f_* X^{(k)} \right), \end{aligned} \quad (5.1.5)$$

where  $i = 0, \dots, k-1$ . For  $i < k-1$ , we have  $du_{i+1} \wedge dx = \omega_{i+1} \wedge dx$  and (5.1.5) gives

$$\begin{aligned} \omega_i \left( f_* \frac{\partial}{\partial u_i} \right) &= 2\omega_{i+1} \wedge dx \left( f_* \frac{\partial}{\partial u_{i+1}}, f_* X^{(k)} \right) \\ &= \omega_{i+1} \left( f_* \frac{\partial}{\partial u_{i+1}} \right) dx(f_* X^{(k)}). \end{aligned} \quad (5.1.6)$$

If  $i = k-1$ , then (5.1.5) and (5.1.3) gives

$$\begin{aligned} \omega_{k-1} \left( f_* \frac{\partial}{\partial u_{k-1}} \right) &= 2du_k \wedge dx \left( f_* \frac{\partial}{\partial u_k}, f_* X^{(k)} \right) \\ &= \frac{\partial \eta_k}{\partial u_k} X^{(k)} \xi. \end{aligned} \quad (5.1.7)$$

From (5.1.6) and (5.1.7), it follows that

$$\omega_{k-j} \left( f_* \frac{\partial}{\partial u_{k-j}} \right) = \frac{\partial \eta_k}{\partial u_k} (X^{(k)} \xi)^j, \quad j = 1, \dots, k.$$

□

By Bäcklund's theorem and Lemma 5.1.1, the invariant differential takes the form

$$Df = \begin{pmatrix} X^{(k)} \xi & 0 & 0 & \cdots & 0 & \frac{\partial \xi}{\partial u_1} & \frac{\partial \xi}{\partial u_0} \\ X^{(k)} \eta_k & \frac{\partial \eta_k}{\partial u_k} & \frac{\partial \eta_k}{\partial u_{k-1}} & \cdots & \frac{\partial \eta_k}{\partial u_2} & \frac{\partial \eta_k}{\partial u_1} & \frac{\partial \eta_k}{\partial u_0} \\ 0 & 0 & a_{k-1,k-1} & \cdots & a_{k-1,2} & a_{k-1,1} & a_{k-1,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{2,2} & a_{2,1} & a_{2,0} \\ 0 & 0 & 0 & \cdots & 0 & a_{1,1} & a_{2,0} \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{0,0} \end{pmatrix} \quad (5.1.8)$$

where  $a_{i,j} = \omega_i \left( f_* \frac{\partial}{\partial u_j} \right)$  and  $a_{i,i} = \frac{\partial \eta_k}{\partial u_k} (X^{(k)} \xi)^{k-i}$ .

## 5.2 Quasiconformal preliminaries

From (5.1.8), the Pansu derivative is given by the matrix

$$\phi_p \sim \begin{pmatrix} X^{(k)} \xi & 0 & 0 & 0 & \cdots & 0 \\ X^{(k)} \eta_k & A & 0 & 0 & \cdots & 0 \\ 0 & 0 & X^{(k)} \xi A & 0 & \cdots & 0 \\ 0 & 0 & 0 & (X^{(k)} \xi)^2 A & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (X^{(k)} \xi)^k A \end{pmatrix}$$

where all entries of the matrix are evaluated at  $p$  and  $A = \frac{\partial \eta_k}{\partial u_k}$ . It follows that

$\|\phi_p\|_g^2$  is the largest eigenvalue of  $M^{\text{tr}} M$  where

$$M = \begin{pmatrix} X^{(k)} \xi & 0 \\ X^{(k)} \eta_k & A \end{pmatrix}.$$

The eigenvalues of  $M^{\text{tr}}M$  are

$$\frac{1}{2} \left( \Lambda(f) \pm \sqrt{(\Lambda(f))^2 - 4(\det M)^2} \right), \quad (5.2.1)$$

where

$$\Lambda(f) = (X^{(k)}\xi)^2 + (X^{(k)}\eta_k)^2 + A^2.$$

The homogeneous dimension is

$$Q = 1 + (k+2)(k+1)/2$$

and, by (2.1.5),

$$\det Jf = (X^{(k)}\xi)^{Q-(k+1)}(A)^{k+1}.$$

From the analytic definition,  $f$  is  $K$ -quasiconformal if

$$\Delta(f) = \Delta_1(f) + \sqrt{(\Delta_1(f))^2 - 4\Delta_2(f)} \leq 2K^{2/Q} \quad (5.2.2)$$

where

$$\Delta_1(f) = \frac{\Lambda(f)}{|\det Jf|^{2/Q}} \quad \text{and} \quad \Delta_2(f) = \frac{(\det M)^2}{|\det Jf|^{4/Q}}.$$

If

$$\alpha(k) = \frac{2 + (k+1)k}{Q}, \quad \beta(k) = \frac{2(k+1)}{Q}$$

and

$$\gamma(k) = 2(\alpha(k) - 1) = 2(1 - \beta(k)),$$

then in the case where  $X^{(k)}\xi > 0$  and  $A > 0$ , we have

$$\Delta_1(f) = \frac{(X^{(k)}\xi)^2 + (X^{(k)}\eta_k)^2 + A^2}{(X^{(k)}\xi)^{\alpha(k)} A^{\beta(k)}}$$

and

$$\Delta_2(f) = \left( \frac{A}{X^{(k)}\xi} \right)^{\gamma(k)}.$$

We introduce the 1-quasiconformal maps which we call the *switch map*, denoted  $\sigma$ , and *flip map*, denoted  $\delta_{-1}$ . The switch map is given by

$$\sigma(x, u^{(k)}) = (-x, u_k, -u_{k-1}, \dots, (-1)^{k-j}u_j, \dots, (-1)^k u_0)$$

and the flip map is given by

$$\delta_{-1}(x, u^{(k)}) = (-x, -u_k, u_{k-1}, \dots, (-1)^{k-j+1}u_j, \dots, (-1)^{k+1}u_0).$$

Note that  $\sigma^2 = \delta_{-1}^2 = \text{Id}$ ,

$$\det J\sigma = \begin{cases} 1 & \text{if } k = 1, 2, 5, 6, 9, 10, \dots \\ -1 & \text{if } k = 3, 4, 7, 8, 11, 12, \dots \end{cases}$$

and

$$\det J\delta_{-1} = \begin{cases} 1 & \text{if } k = 1, 4, 5, 8, 9, 12, 13, \dots \\ -1 & \text{if } k = 2, 3, 6, 7, 10, 11, \dots \end{cases}$$

The switch map commutes with the flip map giving

$$\delta_{-1} \circ \sigma(x, u^{(k)}) = \sigma \circ \delta_{-1}(x, u^{(k)}) = (x, -u_k, \dots, -u_i, \dots, -u_0),$$

moreover, the switch map, the flip map and their compositions are 1-quasiconformal since

$$\Delta(\sigma) = \Delta(\delta_{-1}) = \Delta(\sigma \circ \delta_{-1}) = \Delta(\delta_{-1} \circ \sigma) = 2.$$

It follows that every quasiconformal map can, if necessary, be “flipped”, “switched” or “flipped and switched”, to a quasiconformal map satisfying the conditions

$$X^{(k)}\xi > 0 \quad \text{and} \quad A > 0. \tag{5.2.3}$$

**Lemma 5.2.1** *Let  $f$  be a  $K$ -quasiconformal map of  $\Omega \subseteq G$  satisfying (5.2.3), then*

$$\left( \frac{A}{X^{(k)}\xi} \right)^2 \leq K^{2P},$$

where  $P = 2/(Q - 2(k + 1))$ .

Proof. Let  $\lambda_{\min}, \lambda_{\max}$  denote the eigenvalues of  $M^{\text{tr}}M$  and observe that

$$(X^{(k)}\xi)^2 A^2 = (\det M)^2 = \det M^{\text{tr}}M = \lambda_{\max}\lambda_{\min}. \tag{5.2.4}$$

Since  $f$  is  $K$ -quasiconformal, we have

$$\begin{aligned}\lambda_{\max}^{Q/2} &\leq K \det Jf = K(X^{(k)}\xi)^{Q-(k+1)} A^{k+1} \\ &= K(X^{(k)}\xi)^{Q-2(k+1)} ((X^{(k)}\xi)^2 A^2)^{(k+1)/2} \\ &= K(X^{(k)}\xi)^{Q-2(k+1)} (\lambda_{\max}\lambda_{\min})^{(k+1)/2}\end{aligned}$$

which, since  $\lambda_{\min} \leq \lambda_{\max}$ , implies that

$$\lambda_{\max}^{Q/2} \leq K(X^{(k)}\xi)^{Q-2(k+1)} \lambda_{\max}^{k+1}.$$

If  $P = 2/(Q - 2(k + 1))$ , then the previous inequality becomes

$$\lambda_{\max} \leq K^P (X^{(k)}\xi)^2, \quad (5.2.5)$$

which, together with (5.2.4), gives

$$A^2 \leq K^P \lambda_{\min}. \quad (5.2.6)$$

Combining (5.2.5) and (5.2.6) gives the desired inequality  $\square$

### 5.3 1-quasiconformal maps

In this section we prove a Liouville type theorem showing that a 1-quasiconformal map is necessarily a composition of some or all of the four fundamental types of 1-quasiconformal maps, i.e., left translation, dilation, switch and flip. It should be noted that there is no loss of generality in assuming 1-quasiconformal maps to be smooth since in general this is always the case, see [6].

**Lemma 5.3.1** *If  $f$  is 1-quasiconformal satisfying (5.2.3), then*

$$(1) X^{(k)}\xi = A \quad \text{and} \quad (2) X^{(k)}\eta_k = 0.$$

Proof. From Lemma 5.2 of [6], it follows that for all  $V \in \mathfrak{g}_1$  we have

$$\|\phi_p(V)\| = \text{Lip}_f(p)\|V\|,$$

where

$$\text{Lip}_f(p) = \limsup_{q \rightarrow p} \frac{d(f(q), f(p))}{d(p, q)}.$$

It follows that if  $V = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial u_k}$ , then

$$(X^{(k)}\xi)^2 v_1^2 + (X^{(k)}\eta_k v_1 + Av_2)^2 = \text{Lip}_f(p)^2 (v_1^2 + v_2^2)$$

from which (1) and (2) immediately follow. □

**Corollary 5.3.2** *If  $f$  is 1-quasiconformal satisfying (5.2.3), then  $\xi$  is a function of  $x$  only and*

$$\eta_0(x, u^{(k)}) = \xi'(x)^{k+1} u_0 + B_0(x).$$

Proof. From Lemma 5.1.1 and Lemma 5.3.1

$$\omega_0(f_* \frac{\partial}{\partial u_0}) = A(X^{(k)}\xi)^k = (X^{(k)}\xi)^{k+1},$$

and by Bäcklund's Theorem,  $\omega_0(f_* \frac{\partial}{\partial u_0})$  depends on  $x, u_0, u_1$  only. It follows from the previous expression that

$$\frac{\partial}{\partial u_2}(X^{(k)}\xi) = 0,$$

giving

$$\frac{\partial \xi}{\partial u_1} = 0, \tag{5.3.1}$$

which, by Bäcklund's theorem, forces

$$\frac{\partial \eta_0}{\partial u_1} = 0. \tag{5.3.2}$$

From Lemma 5.1.1,

$$\omega_0(f_* \frac{\partial}{\partial u_0}) = \frac{\partial \eta_0}{\partial u_0} - \eta_1 \frac{\partial \xi}{\partial u_0} = \frac{\partial \eta_k}{\partial u_k} \left( \frac{\partial \xi}{\partial x} + u_1 \frac{\partial \xi}{\partial u_0} \right)^k,$$

which upon applying  $\frac{\partial}{\partial u_1}$ , keeping in mind (5.3.1) and (5.3.2), gives

$$-\frac{\partial \eta_1}{\partial u_1} \frac{\partial \xi}{\partial u_0} = k(X^{(k)}\xi)^{k-1} \frac{\partial \xi}{\partial u_0} \frac{\partial \eta_k}{\partial u_k} + (X^{(k)}\xi)^k \frac{\partial^2 \eta_k}{\partial u_1 \partial u_k}. \tag{5.3.3}$$

By (5.3.1) and Lemma 5.1.1,

$$\omega_1(f_* \frac{\partial}{\partial u_1}) = \frac{\partial \eta_1}{\partial u_1} = \frac{\partial \eta_k}{\partial u_k} (X^{(k)} \xi)^{k-1},$$

thus (5.3.3) becomes

$$0 = (k+1)(X^{(k)} \xi)^{k-1} \frac{\partial \xi}{\partial u_0} \frac{\partial \eta_k}{\partial u_k} + (X^{(k)} \xi)^k \frac{\partial^2 \eta_k}{\partial u_1 \partial u_k}. \quad (5.3.4)$$

From Lemma 5.3.1,

$$\frac{\partial \eta_k}{\partial u_k} = X^{(k)} \xi = \left( \frac{\partial \xi}{\partial x} + u_1 \frac{\partial \xi}{\partial u_0} \right)$$

which by (5.3.1) gives

$$\frac{\partial^2 \eta_k}{\partial u_1 \partial u_k} = \frac{\partial \xi}{\partial u_0}.$$

This last expression and (5.3.4) show that

$$0 = (k+2)(X^{(k)} \xi)^k \frac{\partial \xi}{\partial u_0},$$

and it follows that

$$\frac{\partial \xi}{\partial u_0} = 0 \quad (5.3.5)$$

since  $X^{(k)} \xi > 0$ . By Lemma 5.1.1 and Lemma 5.3.1,

$$\frac{\partial \eta_0}{\partial u_0} = (\xi')^{k+1},$$

hence

$$\eta_0(x, u^{(k)}) = \xi'(x)^{k+1} u_0 + B_0(x).$$

□

**Corollary 5.3.3** *If  $f$  is 1-quasiconformal and satisfies (5.2.3), then*

$$\xi(x) = ax + b \quad \text{and} \quad \eta_k(x, u^{(k)}) = au_k + C_k$$

where  $a > 0$  and  $C_k$  is constant.

Proof. From (1) of Lemma 5.3.1 and (5.3.5) we have

$$\frac{\partial \eta_k}{\partial u_k} = \xi'(x). \quad (5.3.6)$$

From the commutation relations,

$$\begin{aligned} \frac{\partial}{\partial u_{k-j}} &= (-1)^j (\text{ad } X^{(k)})^j \frac{\partial}{\partial u_k} \\ &= (-1)^j \sum_{s=0}^j (-1)^s \binom{j}{s} (X^{(k)})^{j-s} \frac{\partial}{\partial u_k} (X^{(k)})^s, \end{aligned}$$

where  $j = 0, \dots, k$ . Using (2) of Lemma 5.3.1 we then have

$$\frac{\partial \eta_k}{\partial u_{k-j}} = (-1)^j (X^{(k)})^j \frac{\partial \eta_k}{\partial u_k} = (-1)^j \frac{\partial^{j+1} \xi}{\partial x^{j+1}},$$

hence

$$\eta_k(x, u^{(k)}) = \sum_{j=0}^k (-1)^j \xi^{(j+1)}(x) u_{k-j} + C_k(x), \quad (5.3.7)$$

and

$$X^{(k)} \eta_k(x, u^{(k)}) = (-1)^k \xi^{(k+2)}(x) u_0 + C'_k(x).$$

It follows from (2) of Lemma 5.3.1 and the previous expression that

$$\xi^{(k+2)}(x) = 0 \quad \text{and} \quad C'_k(x) = 0. \quad (5.3.8)$$

From Lemma 5.1.1 and (5.3.6) we have

$$\frac{\partial \eta_{k-j}}{\partial u_{k-j}} = (\xi'(x))^{j+1}, \quad j = 0, \dots, k-1,$$

hence

$$\eta_{k-j}(x, u^{(k)}) = (\xi'(x))^{j+1} u_{k-j} + B_{k-j}(x, u_{k-(j+1)}, \dots, u_0), \quad (5.3.9)$$

where  $j = 0, \dots, k-1$ . Furthermore, Corollary 5.3.2 gives the case  $j = k$ , that

is

$$\eta_0(x, u^{(k)}) = \xi'(x)^{k+1} u_0 + B_0(x). \quad (5.3.10)$$

In particular, (5.3.7) and (5.3.8) give

$$B_k(x, u_{k-1}, \dots, u_0) = \sum_{j=1}^k (-1)^j \xi^{(j+1)}(x) u_{k-j} + C_k.$$

For  $j = 1, \dots, k-1$ , we substitute the expressions for  $\eta_{k-j}$  and  $\eta_{k-j+1}$  which come from (5.3.9), into the contact condition

$$X^{(k)} \eta_{k-j} = \eta_{k-j+1} \xi'(x),$$

and apply  $\frac{\partial}{\partial u_{k-j}}$  to obtain

$$(j+1)(\xi')^j \xi'' + \frac{\partial B_{k-j}}{\partial u_{k-(j+1)}} = \frac{\partial B_{k-(j-1)}}{\partial u_{k-j}} \xi', \quad j = 1, \dots, k-1. \quad (5.3.11)$$

Since  $\frac{\partial B_k}{\partial u_{k-1}} = -\xi''$ , we can solve (5.3.11), that is

$$\frac{\partial B_{k-j}}{\partial u_{k-(j+1)}} = -\frac{(j+1)(j+2)}{2} (\xi')^j \xi'', \quad j = 1, \dots, k-1. \quad (5.3.12)$$

The case where  $j = k-1$  in (5.3.12) and (5.3.9) gives

$$\eta_1(x, u^{(k)}) = (\xi'(x))^k u_1 - \frac{k(k+1)}{2} (\xi')^{k-1} \xi''(x) u_0 + C_1(x). \quad (5.3.13)$$

Substituting (5.3.13) and (5.3.10) into the contact condition

$$X^{(k)} \eta_0 = \eta_1 \xi'(x),$$

and applying  $\frac{\partial}{\partial u_0}$ , we obtain

$$\frac{(k+1)(k+2)}{2} (\xi')^k \xi'' = 0, \quad (5.3.14)$$

which forces  $\xi'' = 0$ . It follows that  $\xi(x) = ax + b$  and, from (5.3.7) and (5.3.8) we have  $\eta_k(x, u^{(k)}) = au_k + C_k$ .  $\square$

**Theorem 5.3.4** *Let  $f$  be a 1-quasiconformal map on some open set  $\Omega \subseteq J^k(\mathbb{R}, \mathbb{R})$ . Then  $f$  is a composite of some or all of the four fundamental 1-quasiconformal maps, i.e., left translation, dilation, switch map and flip map.*

Proof. Using the switch and flip maps, we can assume that  $f$  satisfies (5.2.3). From Corollary 5.3.2 and Corollary 5.3.3 we have  $\eta_0(x, u^{(k)}) = a^{k+1}u_0 + B_0(x)$  where  $a > 0$ . The contact conditions become

$$\eta_{j+1} = \frac{X^{(k)}\eta_j}{X^{(k)}\xi} = \frac{1}{a}X^{(k)}\eta_j,$$

and so

$$\eta_{j+1}(x, u^{(k)}) = \frac{1}{a}X^{(k)}\eta_j(x, u^{(k)}) = a^{k-j}u_{j+1} + \frac{1}{a^{j+1}}B_0^{(j+1)}(x).$$

By composing with a left translation, we may assume  $f(0) = 0$ . It then follows from (5.3.7) and (5.3.8) that

$$B_0^{(k)}(x) = a^k C_k = 0$$

and so inductively  $B_0^{(j+1)}(x) = 0$ . Furthermore, by composing with the dilation  $\delta_{1/a}$ , we may assume that

$$\xi(x, u^{(k)}) = x, \quad \text{and} \quad \eta_j(x, u^{(k)}) = u_j, \quad j = 0, \dots, k.$$

Hence  $f$  is the identity. □

## 5.4 Contact automorphisms

**Lemma 5.4.1** *If  $f$  is a diffeomorphic contact map of  $J^k(\mathbb{R}, \mathbb{R})$ , where  $k \geq 2$ , then  $\xi(x, u^{(k)})$  depends on  $x$  only, and*

$$\frac{\partial \eta_0}{\partial u_1} = 0.$$

*Proof.* Since  $Df$  is nonsingular, we have

$$X^{(k)}\xi = \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial u_0}u_1 + \frac{\partial \xi}{\partial u_1}u_2 \neq 0,$$

forcing

$$\frac{\partial \xi}{\partial u_1} = 0 \quad \text{and} \quad \frac{\partial \xi}{\partial u_0} = 0.$$

Furthermore, from (5.1.1), we have

$$\frac{\partial}{\partial u_2}(X^{(k)}\eta_0) = \frac{\partial}{\partial u_2}(\eta_1 X^{(k)}\xi),$$

which implies that

$$\frac{\partial \eta_0}{\partial u_1} = 0.$$

□

In preparation for what is to come, we establish some formulae which result from the previous Lemma.

From Lemma 5.1.1 and Lemma 5.4.1,

$$\frac{\partial \eta_i}{\partial u_i} = \frac{\partial \eta_k}{\partial u_k} (\xi')^{k-i}, \quad i = 0, \dots, k-1,$$

and the case where  $i = 0$  gives

$$\frac{\partial \eta_k}{\partial u_k} = \frac{1}{(\xi')^k} \frac{\partial \eta_0}{\partial u_0} = A(x, u_0). \quad (5.4.1)$$

In the rest of this chapter we will abuse notation in the following way: for  $t < k$ , we use  $X^{(t)}$  to denote the truncation of  $X^{(k)}$  still acting on  $J^k(\mathbb{R}, \mathbb{R})$ , that is

$$X^{(t)} = \frac{\partial}{\partial x} + \sum_{j=0}^{t-1} u_{j+1} \frac{\partial}{\partial u_j}.$$

Since  $A$  depends on  $(x, u_0)$  only, we have

$$X^{(j)} \dots X^{(1)} A = \dots + u_j \frac{\partial A}{\partial u_0}, \quad (5.4.2)$$

and by induction

$$\frac{\partial}{\partial u_j} X^{(j)} X^{(j)} \dots X^{(1)} A = (j+1) X^{(1)} \frac{\partial A}{\partial u_0}, \quad j = 1, \dots, k. \quad (5.4.3)$$

Furthermore, since

$$\eta_s = \frac{1}{\xi'} X^{(s)} \eta_{s-1},$$

we have

$$\begin{aligned}
\frac{\partial \eta_s}{\partial u_{s-1}} &= \frac{1}{\xi'} \left( \frac{\partial}{\partial u_{s-1}} X^{(s)} \eta_{s-1} \right) \\
&= \frac{1}{\xi'} \left( X^{(s)} \frac{\partial \eta_{s-1}}{\partial u_{s-1}} + \frac{\partial \eta_{s-1}}{\partial u_{s-2}} \right) \\
&= \frac{1}{\xi'} X^{(s)} \frac{\partial \eta_{s-1}}{\partial u_{s-1}} + \frac{1}{(\xi')^2} X^{(s-1)} \frac{\partial \eta_{s-2}}{\partial u_{s-2}} + \frac{1}{(\xi')^2} \frac{\partial \eta_{s-2}}{\partial u_{s-3}} \\
&\quad \vdots \\
&= \sum_{t=1}^s \frac{1}{(\xi')^t} X^{(s-t+1)} \frac{\partial \eta_{s-t}}{\partial u_{s-t}} \\
&= \sum_{t=1}^s \frac{1}{(\xi')^t} X^{(s-t+1)} ((\xi')^{k-s+t} A).
\end{aligned} \tag{5.4.4}$$

## 5.5 Quasiconformal automorphisms

If  $f$  is a  $K$ -quasiconformal automorphism of  $J^k(\mathbb{R}, \mathbb{R})$ , where  $k \geq 2$ , then (5.2.2) and Lemma 5.2.1 give

$$\Delta_1(f)^2 - 4K^{P\gamma(k)} \leq \Delta_1(f)^2 - 4\Delta_2(f) \leq (2K^{2/Q} - \Delta_1(f))^2,$$

and it follows that  $\Delta_1(f) \leq K^{Q/2} + K^{4/Q-Q/2} = C(K)$ . If  $f$  satisfies (5.2.3), then

$$\Delta_1(f) = \frac{(\xi')^2 + (X^{(k)}\eta_k)^2 + A^2}{(\xi')^{\alpha(k)} A^{\beta(k)}} \leq C(K)$$

or equivalently

$$(X^{(k)}\eta_k)^2 \leq C(K)(\xi')^{\alpha(k)} A^{\beta(k)} - (\xi')^2 - A^2. \tag{5.5.1}$$

Since  $\xi$  depends on  $x$  only, and  $A$  depends on  $x$  and  $u_0$  only, it follows from (5.5.1) that  $(X^{(k)}\eta_k)^2$  is bounded with respect to the variables  $u_1, \dots, u_k$ .

The polynomial nature of  $f$  in the dependent variables  $u_1, \dots, u_k$ , together with (5.5.1), give enough information to determine explicitly the quasiconformal automorphisms of  $J^k(\mathbb{R}, \mathbb{R})$ , assuming suitable smoothness conditions. We introduce the canonical quasiconformal automorphisms. Let  $C^\infty(\mathbb{R})_0$  denote

the smooth real-valued functions  $\mu$  on  $\mathbb{R}$  such that  $\mu(0) = 0$  and  $\|\mu'\|_\infty < \infty$ .

Let

$$\mathbb{R}^+ \times C^\infty(\mathbb{R})_0$$

denote the group with multiplication

$$(\lambda_1, \mu_1)(\lambda_2, \mu_2) = (\lambda_1\lambda_2, \lambda_1\mu_2 + \mu_1).$$

A canonical quasiconformal automorphism  $f^{\lambda, \mu}$  is defined in coordinates by

$$\begin{aligned} \xi^{\lambda, \mu}(x, u^{(k)}) &= x, \\ \eta_k^{\lambda, \mu}(x, u^{(k)}) &= \lambda u_k + \mu(x), \\ \eta_{k-1}^{\lambda, \mu}(x, u^{(k)}) &= \lambda u_{k-1} + \int_0^x \mu(t_1) dt_1, \\ &\vdots \\ \eta_{k-i}^{\lambda, \mu}(x, u^{(k)}) &= \lambda u_{k-i} + \int_0^x \int_0^{t_i} \cdots \int_0^{t_2} \mu(t_1) dt_1 dt_2 \cdots dt_i. \end{aligned}$$

and the map  $(\lambda, \mu) \mapsto f^{\lambda, \mu}$  is an isomorphism. The canonical quasiconformal automorphisms are normalised, that is

$$f(0) = 0 \quad \text{and} \quad \frac{\partial \xi}{\partial x}(0) = 1.$$

Note that  $f^{\lambda, \mu}$  defines a quasiconformal map if  $\mu$  is Lipschitz. To avoid the extra complexity of counting derivatives, we state and prove the following theorem assuming  $\mu$  is smooth.

**Theorem 5.5.1** *Every smooth normalised quasiconformal automorphism of  $J^k(\mathbb{R}, \mathbb{R})$ , where  $k \geq 2$ , is canonical. Moreover, every quasiconformal automorphism is a composition*

$$h(\tau_g, \delta_s, \sigma, \delta_{-1}) \circ f^{\lambda, \mu},$$

where  $h(\tau_g, \delta_s, \sigma, \delta_{-1})$  is a composition of  $\tau_g$ ,  $\delta_s$ ,  $\sigma$  and  $\delta_{-1}$ .

The proof proceeds by first showing that

$$\eta_k(x, u^{(k)}) = \sum_{i=0}^k (-1)^i q^{(i)}(x) u_{k-i} + B_{k+1}(x),$$

where  $q(x)$  is a polynomial in the variable  $x$  with  $\deg(q) \leq k$ . The cases where  $k = 2$  and  $3$  are established first, as the general argument will require the assumption that  $k \geq 4$ .

**CASE** ( $k = 2$ ). From (5.4.1),  $\eta_2 = Au_2 + B_1(x, u_1, u_0)$ , hence

$$X^{(2)}\eta_2 = (X^{(1)}A)u_2 + X^{(2)}B_1 = \left( X^{(1)}A + \frac{\partial B_1}{\partial u_1} \right) u_2 + X^{(1)}B_1,$$

and (5.5.1) implies that

$$X^{(1)}A + \frac{\partial B_1}{\partial u_1} = 0 \quad \text{and} \quad X^{(2)}\eta_2 = X^{(1)}B_1.$$

It follows that

$$B_1 = - \left( \frac{\partial A}{\partial x} u_1 + \frac{\partial A}{\partial u_0} \frac{u_1^2}{2} \right) + B_2(x, u_0),$$

and

$$\begin{aligned} X^{(2)}\eta_2 &= - \left( \frac{\partial^2 A}{\partial x^2} + u_1 \frac{\partial^2 A}{\partial x \partial u_0} \right) u_1 - \left( \frac{\partial^2 A}{\partial x \partial u_0} + u_1 \frac{\partial^2 A}{\partial u_0^2} \right) \frac{u_1^2}{2} + \frac{\partial B_2}{\partial x} + u_1 \frac{\partial B_2}{\partial u_0} \\ &= \left( -\frac{\partial^2 A}{\partial x^2} + \frac{\partial B_2}{\partial u_0} \right) u_1 - \frac{3}{2} \frac{\partial^2 A}{\partial x \partial u_0} u_1^2 - \frac{\partial^2 A}{\partial u_0^2} \frac{u_1^3}{2} + \frac{\partial B_2}{\partial x}. \end{aligned}$$

By (5.5.1) we have

$$\frac{\partial^2 A}{\partial u_0^2} = 0, \quad \frac{\partial^2 A}{\partial x \partial u_0} = 0, \quad \frac{\partial B_2}{\partial u_0} = \frac{\partial^2 A}{\partial x^2} \quad \text{and} \quad X^{(2)}\eta_2 = \frac{\partial B_2}{\partial x}.$$

It follows that

$$A = \rho u_0 + q(x), \quad B_2 = q''(x)u_0 + B_3(x) \quad \text{and} \quad X^{(2)}\eta_2 = q'''(x)u_0 + B_3'(x).$$

By Lemma (5.2.1),  $\rho = 0$ , hence the right hand side of (5.5.1) depends on  $x$  only, which implies that  $q'''(x) = 0$ . Moreover

$$\eta_2 = q(x)u_2 - q'(x)u_1 + q''(x)u_0 + B_3(x). \quad (5.5.2)$$

**CASE** ( $k = 3$ ). From (5.4.1),  $\eta_3 = Au_3 + B_1(x, u_2, u_1, u_0)$ , hence

$$X^{(3)}\eta_3 = (X^{(1)}A)u_3 + X^{(3)}B_1 = \left( X^{(1)}A + \frac{\partial B_1}{\partial u_2} \right) u_3 + X^{(2)}B_1,$$

and (5.5.1) implies that

$$X^{(1)}A + \frac{\partial B_1}{\partial u_2} = 0 \quad \text{and} \quad X^{(3)}\eta_3 = X^{(2)}B_1.$$

It follows that

$$B_1 = -(X^{(1)}A)u_2 + B_2(x, u_1, u_0)$$

and

$$\begin{aligned} X^{(3)}\eta_3 &= \left( -X^{(2)}X^{(1)}A + \frac{\partial B_2}{\partial u_1} \right) u_2 + X^{(1)}B_2 \\ &= \left( -u_2 \frac{\partial}{\partial u_1} X^{(1)}A - X^{(1)}X^{(1)}A + \frac{\partial B_2}{\partial u_1} \right) u_2 + X^{(1)}B_2 \\ &= - \left( \frac{\partial}{\partial u_1} X^{(1)}A \right) u_2^2 + \left( -X^{(1)}X^{(1)}A + \frac{\partial B_2}{\partial u_1} \right) u_2 + X^{(1)}B_2. \end{aligned}$$

By (5.5.1),

$$\frac{\partial}{\partial u_1} X^{(1)}A = 0, \quad -X^{(1)}X^{(1)}A + \frac{\partial B_2}{\partial u_1} = 0 \quad \text{and} \quad X^{(3)}\eta_3 = X^{(1)}B_2.$$

Hence

$$A = q(x), \quad B_2 = q''(x)u_1 + B_3(x, u_0)$$

and

$$X^{(3)}\eta_3 = \left( q'''(x) + \frac{\partial B_3}{\partial u_0} \right) u_1 + \frac{\partial B_3}{\partial x}.$$

By (5.5.1),

$$B_3 = -q'''(x)u_0 + B_4(x) \quad \text{and} \quad X^{(3)}\eta_3 = q''''(x)u_0 + B_4'(x),$$

where, again by (5.5.1),  $q''''(x) = 0$ . Hence

$$\eta_3 = q(x)u_3 - q'(x)u_2 + q''(x)u_1 - q'''(x)u_0 + B_4(x). \quad (5.5.3)$$

To obtain (5.5.2) and (5.5.3) for general  $k$ , we first prove the following Lemma.

**Lemma 5.5.2** *If  $k \geq 4$  and  $f : J^k(\mathbb{R}, \mathbb{R}) \rightarrow J^k(\mathbb{R}, \mathbb{R})$  is a quasiconformal automorphism, then*

$$\eta_k = Au_k + \sum_{i=1}^{j-1} (-1)^i (X^{(i)} \cdots X^{(1)}A)u_{k-i} + B_j(x, u_{k-j}, \dots, u_0),$$

where  $k \geq 2j - 1$ .

*Proof.* From (5.4.1),

$$\eta_k = Au_k + B_1(x, u_{k-1}, \dots, u_0), \quad k \geq 2,$$

hence

$$X^{(k)}\eta_k = (X^{(1)}A)u_k + u_k \frac{\partial B_1}{\partial u_{k-1}} + X^{(k-1)}B_1.$$

By (5.5.1),

$$X^{(1)}A + \frac{\partial B_1}{\partial u_{k-1}} = 0, \quad k \geq 2,$$

and

$$B_1 = -(X^{(1)}A)u_{k-1} + B_2(x, u_{k-2}, \dots, u_0), \quad k \geq 3.$$

It follows that

$$\eta_k = Au_k - (X^{(1)}A)u_{k-1} + B_2(x, u_{k-2}, \dots, u_0), \quad k \geq 3,$$

and

$$X^{(k)}\eta_k = X^{(k-1)}B_1 = -(X^{(2)}X^{(1)}A)u_{k-1} + u_{k-1} \frac{\partial B_2}{\partial u_{k-2}} + X^{(k-2)}B_2.$$

By (5.5.1),

$$-(X^{(2)}X^{(1)}A) + \frac{\partial B_2}{\partial u_{k-2}} = 0 \quad \text{and} \quad X^{(k)}\eta_k = X^{(k-2)}B_2, \quad k \geq 3.$$

It follows that

$$B_2 = (X^{(2)}X^{(1)}A)u_{k-2} + B_3(x, u_{k-3}, \dots, u_0), \quad k \geq 5,$$

and

$$\eta_k = Au_k - (X^{(1)}A)u_{k-1} + (X^{(2)}X^{(1)}A)u_{k-2} + B_3(x, u_{k-3}, \dots, u_0), \quad k \geq 5.$$

Iterating this procedure gives the desired result.  $\square$

Assume that  $j$  is maximal, i.e.,  $j = [(k+1)/2]$ , then

$$\begin{aligned} X^{(k)}\eta_k &= (-1)^{j-1}(X^{(j)}X^{(j-1)} \dots X^{(1)}A)u_{k-j+1} + X^{(k-j+1)}B_j(x, u_{k-j}, \dots, u_0) \\ &= \left( (-1)^{j-1}(X^{(j)}X^{(j-1)} \dots X^{(1)}A) + \frac{\partial B_j}{\partial u_{k-j}} \right) u_{k-j+1} + X^{(k-j)}B_j, \end{aligned}$$

and by quasiconformality,

$$\frac{\partial B_j}{\partial u_{k-j}} = (-1)^j (X^{(j)} X^{(j-1)} \dots X^{(1)} A), \quad (5.5.4)$$

where the right hand side depends on  $u_{k-j}$ .

**Lemma 5.5.3** *If  $k$  is even, say  $k = 2\alpha$ , with  $\alpha \geq 2$ , then  $A = q(x)$ , where  $q(x)$  is a polynomial in the variable  $x$  with  $\deg(q) \leq k$ , and*

$$\eta_k = \sum_{i=0}^k (-1)^i q^{(i)}(x) u_{k-i} + B_{k+1}(x). \quad (5.5.5)$$

*Proof.* Letting  $j = \alpha$  in (5.5.4) and using

$$X^{(\alpha)} = u_\alpha \frac{\partial}{\partial u_{\alpha-1}} + X^{(\alpha-1)},$$

we have

$$\begin{aligned} \frac{\partial B_\alpha}{\partial u_\alpha} &= (-1)^\alpha (X^{(\alpha)} X^{(\alpha-1)} \dots X^{(1)} A) \\ &= (-1)^\alpha u_\alpha \left( \frac{\partial}{\partial u_{\alpha-1}} X^{(\alpha-1)} \dots X^{(1)} A \right) + (-1)^\alpha (X^{(\alpha-1)} X^{(\alpha-1)} \dots X^{(1)} A) \\ &= (-1)^\alpha u_\alpha \left( \frac{\partial A}{\partial u_0} \right) + (-1)^\alpha (X^{(\alpha-1)} X^{(\alpha-1)} \dots X^{(1)} A), \end{aligned}$$

where the last line follows from (5.4.2).

Since  $A$  is a function of  $(x, u_0)$ , we obtain

$$\begin{aligned} B_\alpha &= (-1)^\alpha \left( \frac{\partial A}{\partial u_0} \right) \frac{u_\alpha^2}{2} + (-1)^\alpha (X^{(\alpha-1)} X^{(\alpha-1)} \dots X^{(1)} A) u_\alpha \\ &\quad + B_{\alpha+1}(x, u_{\alpha-1}, \dots, u_0), \end{aligned} \quad (5.5.6)$$

and it follows that

$$\begin{aligned} X^{(k)} \eta_k &= X^{(\alpha)} B_\alpha = (-1)^\alpha \left( X^{(\alpha)} \frac{\partial A}{\partial u_0} \right) \frac{u_\alpha^2}{2} \\ &\quad + (-1)^\alpha (X^{(\alpha)} X^{(\alpha-1)} \dots X^{(1)} A) u_\alpha \\ &\quad + u_\alpha \frac{\partial B_{\alpha+1}}{\partial u_{\alpha-1}} + X^{(\alpha-1)} B_{\alpha+1}. \end{aligned}$$

By (5.4.2) and (5.4.3),

$$\begin{aligned}
X^{(k)}\eta_k &= X^{(\alpha)}B_\alpha = (-1)^\alpha \left( X^{(\alpha)} \frac{\partial A}{\partial u_0} \right) \frac{u_\alpha^2}{2} \\
&\quad + (-1)^\alpha \left( (X^{(\alpha-1)} X^{(\alpha-1)} \dots X^{(1)} A) + u_\alpha \alpha X^{(1)} \frac{\partial A}{\partial u_0} \right) u_\alpha \\
&\quad + u_\alpha \frac{\partial B_{\alpha+1}}{\partial u_{\alpha-1}} + X^{(\alpha-1)} B_{\alpha+1} \\
&= (-1)^\alpha \left( \alpha + \frac{1}{2} \right) \left( X^{(1)} \frac{\partial A}{\partial u_0} \right) u_\alpha^2 \\
&\quad + \left( (-1)^\alpha (X^{(\alpha-1)} X^{(\alpha-1)} X^{(\alpha-1)} \dots X^{(1)} A) + \frac{\partial B_{\alpha+1}}{\partial u_{\alpha-1}} \right) u_\alpha \\
&\quad + X^{(\alpha-1)} B_{\alpha+1}.
\end{aligned}$$

By (5.5.1),

$$X^{(1)} \frac{\partial A}{\partial u_0} = 0 \quad (5.5.7)$$

$$\frac{\partial B_{\alpha+1}}{\partial u_{\alpha-1}} = (-1)^{\alpha+1} (X^{(\alpha-1)} X^{(\alpha-1)} X^{(\alpha-1)} \dots X^{(1)} A) \quad (5.5.8)$$

$$X^{(k)}\eta_k = X^{(\alpha-1)} B_{\alpha+1}. \quad (5.5.9)$$

From (5.5.7),

$$\frac{\partial^2 A}{\partial u_0^2} = 0 \quad \text{and} \quad \frac{\partial^2 A}{\partial x \partial u_0} = 0,$$

implying that

$$A = \rho u_0 + q(x),$$

and by Lemma (5.2.1),  $\rho = 0$ . Furthermore

$$X^{(i)} \dots X^{(1)} A = q^{(i)}(x),$$

and (5.5.6) gives

$$B_\alpha = (-1)^\alpha q^{(\alpha)}(x) u_\alpha + B_{\alpha+1}(x, u_{\alpha-1}, \dots, u_0).$$

It also follows that (5.5.8) becomes

$$\frac{\partial B_{\alpha+1}}{\partial u_{\alpha-1}} = (-1)^{\alpha+1} q^{(\alpha+1)}(x),$$

giving

$$B_{\alpha+1} = (-1)^{\alpha+1} q^{(\alpha+1)}(x) u_{\alpha-1} + B_{\alpha+2}(x, u_{\alpha-2}, \dots, u_0).$$

By (5.5.9),

$$\begin{aligned} X^{(k)} \eta_k &= X^{(\alpha-1)} B_{\alpha+1} \\ &= (-1)^{\alpha+1} q^{(\alpha+2)}(x) u_{\alpha-1} + X^{(\alpha-1)} B_{\alpha+2}(x, u_{\alpha-2}, \dots, u_0) \\ &= \left( (-1)^{\alpha+1} q^{(\alpha+2)}(x) + \frac{\partial B_{\alpha+2}}{\partial u_{\alpha-2}} \right) u_{\alpha-1} + X^{(\alpha-2)} B_{\alpha+2}(x, u_{\alpha-2}, \dots, u_0), \end{aligned}$$

and by (5.5.1),

$$B_{\alpha+2} = (-1)^{\alpha+2} q^{(\alpha+2)}(x) u_{\alpha-2} + B_{\alpha+3}(x, u_{\alpha-3}, \dots, u_0).$$

Iterating this procedure, keeping in mind that the right hand side of (5.5.1) depends on  $x$  only, gives (5.5.5) and  $q^{(2\alpha+1)}(x) = 0$ .  $\square$

**Lemma 5.5.4** *If  $k$  is odd, say  $k = 2\beta + 1$ , with  $\beta > 1$ , then  $A = q(x)$ , where  $q(x)$  is a polynomial in the variable  $x$  with  $\deg(q) \leq k$ , and*

$$\eta_k = \sum_{i=0}^k (-1)^i q^{(i)}(x) u_{k-i} + B_{k+1}(x). \quad (5.5.10)$$

*Proof.* Letting  $j = \beta + 1$  in (5.5.4) and using

$$X^{(\beta+1)} = X^{(\beta)} + u_{\beta+1} \frac{\partial}{\partial u_{\beta}} \quad \text{and} \quad X^{(\beta)} = X^{(\beta-1)} + u_{\beta} \frac{\partial}{\partial u_{\beta-1}},$$

we have

$$\begin{aligned} \frac{\partial B_{\beta+1}}{\partial u_{\beta}} &= (-1)^{\beta+1} (X^{(\beta+1)} X^{(\beta)} \dots X^{(1)} A) \\ &= (-1)^{\beta+1} (X^{(\beta)} X^{(\beta)} \dots X^{(1)} A) + (-1)^{\beta+1} u_{\beta+1} \left( \frac{\partial}{\partial u_{\beta}} X^{(\beta)} \dots X^{(1)} A \right). \end{aligned}$$

For  $\beta > 1$ , we have  $X^{(\beta)} = X^{(\beta-1)} + u_{\beta} \frac{\partial}{\partial u_{\beta-1}}$ , giving

$$\begin{aligned} X^{(\beta)} X^{(\beta)} \dots X^{(1)} A &= X^{(\beta-1)} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A \\ &\quad + u_{\beta} \left( X^{(\beta-1)} \frac{\partial}{\partial u_{\beta-1}} X^{(\beta-1)} \dots X^{(1)} A \right) \\ &\quad + u_{\beta} \left( \frac{\partial}{\partial u_{\beta-1}} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A \right) \\ &\quad + u_{\beta}^2 \left( \frac{\partial}{\partial u_{\beta-1}} \frac{\partial}{\partial u_{\beta-1}} X^{(\beta-1)} \dots X^{(1)} A \right). \end{aligned}$$

By (5.4.2),  $\frac{\partial}{\partial u_\beta} X^{(\beta)} \dots X^{(1)} A = \frac{\partial A}{\partial u_0}$ , giving

$$\begin{aligned} X^{(\beta)} X^{(\beta)} \dots X^{(1)} A &= X^{(\beta-1)} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A \\ &\quad + u_\beta \left( X^{(\beta-1)} \frac{\partial A}{\partial u_0} \right) \\ &\quad + u_\beta \left( \frac{\partial}{\partial u_{\beta-1}} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A \right) \\ &\quad + u_\beta^2 \left( \frac{\partial}{\partial u_{\beta-1}} \frac{\partial A}{\partial u_0} \right). \end{aligned}$$

For  $\beta > 1$ , we have  $\frac{\partial}{\partial u_{\beta-1}} \frac{\partial A}{\partial u_0} = 0$ , and by (5.4.3),

$$\frac{\partial}{\partial u_{\beta-1}} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A = \beta X^{(1)} \frac{\partial A}{\partial u_0},$$

hence

$$X^{(\beta)} X^{(\beta)} \dots X^{(1)} A = X^{(\beta-1)} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A + u_\beta (\beta + 1) \left( X^{(1)} \frac{\partial A}{\partial u_0} \right).$$

It follows that

$$\begin{aligned} \frac{\partial B_{\beta+1}}{\partial u_\beta} &= (-1)^{\beta+1} \left( X^{(\beta-1)} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A + u_\beta (\beta + 1) \left( X^{(1)} \frac{\partial A}{\partial u_0} \right) \right) \\ &\quad + (-1)^{\beta+1} u_{\beta+1} \left( \frac{\partial A}{\partial u_0} \right), \end{aligned}$$

hence

$$\begin{aligned} B_{\beta+1} &= (-1)^{\beta+1} (X^{(\beta-1)} X^{(\beta-1)} X^{(\beta-1)} \dots X^{(1)} A) u_\beta \\ &\quad + (-1)^{\beta+1} (\beta + 1) \left( X^{(1)} \frac{\partial A}{\partial u_0} \right) \frac{u_\beta^2}{2} \\ &\quad + (-1)^{\beta+1} \left( \frac{\partial A}{\partial u_0} \right) u_{\beta+1} u_\beta \\ &\quad + B_{\beta+2}(x, u_{\beta-1}, \dots, u_0), \end{aligned}$$

and

$$\begin{aligned}
X^{(k)}\eta_k &= X^{(\beta)}B_{\beta+1} \\
&= (-1)^{\beta+1}(X^{(\beta)}X^{(\beta-1)}X^{(\beta-1)}X^{(\beta-1)}\dots X^{(1)}A)u_\beta \\
&\quad + (-1)^{\beta+1}(\beta+1)\left(X^{(2)}X^{(1)}\frac{\partial A}{\partial u_0}\right)\frac{u_\beta^2}{2} \\
&\quad + (-1)^{\beta+1}\left(X^{(1)}\frac{\partial A}{\partial u_0}\right)u_{\beta+1}u_\beta \\
&\quad + \frac{\partial B_{\beta+2}}{\partial u_{\beta-1}}u_\beta + X^{(\beta-1)}B_{\beta+2}.
\end{aligned}$$

By (5.5.1),

$$X^{(1)}\frac{\partial A}{\partial u_0} = 0 \tag{5.5.11}$$

$$\frac{\partial B_{\beta+2}}{\partial u_{\beta-1}} = (-1)^{\beta+2}(X^{(\beta)}X^{(\beta-1)}X^{(\beta-1)}X^{(\beta-1)}\dots X^{(1)}A) \tag{5.5.12}$$

$$X^{(k)}\eta_k = X^{(\beta)}B_{\beta+1} = X^{(\beta-1)}B_{\beta+2}. \tag{5.5.13}$$

From (5.5.11),

$$\frac{\partial^2 A}{\partial u_0^2} = 0 \quad \text{and} \quad \frac{\partial^2 A}{\partial x \partial u_0} = 0$$

implying that

$$A = \rho u_0 + q(x)$$

and by Lemma (5.2.1),  $\rho = 0$ . Furthermore

$$X^{(i)}\dots X^{(1)}A = q^{(i)}(x)$$

and

$$B_{\beta+1} = (-1)^{\beta+1}q^{(\beta+1)}(x)u_\beta + B_{\beta+2}(x, u_{\beta-1}, \dots, u_0).$$

Furthermore, (5.5.12) becomes

$$\frac{\partial B_{\beta+2}}{\partial u_{\beta-1}} = (-1)^{\beta+2}q^{(\beta+1)}(x),$$

giving

$$B_{\beta+2} = (-1)^{\beta+2}q^{(\beta+1)}(x)u_{\beta-1} + B_{\beta+3}(x, u_{\beta-2}, \dots, u_0).$$

By (5.5.13),

$$X^{(k)}\eta_k = X^{(\beta-1)}B_{\beta+2} = (-1)^{\beta+2}q^{(\beta+2)}(x)u_{\beta-1} + X^{(\beta-1)}B_{\beta+3},$$

and by (5.5.1),

$$B_{\beta+3} = (-1)^{\beta+3}q^{(\beta+2)}(x)u_{\beta-2} + B_{\beta+4}(x, u_{\beta-3}, \dots, u_0).$$

Iterating the procedure, keeping in mind that the right hand side of (5.5.1) depends on  $x$  only, gives (5.5.10) and  $q^{(2\beta+2)}(x) = 0$ .  $\square$

**Lemma 5.5.5** *The polynomial  $q(x)$  is constant, giving*

$$\xi(x) = ax + b \quad \text{and} \quad \eta_k = qu_k + \mu(x).$$

Proof. By (5.5.5),

$$\frac{\partial \eta_k}{\partial u_{k-1}} = -q'(x)$$

and by (5.4.4),

$$\begin{aligned} \frac{\partial \eta_k}{\partial u_{k-1}} &= \sum_{t=1}^k \frac{1}{(\xi')^t} X^{(k-t+1)} ((\xi')^t q) \\ &= \sum_{t=1}^k \frac{1}{(\xi')^t} (t(\xi')^{t-1} \xi'' q + (\xi')^t q') \\ &= \sum_{t=1}^k \left( t \frac{\xi''}{\xi'} q + q' \right) \\ &= \frac{k(k+1)}{2} \frac{\xi''}{\xi'} q + kq'. \end{aligned}$$

Equating these two expressions gives

$$\frac{k}{2} \frac{\xi''}{\xi'} q(x) + q'(x) = 0$$

and it follows that  $|q| = C|\xi'|^{-k/2}$ , where  $C > 0$ . From Lemma (5.2.1),

$$|q|^{2+4/k} \leq C^{4/k} K^{2P}, \quad (5.5.14)$$

which forces  $q$  to be constant and consequently  $\xi'$  is also constant.  $\square$

We can now prove Theorem (5.5.1). Using suitable 1-quasiconformal maps, we normalise  $f$  so that

$$\xi(x) = x \quad \text{and} \quad \eta_k = \lambda u_k + \mu(x)$$

where  $\mu(0) = 0$  and  $(\mu'(x))^2$  is bounded. From (5.1.1) we have

$$X^{(j+1)}\eta_j = \eta_{j+1}, \quad j = 0, \dots, k-1. \quad (5.5.15)$$

In particular,

$$X^{(k)}\eta_{k-1} = \lambda u_k + \mu(x)$$

implying that

$$\frac{\partial \eta_{k-1}}{\partial u_{k-1}} = \lambda, \quad \frac{\partial \eta_{k-1}}{\partial x} = \mu(x), \quad \frac{\partial \eta_{k-1}}{\partial u_\ell} = 0 \quad \text{and} \quad \ell = 0, \dots, k-2.$$

It follows that

$$\eta_{k-1} = \lambda u_{k-1} + \int_0^x \mu(t_1) dt_1.$$

From (5.5.15),

$$X^{(k-1)}\eta_{k-2} = \eta_{k-1},$$

giving

$$\eta_{k-2} = \lambda u_{k-2} + \int_0^x \int_0^{t_2} \mu(t_1) dt_1 dt_2,$$

and so on.

# Chapter 6

## The Heisenberg Group

### 6.1 Preliminaries

In contrast to the jet spaces  $J^k(\mathbb{R}, \mathbb{R})$ , where  $k \geq 2$ , the contact and quasiconformal theory of the Heisenberg group  $J^1(\mathbb{R}, \mathbb{R})$  is significantly different. This is a direct consequence of Bäcklund's theorem and apparent from the Liouville theorem for  $J^1(\mathbb{R}, \mathbb{R})$ , due to A. Korányi and H. M. Reimann in [15].

From (4.2.1) we have

$$(x, u_1, u) \odot (y, v_1, v) = (x + y, u_1 + v_1, u + v + u_1 y)$$

giving

$$\left\{ X = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u} \right\}$$

as a basis of the left-invariant fields with horizontal space

$$\text{span} \left\{ X, \frac{\partial}{\partial u_1} \right\} = \text{Ker}(\omega), \quad \omega = du - u_1 dx.$$

If  $f = (\xi(x, u_1, u), \eta_1(x, u_1, u), \eta_0(x, u_1, u))$  is a contact map on  $J^1(\mathbb{R}, \mathbb{R})$  then the differential is given by

$$Df = \begin{pmatrix} X\xi & \frac{\partial \xi}{\partial u_1} & \frac{\partial \xi}{\partial u} \\ X\eta_1 & \frac{\partial \eta_1}{\partial u_1} & \frac{\partial \eta_1}{\partial u} \\ 0 & 0 & \omega(f_* \frac{\partial}{\partial u}) \end{pmatrix} = \begin{pmatrix} M & \star \\ 0 & \omega(f_* \frac{\partial}{\partial u}) \end{pmatrix}.$$

Using Cartan's formula:  $\omega([X, Y]) = X\omega(Y) - Y\omega(X) - 2d\omega(X, Y)$ , and the contact conditions, we have

$$\det M = \omega\left(f_* \frac{\partial}{\partial u}\right), \quad (6.1.1)$$

hence

$$\det Df = (\det M)^2.$$

The eigenvalues of  $M^{\text{tr}}M$  are

$$\frac{1}{2} \left( \Lambda(f) \pm \sqrt{(\Lambda(f))^2 - 4(\det M)^2} \right)$$

where

$$\Lambda(f) = (X\xi)^2 + \left(\frac{\partial\xi}{\partial u_1}\right)^2 + (X\eta_1)^2 + \left(\frac{\partial\eta_1}{\partial u_1}\right)^2.$$

The homogeneous dimension is 4, hence by Pansu's definition,  $f$  is  $K$ -quasiconformal if

$$\Delta(f) + \sqrt{(\Delta(f))^2 - 4} \leq 2K^{1/2},$$

where

$$\Delta(f) = \frac{(X\xi)^2 + \left(\frac{\partial\xi}{\partial u_1}\right)^2 + (X\eta_1)^2 + \left(\frac{\partial\eta_1}{\partial u_1}\right)^2}{|\det M|}.$$

The switch and flip maps are defined on  $J^1(\mathbb{R}, \mathbb{R})$  by

$$\sigma(x, u^{(1)}) = (-x, u_1, -u_0)$$

and

$$\delta_{-1}(x, u^{(1)}) = (-x, -u_1, u_0).$$

The switch map commutes with the flip map giving

$$\delta_{-1} \circ \sigma(x, u^{(1)}) = \sigma \circ \delta_{-1}(x, u^{(1)}) = (x, -u_1, -u_0),$$

moreover, the switch map, the flip map and their compositions are 1-quasiconformal since

$$\Delta(\sigma) = \Delta(\delta_{-1}) = \Delta(\sigma \circ \delta_{-1}) = \Delta(\delta_{-1} \circ \sigma) = 2.$$

It follows that every quasiconformal map can, if necessary, be “switched” or “flipped and switched”, to a quasiconformal map such that  $\det M > 0$ .

## 6.2 1-Quasiconformal maps

The 1-quasiconformal maps of  $J^k(\mathbb{R}, \mathbb{R})$ , where  $k \geq 2$ , are point transformations, however this is not the case for  $J^1(\mathbb{R}, \mathbb{R})$ . In [15], it is shown that with some regularity assumptions, the 1-quasiconformal maps of the Heisenberg group, satisfying a condition equivalent to  $\det M > 0$ , are the actions of group elements in  $SU(1, 2)$ . Later in [5], it was shown that 1-quasiconformal maps must be smooth, hence the 1-quasiconformal maps of the Heisenberg group, satisfying a condition equivalent to  $\det M > 0$ , are exactly the actions of group elements in  $SU(1, 2)$ .

The following discussion reviews the results on 1-quasiconformal maps of the Heisenberg group in [15]. The jet space expressions for these results give the 1-quasiconformal maps of  $J^1(\mathbb{R}, \mathbb{R})$ .

The group  $SU(1, 2)$ , is the subgroup of  $SL(3, \mathbb{C})$  consisting of elements  $g$  which satisfy  $g^*Jg = J$ , where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The condition  $g^*Jg = J$  is equivalent to  $g$  leaving invariant the form

$$\langle y, y \rangle = |y_0|^2 - |y_1|^2 - |y_2|^2.$$

Let

$$C = \{y \in \mathbb{C}^3 \mid \langle y, y \rangle > 0\},$$

$$B = \{w \in \mathbb{C}^2 \mid |w| < 1\},$$

$$S = \partial B,$$

and define  $\pi : C^c - \{0\} \rightarrow B^c$  by  $\pi(y_0, y_1, y_2) = (y_1/y_0, y_2/y_0)$ . If  $g = (g_{ij})$ , then the projected action of  $SU(1, 2)$  on  $B$  or  $S$  is given by

$$\rho(g, w) = \left( \frac{g_{10} + g_{11}w_1 + g_{12}w_2}{g_{00} + g_{01}w_1 + g_{02}w_2}, \frac{g_{20} + g_{21}w_1 + g_{22}w_2}{g_{00} + g_{01}w_1 + g_{02}w_2} \right)$$

and  $PSU(1, 2) = SU(1, 2)/\{-\text{id}, \text{id}\}$ .

An element  $g \in SU(1, 2)$  determines an element of the isotropy subgroup  $PSU(1, 2)_{\{-e_2\}}$ , where  $-e_2 = (0, -1) \in S$ , if and only if it takes the form

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} \cosh(s) & 0 & \sinh(s) \\ 0 & 1 & 0 \\ \sinh(s) & 0 & \cosh(s) \end{pmatrix} \begin{pmatrix} 1 + \frac{-it+|z|^2}{2} & i\bar{z} & \frac{-it+|z|^2}{2} \\ -iz & 1 & -iz \\ \frac{it-|z|^2}{2} & -i\bar{z} & 1 + \frac{it-|z|^2}{2} \end{pmatrix},$$

and we write  $g = m(e^{i\theta})a(s)n(z, t)$ . Note that  $m(e^{i\theta_1})m(e^{i\theta_2}) = m(e^{i\theta_1}e^{i\theta_2})$ ,  $a(s_1)a(s_2) = a(s_1 + s_2)$  and

$$n(z, t)n(z', t') = n(z + z', t + t' + 2\text{Im}(z\bar{z}')).$$

The space of pairs  $[z, t] \in \mathbb{C} \times \mathbb{R}$  with group multiplication

$$[z, t][z', t'] = z + z', t + t' + 2\text{Im}(z\bar{z}') \quad (6.2.1)$$

is a model of the Heisenberg group which we denote  $\mathbb{H}^1$ . The map

$$(x, u^{(1)}) = (x, u_1, u) \mapsto [x + iu_1, 4u - 2xu_1] \quad (6.2.2)$$

defines an isomorphism  $J^1(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{H}^1$ .

The Cayley transform

$$(z_1, z_2) = \mathcal{C}(w_1, w_2) = \left( \frac{iw_1}{1+w_2}, i\frac{1-w_2}{1+w_2} \right)$$

maps

$$B = \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1|^2 + |w_2|^2 < 1\}$$

biholomorphically onto

$$D = \{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Im } z_2 > |z_1|^2\},$$

with

$$\mathcal{C}^{-1}(z_1, z_2) = \left( \frac{-2iz_1}{1-iz_2}, \frac{1+iz_2}{1-iz_2} \right).$$

The generalized stereographic projection is the map  $\Pi : S \setminus \{-e_2\} \rightarrow \mathbb{H}^1$  given by  $\pi \circ \mathcal{C}$  where  $\pi(z_1, z_2) = [z_1, \text{Re } z_2]$ .

$$\Pi(w_1, w_2) = \left[ \frac{iw_1}{1+w_2}, \frac{2\text{Im } w_2}{1+2\text{Re } w_2 + |w_2|^2} \right]$$

$$\Pi^{-1}[z_1, t] = \left( \frac{-2iz_1}{1 + |z_1|^2 - it}, \frac{1 - |z_1|^2 + it}{1 + |z_1|^2 - it} \right).$$

If  $w \in S \setminus \{-e_2\}$  and  $\Pi(w) = [z_1, t]$ , then

- (1)  $\Pi(\rho(m(e^{i\theta}), w)) = [e^{-3i\theta} z_1, t]$
- (2)  $\Pi(\rho(a(s), w)) = [e^{-s} z_1, e^{-2s} t]$
- (3)  $\Pi(\rho(n(z_0, t_0), w)) = [z_0, t_0][z_1, t].$

The inversion  $j : S \rightarrow S$  given by  $j(w_1, w_2) = (-w_1, -w_2)$  induces the Heisenberg inversion

$$\Pi(j(w)) = \left[ \frac{z}{it - |z|^2}, \frac{-t}{t^2 + |z|^4} \right].$$

As stated earlier, due to [15] and [5], the group actions generated by (1), (2) and (3), and the Heisenberg inversion are exactly the 1-quasiconformal maps of  $\mathbb{H}^1$ .

Using (6.2.2) we obtain the corresponding action of  $SU(1, 2)$  on  $J^1(\mathbb{R}, \mathbb{R})$ . In particular, (2) and (3) are simply dilation and left translation, whereas (1) becomes

$$\begin{aligned} \xi(x, u^{(1)}) &= x \cos(3\theta) - u_1 \sin(3\theta) \\ \eta_1(x, u^{(1)}) &= u_1 \cos(3\theta) + x \sin(3\theta) \\ \eta(x, u^{(1)}) &= u - \frac{1}{2}xu_1 + \frac{1}{2}(x \cos(3\theta) - u_1 \sin(3\theta))(u_1 \cos(3\theta) + x \sin(3\theta)). \end{aligned} \tag{6.2.3}$$

Note that when  $\sin(3\theta) \neq 0$ , (6.2.3) is not a point transformation, and the case when  $\cos(3\theta) = 0$  and  $\sin(3\theta) = -1$  is the Legendre transformation. The Heisenberg inversion becomes

$$\begin{aligned} \xi(x, u^{(1)}) &= -\frac{x^3 + 3xu_1^2 - 4u_1u}{16u^2 - 16uxu_1 + 6x^2u_1^2 + x^4 + u_1^4} \\ \eta_1(x, u^{(1)}) &= \frac{-4xu + x^2u_1 - u_1^3}{16u^2 - 16uxu_1 + 6x^2u_1^2 + x^4 + u_1^4} \\ \eta(x, u^{(1)}) &= \frac{-16u^3 + 16u^2xu_1 - 6ux^2u_1^2 + ux^4 - 3uu_1^4 + 2x^3u_1^3 + 2xu_1^5}{(16u^2 - 16uxu_1 + 6x^2u_1^2 + x^4 + u_1^4)^2}. \end{aligned}$$

### 6.3 Quasiconformal point automorphisms

As was demonstrated in the previous chapter, under suitable smoothness assumptions, the quasiconformal automorphisms of  $J^k(\mathbb{R}, \mathbb{R})$ , where  $k \geq 2$ , are point transformations with a canonical form. For the Heisenberg group, we can construct a subgroup of the quasiconformal automorphisms with canonical form, however the quasiconformal automorphisms need not be point transformations as the 1-quasiconformal maps of the Heisenberg group need not be point transformations. First we obtain the global quasiconformal point transformations of  $J^1(\mathbb{R}, \mathbb{R})$ .

Assume that

$$f = (\xi(x, u), \eta_1(x, u_1, u), \eta(x, u))$$

is a global quasiconformal point transformation of  $J^1(\mathbb{R}, \mathbb{R})$ , satisfying  $\det M > 0$ . Since

$$\eta_1(x, u_1, u) = \frac{X\eta}{X\xi} = \frac{\eta_x + u_1\eta_u}{\xi_x + u_1\xi_u}$$

we have  $\xi_u = 0$  so

$$f = (\xi(x), \eta_1(x, u_1, u), \eta(x, u))$$

and

$$\eta_1(x, u_1, u) = \frac{\eta_x + u_1\eta_u}{\xi_x}.$$

It follows that

$$\Delta(f) = \frac{(\xi_x)^2 + (X\eta_1)^2 + (\eta_u/\xi_x)^2}{\eta_u}$$

and

$$X\eta_1 = \left( u_1^2 \eta_{uu} \xi_x + u_1 (2\eta_{xu} \xi_x - \eta_u \xi_{xx}) + (\eta_{xx} \xi_x - \eta_x \xi_{xx}) \right) / \xi_x^2.$$

Since  $\Delta(f)$  is bounded we have

$$(1) \quad \eta_{uu} \xi_x = 0 \quad \text{and} \quad (2) \quad 2\eta_{xu} \xi_x - \eta_u \xi_{xx} = 0.$$

Since  $\xi_x \neq 0$ , (1) implies that

$$\eta(x, u) = a(x)u + b(x) \quad \text{and} \quad \eta_1(x, u_1, u) = \frac{a_x u + b_x + u_1 a(x)}{\xi_x}.$$

Since  $\det M > 0$  we have  $a(x) > 0$ , furthermore, (2) becomes

$$2\xi_x \eta_{xu} - \eta_u \xi_{xx} = 2\xi_x a_x - a(x) \xi_{xx} = 0, \quad (6.3.1)$$

and

$$\Delta(f) = \frac{(\xi_x)^2 + (X\eta_1)^2 + (a(x)/\xi_x)^2}{a(x)}$$

where

$$X\eta_1 = \left( u(a_{xx}\xi_x - a_x\xi_{xx}) + (b_{xx}\xi_x - b_x\xi_{xx}) \right) / \xi_x^2.$$

Since  $\Delta(f)$  is bounded, we have

$$a_{xx}\xi_x - a_x\xi_{xx} = 0 \quad (6.3.2)$$

and

$$\Delta(f) = \frac{\xi_x^2}{a(x)} + \frac{(\xi_x b_{xx} - \xi_{xx} b_x)^2}{\xi_x^4 a(x)} + \frac{a(x)}{\xi_x^2}. \quad (6.3.3)$$

Assume  $a(x)$  is not constant, then (6.3.1) implies that

$$\frac{\xi_{xx}}{\xi_x} - 2\frac{a_x}{a(x)} = 0,$$

hence

$$\frac{d}{dx} \ln \left( \frac{a(x)^2}{|\xi_x|} \right) = 0$$

and

$$\xi_x = C_1 a(x)^2.$$

Substituting the previous identity into (6.3.2) gives

$$a_{xx}a(x) - 2a_x^2 = 0,$$

which, if we write  $y = a_x/a$ , becomes

$$y' = y^2.$$

By the standard uniqueness theory of ordinary differential equations,

$$y = \frac{1}{C_2 - x}$$

for some constant  $C_2$ , and it follows that

$$a(x) = \frac{C_3}{C_2 - x}$$

for some constant  $C_3$ . However, (6.3.3) shows that

$$\frac{\xi_x^2}{a(x)} = C_1^2 a(x)^3$$

must be bounded so  $a$  must be constant. Moreover, (6.3.3) implies  $\xi_{xx} = 0$  and it follows that

$$\xi(x) = \alpha x + \beta, \quad \eta_1(x, u_1, u) = \frac{a}{\alpha} u_1 + \frac{b_x}{\alpha} \quad \text{and} \quad \eta(x, u) = au + b(x)$$

where  $|b_{xx}|$  is bounded.

Normalising by the dilation  $\delta_{1/\alpha}$  and a suitable left translation  $\tau_g$ , we can assume that  $f = \delta_\alpha \circ \tau_{g^{-1}} \circ f^{\lambda, \mu}$  where  $f^{\lambda, \mu}$  is given by

$$\begin{aligned} \xi^{\lambda, \mu}(x, u^{(1)}) &= x \\ f_1^{\lambda, \mu}(x, u^{(1)}) &= \lambda u_1 + \mu(x) \\ f_0^{\lambda, \mu}(x, u^{(1)}) &= \lambda u + \int_0^x \mu(t) dt \end{aligned} \tag{6.3.4}$$

and  $|\mu'(x)|$  is bounded.

Together, dilations, left translations, switch maps, flip maps, the point automorphisms (6.3.4) and rotations (6.2.3), generate a subgroup  $\mathcal{G}$  of the quasiconformal automorphism group of  $J^1(\mathbb{R}, \mathbb{R})$ .

## 6.4 Equivariant lifts

There is another method of constructing quasiconformal automorphisms of  $J^1(\mathbb{R}, \mathbb{R})$  due to Tang in [29], also used in [7]. There is a free action of  $\mathbb{R}$  on  $J^1(\mathbb{R}, \mathbb{R})$  defined by

$$a_r(x, u_1, u) = (0, 0, r) \odot (x, u_1, u) = (x, u_1, u + r), \quad r \in \mathbb{R}.$$

The quotient space  $J^1(\mathbb{R}, \mathbb{R})/\mathbb{R}$  of this action identifies with  $\mathbb{R}^2$  and the projection  $\pi$  takes the form  $\pi(x, u_1, u) = (x, u_1)$ . The Carnot–Carathéodory metric is

invariant under this action of  $\mathbb{R}$  and induces the Euclidean metric on the quotient space. In [29] and [7], it is shown that a rectifiable curve in  $J^1(\mathbb{R}, \mathbb{R})/\mathbb{R}$  has a rectifiable horizontal lift in  $J^1(\mathbb{R}, \mathbb{R})$ , which is unique up to choosing the initial point. From this result, [29] and [7] show that a quasiconformal map of  $J^1(\mathbb{R}, \mathbb{R})/\mathbb{R} \cong \mathbb{R}^2$ , which distorts area by a constant factor, must be bilipschitz and lifts to an equivariant map of  $J^1(\mathbb{R}, \mathbb{R})$  which is bilipschitz with respect to the Carnot–Carathéodory metric. Moreover the lift is unique up to composition with  $a_r$  for some  $r$ . Since the lift is bilipschitz with respect to the Carnot–Carathéodory metric, it is quasiconformal by the geometric definition.

In particular, dilation, left translation, and the rotations (6.2.3) all project to quasiconformal maps of  $\mathbb{R}^2$  which distort area by a constant factor. For the point transformations (6.3.4), the projection has the form

$$F(x + iu_1) = x + i(\lambda u_1 + \mu(x)).$$

Recall that a diffeomorphism  $h$ , of a domain in  $\mathbb{C}$ , is quasiconformal if and only if  $\|\bar{\partial}h/\partial h\|_\infty \leq c < 1$ , where  $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial u_1})$  and  $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial u_1})$ . Since

$$\frac{\bar{\partial}F}{\partial F} = \frac{1 - \lambda + i\mu'(x)}{1 + \lambda + i\mu'(x)},$$

and  $\lambda > 0$ , it follows that  $F$  is quasiconformal. Moreover  $F$  distorts area by the constant factor  $\lambda$ .

It follows from uniqueness that  $\mathcal{G}$  is lifted from a subgroup of the quasiconformal maps of  $\mathbb{R}^2$  which distort area by a constant factor. As is pointed out in [7], the point transformations (6.3.4) define quasiconformal automorphisms if  $\mu(x)$  is bilipschitz.

If  $f$  is a smooth contact map on  $J^1(\mathbb{R}, \mathbb{R})$  such that

$$f(x, u_1, u) = (\xi(x, u_1), \eta_1(x, u_1), \eta(x, u_1, u)),$$

then the contact conditions give

$$X\eta = \eta_1 \frac{\partial \xi}{\partial x} \quad \text{and} \quad \frac{\partial \eta}{\partial u_1} = \eta_1 \frac{\partial \xi}{\partial u_1}. \quad (6.4.1)$$

Applying  $\frac{\partial}{\partial u}$  to these contact conditions gives

$$\frac{\partial^2 \eta}{\partial u \partial x} + u_1 \frac{\partial^2 \eta}{\partial u^2} = 0 \quad \text{and} \quad \frac{\partial^2 \eta}{\partial u \partial u_1} = 0.$$

It follows that

$$\eta(x, u_1, u) = au + b(x, u_1),$$

and (6.1.1) shows that  $\det JF = a$  which implies that  $F$  distorts area by a constant factor.

If  $F = \xi + i\eta_1$ , then

$$\frac{|\partial F|^2 + |\bar{\partial} F|^2}{|\partial F|^2 - |\bar{\partial} F|^2} = \frac{1}{2} \Delta(f) = \frac{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial u_1}\right)^2 + \left(\frac{\partial \eta_1}{\partial x}\right)^2 + \left(\frac{\partial \eta_1}{\partial u_1}\right)^2}{2 \left| \frac{\partial \xi}{\partial x} \frac{\partial \eta_1}{\partial u_1} - \frac{\partial \xi}{\partial u_1} \frac{\partial \eta_1}{\partial x} \right|}.$$

If  $f$  is quasiconformal, then Fubini's theorem implies that  $\partial F, \bar{\partial} F \in L^4_{\text{loc}}$ . Since  $F$  is an injective open map, we conclude that  $F$  is quasiconformal.

# Chapter 7

## Rigidity and Strata Dimension

### 7.1 Introduction

As was discussed in chapter 1, the folklore rule of thumb is that noncommutativity should force rigidity in the sense that a high degree of noncommutativity should imply more rigidity. A measure of noncommutativity or rigidity might be the “growth vector” which records the step and strata dimensions of the group. However, such a measure must depend on more than the growth vector data. For example, from [34], there is a Carnot group  $G_{HC}$  of dimension 6, associated to the Hilbert Cartan equation  $v' = (u'')^2$ , such that  $L_1 = \text{span}\{X_1, X_2\}$ ,

$$L_{j+1} = [L_1, L_j] = \text{span}\{e_{j+1}\} \quad \text{where } j = 1, \dots, 4,$$

and  $[e_3, e_4] = e_6$ . Furthermore the Lie algebra of contact vector fields is isomorphic with the 14-dimensional Lie algebra  $G_2$  (see [1]), hence  $G_{HC}$  is rigid. The growth vector for  $G_{HC}$  is  $(2, 1, 1, 1, 1)$  which is the growth vector for  $J^4(\mathbb{R}, \mathbb{R})$ . Hence we have two Carnot groups with the same growth vector but with opposite rigidity.

Similarly, we can construct distinct Carnot groups with growth vectors  $(3, 2, 1)$ , but with opposite rigidity. For example, using the vector field method, as in [15] and [32], it is easy to check that the Carnot group corresponding to the Lie algebra  $\mathfrak{n}(4, \mathbb{R})$ , the strictly upper triangular  $4 \times 4$  real matrices, is rigid

with strata dimensions  $(3, 2, 1)$ . For the nonrigid example we use Grassmanian prolongation (see [17]).

## 7.2 Grassmanian prolongation

Let  $\Sigma(k, M)$  be a distribution of  $k$ -dimensional subspaces of an  $n$ -dimensional manifold  $M$ , i.e., if  $p \in M$ , then  $\Sigma(k, M)_p$  is a  $k$ -dimensional subspaces of  $T_p M$  and for some neighborhood  $U$  of  $p$  there exist smooth vector fields  $X_1, \dots, X_k$  such that

$$\Sigma(k, M)_q = \text{span}\{X_1(q), \dots, X_k(q)\}, \quad q \in U.$$

The study of  $\ell$ -dimensional submanifolds of  $M$  which are tangent to  $\Sigma(k, M)$  gives rise to the bundle  $\text{Gr}(\ell, \Sigma(k, M)) \rightarrow M$  where each fiber  $\text{Gr}(\ell, \Sigma(k, M))_p$  is the Grassmannian of  $\ell$ -dimensional subspaces  $\Lambda_p \subset \Sigma(k, M)_p$ . The elements of  $\text{Gr}(\ell, \Sigma(k, M))_p$  represent the possible tangent spaces of the submanifolds.

A curve through  $(p, \Lambda_p) \in \text{Gr}(\ell, \Sigma(k, M))$  has the form  $(\gamma(t), \Lambda_{\gamma(t)})$ , where  $\gamma(0) = p$ , and is defined to be horizontal at  $(p, \Lambda_p)$  if  $\dot{\gamma}(0) \in \Lambda_p$ . These curves define a subspace of  $T_{(p, \Lambda_p)} \text{Gr}(\ell, \Sigma(k, M))$  and the collection of all these subspaces defines a distribution  $\Sigma(\text{Gr}(\ell, \Sigma(k, M)))$  on  $\text{Gr}(\ell, \Sigma(k, M))$ . The Grassman bundle  $\text{Gr}(\ell, \Sigma(k, M))$ , together with the distribution  $\Sigma(\text{Gr}(\ell, \Sigma(k, M)))$ , is called the Grassman prolongation of  $\Sigma(k, M)$ . A contact map of  $M$  lifts to a contact map of  $\text{Gr}(\ell, \Sigma(k, M))$  via  $f(p, \Lambda_p) = (f(p), f_*\Lambda_p)$  so that  $M$  and  $\text{Gr}(\ell, \Sigma(k, M))$  share the same rigidity.

## 7.3 Nonrigid (3,2,1)

Consider the Carnot group  $G$  with Lie algebra given by  $\text{span}\{X_1, X_2, X_3, X_4\}$  and nontrivial brackets  $[X_1, X_2] = [X_1, X_3] = X_4$ . The horizontal space is  $\mathcal{H} = \text{span}\{X_1, X_2, X_3\}$  and the strata dimensions are  $(3, 1)$ . In second kind

coordinates, we have

$$X_1 = \frac{\partial}{\partial x_1} - (x_2 + x_3) \frac{\partial}{\partial x_4} \quad X_2 = \frac{\partial}{\partial x_2} \quad X_3 = \frac{\partial}{\partial x_3} \quad X_4 = \frac{\partial}{\partial x_4}$$

with corresponding dual forms  $dx_1$ ,  $dx_2$ ,  $dx_3$ , and  $dx_4 + (x_2 + x_3)dx_1$ , and  $\mathcal{H} = \Sigma(3, G)$ . A vector field  $V = \sum v_i X_i$  induces a contact flow if  $[\mathcal{H}, V] = 0 \pmod{\mathcal{H}}$  which implies that

$$X_1 v_4 + v_2 + v_3 = 0, \quad X_2 v_4 - v_1 = 0 \quad \text{and} \quad X_3 v_4 - v_1 = 0.$$

It follows that

$$V = (X_2 v_4) X_1 + v_2 X_2 - (X_1 v_4 + v_2) X_3 + v_4 X_4$$

with  $v_2$  arbitrary and  $v_4 = P(x_1, x_2 + x_3, x_4)$  for any suitably smooth  $P$ . We conclude that  $G$  is nonrigid. Alternatively,  $G$  can be recognised as the the product of the Heisenberg group and the real line which accounts for it being nonrigid. In particular, with

$$X = X_1, \quad Y = \frac{1}{2}(X_2 + X_3), \quad Z = \frac{1}{2}(X_2 - X_3), \quad \text{and} \quad T = X_4,$$

the commutation relations are given by  $[X, Y] = T$ , and all other commutators vanishing.

We Grassman prolong  $G$  by 1-dimensional subspaces of the form

$$\text{span}\{X_1 + tX_2 + sX_3\} \subset \mathcal{H}.$$

Thus we define  $\gamma = (x_1, x_2, x_3, x_4, t, s)$  to be horizontal if

$$\dot{x}_1 \neq 0, \quad \dot{x}_4 = -(x_2 + x_3)\dot{x}_1 \quad \text{and} \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \lambda(1, t, s),$$

or equivalently if

$$\dot{\gamma} = \dot{x}_1(X_1 + tX_2 + sX_3) + \dot{t} \frac{\partial}{\partial t} + \dot{s} \frac{\partial}{\partial s}.$$

It follows that

$$\tilde{\mathcal{H}} = \Sigma(\text{Gr}(1, \Sigma(3, G))) = \text{span} \left\{ \tilde{X}_1, T, S \right\},$$

where

$$\tilde{X}_1 = X_1 + tX_2 + sX_3, \quad T = \frac{\partial}{\partial t} \quad \text{and} \quad S = \frac{\partial}{\partial s},$$

moreover the nontrivial brackets are

$$\begin{aligned} [T, \tilde{X}_1] &= X_2, & [S, \tilde{X}_1] &= X_3 \\ [\tilde{X}_1, X_2] &= [X_1, X_2] = X_4 \\ [\tilde{X}_1, X_3] &= [X_1, X_3] = X_4. \end{aligned}$$

By construction, the corresponding Carnot group is nonrigid with strata dimensions  $(3, 2, 1)$ .

## 7.4 Rigid (3,2,1)

Denote by  $G$  the subgroup of  $GL(4, \mathbb{R})$  consisting of elements

$$x = \begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ 0 & 1 & x_2 & x_5 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The rigidity of  $G$  was first established in [9] as it is a nilpotent Iwasawa subgroup of  $GL(4, \mathbb{R})$ . The following is an elementary proof.

A basis for the Lie Algebra  $L$ , of left-invariant vector fields, is given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} & X_2 &= \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_4} & X_3 &= \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_5} + x_4 \frac{\partial}{\partial x_6} \\ X_4 &= \frac{\partial}{\partial x_4} & X_5 &= \frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_6} & X_6 &= \frac{\partial}{\partial x_6} \end{aligned}$$

with dual basis

$$\begin{aligned} \omega_1 &= dx_1 & \omega_2 &= dx_2 & \omega_3 &= dx_3 \\ \omega_4 &= dx_4 - x_1 dx_2 & \omega_5 &= dx_5 - x_2 dx_3 & \omega_6 &= dx_6 - x_1 dx_5 - (x_4 - x_1 x_2) dx_3. \end{aligned}$$

The nontrivial brackets are

$$[X_1, X_2] = X_4, \quad [X_2, X_3] = X_5, \quad [X_1, X_5] = X_6, \quad [X_3, X_4] = -X_6,$$

which, upon setting

$$L_1 = \text{span}\{X_1, X_2, X_3\}, \quad L_2 = \text{span}\{X_4, X_5\}, \quad L_3 = \text{span}\{X_6\}$$

give  $L_2 = [L_1, L_1]$  and  $L_3 = [L_1, L_2] = Z(L)$ .

If a vector field  $V = \sum v_i X_i$  is to generate a flow which preserves  $L_1$  it must satisfy  $[V, L_1] \subset L_1$  hence

$$\begin{array}{lll} X_1 v_4 = -v_2 & X_1 v_5 = 0 & X_1 v_6 = -v_5 \\ X_2 v_4 = v_1 & X_2 v_5 = -v_3 & X_2 v_6 = 0 \\ X_3 v_4 = 0 & X_3 v_5 = v_2 & X_3 v_6 = v_4 \end{array}$$

which imply

$$X_1^2 v_6 = 0 \quad X_2 v_6 = 0 \quad X_3^2 v_6 = 0$$

and

$$v_5 = -X_1 v_6 \quad v_4 = X_3 v_6 \quad v_3 = X_2 X_1 v_6 \quad v_2 = -X_3 X_1 v_6 \quad v_1 = X_2 X_3 v_6.$$

Since  $X_1^2 v_6 = 0$  we have  $v_6 = Ax_1 + B$  with  $A$  and  $B$  independent of  $x_1$ , moreover  $X_2 v_6 = 0$  implies

$$(X_2 A)x_1 + X_2 B = \left( \frac{\partial A}{\partial x_2} + x_1 \frac{\partial A}{\partial x_4} \right) x_1 + \frac{\partial B}{\partial x_2} + x_1 \frac{\partial B}{\partial x_4}$$

giving

$$\frac{\partial A}{\partial x_4} = 0 \quad \frac{\partial A}{\partial x_2} + \frac{\partial B}{\partial x_4} = 0 \quad \frac{\partial B}{\partial x_2} = 0.$$

It follows that

$$A = A_1(x_3, x_5, x_6)x_2 + A_2(x_3, x_5, x_6)$$

and

$$B = -A_1(x_3, x_5, x_6)x_4 + B_2(x_3, x_5, x_6)$$

which, by direct calculation, further implies

$$\begin{aligned}
X_3^2 v_6 = & \left( \frac{\partial^2 A_1}{\partial x_5^2} \right) x_1 x_2^3 + \left( 2 \frac{\partial^2 A_1}{\partial x_5 \partial x_3} + \frac{\partial^2 A_2}{\partial x_5^2} \right) x_1 x_2^2 \\
& + 2 \left( \frac{\partial^2 A_1}{\partial x_6 \partial x_5} \right) x_4 x_1 x_2^2 + \left( \frac{\partial^2 A_1}{\partial x_6^2} \right) x_1 x_2 x_4^2 \\
& + 2 \left( \frac{\partial^2 A_2}{\partial x_6 \partial x_5} + \frac{\partial^2 A_1}{\partial x_6 \partial x_3} \right) x_1 x_2 x_4 + \left( 2 \frac{\partial^2 A_2}{\partial x_5 \partial x_3} + \frac{\partial^2 A_1}{\partial x_3^2} \right) x_1 x_2 \\
& + \left( \frac{\partial^2 A_2}{\partial x_6^2} \right) x_1 x_4^2 + 2 \left( \frac{\partial^2 A_2}{\partial x_6 \partial x_3} \right) x_1 x_4 + \left( \frac{\partial^2 A_2}{\partial x_3^2} \right) x_1 + \left( \frac{\partial^2 B_2}{\partial x_5^2} \right) x_2^2 \\
& - \left( \frac{\partial^2 A_1}{\partial x_5^2} \right) x_4 x_2^2 - 2 \left( \frac{\partial^2 A_1}{\partial x_6 \partial x_5} \right) x_2 x_4^2 - 2 \left( \frac{\partial^2 A_1}{\partial x_5 \partial x_3} - \frac{\partial^2 B_2}{\partial x_6 \partial x_5} \right) x_2 x_4 \\
& + 2 \left( \frac{\partial^2 B_2}{\partial x_5 \partial x_3} \right) x_2 - \left( \frac{\partial^2 A_1}{\partial x_6^2} \right) x_4^3 + \left( \frac{\partial^2 B_2}{\partial x_6^2} - 2 \frac{\partial^2 A_1}{\partial x_6 \partial x_3} \right) x_4^2 \\
& + \left( - \frac{\partial^2 A_1}{\partial x_3^2} + 2 \frac{\partial^2 B_2}{\partial x_6 \partial x_3} \right) x_4 + \left( \frac{\partial^2 B_2}{\partial x_3^2} \right).
\end{aligned}$$

Each bracketed term in the previous expression must vanish, which forces  $v_6$  to be a polynomial. In particular, we have

$$\begin{aligned}
\frac{\partial^2 A_1}{\partial x_5^2} = 0 \quad \frac{\partial^2 A_1}{\partial x_6^2} = 0 \quad \frac{\partial^2 A_1}{\partial x_6 \partial x_5} = 0 \\
\frac{\partial^2 A_2}{\partial x_3^2} = 0 \quad \frac{\partial^2 A_2}{\partial x_6^2} = 0 \quad \frac{\partial^2 A_2}{\partial x_6 \partial x_3} = 0 \\
\frac{\partial^2 B_2}{\partial x_3^2} = 0 \quad \frac{\partial^2 B_2}{\partial x_5^2} = 0 \quad \frac{\partial^2 B_2}{\partial x_5 \partial x_3} = 0
\end{aligned}$$

which implies

$$\begin{aligned}
A_1 &= \alpha_1(x_3)x_6 + \beta_1(x_3)x_5 + \gamma_1(x_3) \\
A_2 &= \alpha_2(x_5)x_6 + \beta_2(x_5)x_3 + \gamma_2(x_5) \\
B_2 &= \alpha_3(x_6)x_5 + \beta_3(x_6)x_3 + \gamma_3(x_6).
\end{aligned}$$

The remaining bracketed terms give

1.  $2 \frac{\partial^2 A_1}{\partial x_5 \partial x_3} + \frac{\partial^2 A_2}{\partial x_5^2} = 2\beta_1'(x_3) + \alpha_2''(x_5)x_6 + \beta_2''(x_5)x_3 + \gamma_2''(x_5) = 0$
2.  $\frac{\partial^2 A_2}{\partial x_6 \partial x_5} + \frac{\partial^2 A_1}{\partial x_6 \partial x_3} = \alpha_2'(x_5) + \alpha_1'(x_3) = 0$
3.  $2 \frac{\partial^2 A_2}{\partial x_5 \partial x_3} + \frac{\partial^2 A_1}{\partial x_3^2} = 2\beta_2'(x_5) + \alpha_1''(x_3)x_6 + \beta_1''(x_3)x_5 + \gamma_1''(x_3) = 0$

4.  $\frac{\partial^2 A_1}{\partial x_5 \partial x_3} - \frac{\partial^2 B_2}{\partial x_6 \partial x_5} = \beta_1'(x_3) - \alpha_3'(x_6) = 0$
5.  $\frac{\partial^2 B_2}{\partial x_6^2} - 2\frac{\partial^2 A_1}{\partial x_6 \partial x_3} = \alpha_3''(x_6)x_5 + \beta_3''(x_6)x_3 + \gamma_3''(x_6) - 2\alpha_1'(x_3) = 0$
6.  $-\frac{\partial^2 A_1}{\partial x_3^2} + 2\frac{\partial^2 B_2}{\partial x_6 \partial x_3} = -\alpha_1''(x_3)x_6 - \beta_1''(x_3)x_5 - \gamma_1''(x_3) + 2\beta_3'(x_6) = 0.$

From 1, 3 and 5 we have  $\alpha_2''(x_5) = 0$ ,  $\alpha_1''(x_3) = 0$  and  $\alpha_3''(x_6) = 0$ . It follows using 2, that

$$\alpha_1(x_3) = P_1 x_3 + Q_1 \quad \alpha_2(x_5) = -P_1 x_5 + Q_2 \quad \alpha_3(x_6) = P_3 x_6 + Q_3$$

and, from 4, it follows that

$$\beta_1(x_3) = P_3 x_3 + Q_4.$$

The remaining equations are

1.  $2\frac{\partial^2 A_1}{\partial x_5 \partial x_3} + \frac{\partial^2 A_2}{\partial x_5^2} = 2P_3 + \beta_2''(x_5)x_3 + \gamma_2''(x_5) = 0$
3.  $2\frac{\partial^2 A_2}{\partial x_5 \partial x_3} + \frac{\partial^2 A_1}{\partial x_3^2} = 2\beta_2'(x_5) + \gamma_1''(x_3) = 0$
5.  $\frac{\partial^2 B_2}{\partial x_6^2} - 2\frac{\partial^2 A_1}{\partial x_6 \partial x_3} = \beta_3''(x_6)x_3 + \gamma_3''(x_6) - 2P_1 = 0$
6.  $-\frac{\partial^2 A_1}{\partial x_3^2} + 2\frac{\partial^2 B_2}{\partial x_6 \partial x_3} = -\gamma_1''(x_3) + 2\beta_3'(x_6) = 0.$

From 1, it follows that

$$\beta_2(x_5) = P_4 x_5 + Q_5 \quad \text{and} \quad \gamma_2(x_5) = -P_3 x_5^2 + Q_6 x_5 + Q_7,$$

and from 3, it follows that

$$\gamma_1(x_3) = -P_4 x_3^2 + Q_8 x_3 + Q_9.$$

From 5, it follows that

$$\beta_3(x_6) = P_5 x_6 + Q_{10} \quad \text{and} \quad \gamma_3(x_6) = P_1 x_6^2 + Q_{11} x_6 + Q_{12},$$

and 6 shows that

$$P_5 = -P_4.$$

We conclude that  $v_6$  is a polynomial depending on the 15 coefficients

$$P_1, P_3, P_4, Q_1, \dots, Q_{12},$$

hence  $G$  is rigid.



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