REAL AND $p$-ADIC OSCILLATORY INTEGRALS

A thesis presented to

The University of New South Wales

in fulfillment of the thesis requirement
for the degree of

Doctor of Philosophy

by

KEITH MCKENZIE ROGERS
Abstract

After our introduction in Chapter 1, we consider van der Corput’s lemma in Chapter 2. We find the nodes that minimize divided differences, and use these to find the sharp constant in a related sublevel set estimate. We go on to find the sharp constant in the first instance of the van der Corput lemma using a complex mean value theorem for integrals. With these bounds we improve the constant in the general van der Corput lemma, so that it is asymptotically sharp.

In Chapter 3 we review the $p$-adic numbers and some results from Fourier analysis over the $p$-adics. In Chapter 4 we prove a $p$-adic version of van der Corput’s lemma for polynomials, opening the way for the study of oscillatory integrals over the $p$-adics. In Chapter 5 we apply this result to bound maximal averages. We show that maximal averages over curves defined by $p$-adic polynomials are $L^q$ bounded, where $1 < q < \infty$. 
Acknowledgments

I would like to thank: Michael Cowling for being a great supervisor; the School of Mathematics at the University of New South Wales for their generous support throughout my candidature; Venta Terauds for lots of proof reading; Steve Finch for bringing some results to my attention; my parents; and the numerous anonymous referees.
# Contents

Abstract 

Acknowledgments 

1 Introduction 1

2 Sharp van der Corput estimates 3

2.1 Sublevel set estimates 3

2.2 Divided differences 6

2.3 A complex mean value theorem 8

2.4 Van der Corput lemmas 11

3 Introduction to the $p$-adic numbers 15

3.1 The Hardy–Littlewood maximal theorem 17

3.2 Fourier analysis on the $p$-adic numbers 20

3.3 Bessel potentials on the $p$-adic numbers 28

4 A van der Corput lemma for the $p$-adic numbers 35

4.1 Preliminary lemmas 37

4.2 Proof of Lemma 4.1 40

5 Maximal averages along $p$-adic curves 45

5.1 The Calderón–Zygmund decomposition 46

5.2 $L^2$ boundedness 48

5.3 $L^q$ boundedness 51
5.4 The method of descent ........................................ 55
5.5 Conclusion ..................................................... 57
Chapter 1

Introduction

Oscillatory integral theory has its origins at the very heart of harmonic analysis. The introduction of Fourier’s famous transform, the original and perhaps best example of an oscillatory integral, led to the consideration of more general oscillatory integrals. This work was mainly conducted by Fourier, Airy, Stokes, Lipschitz and Riemann in the nineteenth century, and was undertaken in order to understand the behavior of the Fourier transform.

The subject was given fresh impetus in the early twentieth century when J.G. van der Corput proved his celebrated lemma. He was concerned with applications in number theory, in particular in bounding exponential sums. More recently the focus has shifted to bounding operators which are fashioned from oscillatory integrals. These appear naturally in many areas of harmonic analysis, due to the prevalent use of the Fourier transform.

We will mainly be concerned with van der Corput’s lemma and its applications. It states that if \( n \geq 2 \) and the modulus of the \( n \)th derivative of a real-valued function is bounded away from the origin by \( \lambda \), then

\[
\left| \int_a^b e^{if(x)} \, dx \right| \leq \frac{C_n}{\lambda^{1/n}},
\]

where \( C_n \) is a constant depending only on \( n \). There is also a version when \( n = 1 \). It was recently shown by G.I. Arhipov, A.A. Karacuba and V.N. Čubarikov that \( C_n \) can be taken to depend on \( n \) in a linear fashion, and that this dependence
is sharp. Our contribution in Chapter 2 is to improve the constant so that the bound becomes absolutely sharp as $n$ tends to infinity [20].

An important and largely open question is whether there is a van der Corput lemma for higher dimensions. It is also natural to ask whether similar results hold if we consider functions over other locally compact abelian groups. In Chapter 3 we introduce the $p$-adic numbers, which have importance in number theory. Here intervals are replaced by balls, and the exponential function is replaced by the additive character. Perhaps the key new result in this thesis is in Chapter 4, where we find a version of van der Corput’s lemma for the $p$-adic numbers, with the added constraint that the functions in question are polynomials [21].

Finally, in Chapter 5, we apply our van der Corput lemma for the $p$-adic numbers to maximal averages. Using similar arguments to those employed by E.M. Stein [27] to prove an analogous result in the Euclidean case, we show that maximal averages along curves defined by polynomials are $L^q$ bounded on $L^q$, where $1 < q < \infty$. [22].

The constants $C$ take different values throughout, and a familiarity with Lebesgue integration is assumed.
Chapter 2

Sharp van der Corput estimates

In the first two sections we will find the nodes that minimize divided differences. We will use these to find the sharp constant in a sublevel set estimate relating to van der Corput’s lemma. In the third section we will find the sharp constant in the first instance of the van der Corput lemma using a complex mean value theorem for integrals. With these bounds we will improve the constant in the general van der Corput lemma, so that it is asymptotically sharp.

In this chapter, functions are real-valued unless otherwise stated.

2.1 Sublevel set estimates

The importance of sublevel set estimates in van der Corput lemmas was highlighted by A. Carbery, M. Christ, and J. Wright [5, 6]. We find the sharp constant in the following sublevel set estimate.

Lemma 2.1. Suppose that \( f : (a, b) \to \mathbb{R} \) is \( n \) times differentiable where \( n \geq 1 \), and that \( |f^{(n)}(x)| \geq \lambda > 0 \) on \( (a, b) \). Then

\[
\left| \{x \in (a, b) : |f(x)| \leq \alpha \} \right| \leq C_n \left( \alpha / \lambda \right)^{1/n},
\]

where \( C_n = (n! \ 2^{2n-1})^{1/n} \).
We note that \( C_n \leq 2n \) for all \( n \geq 1 \), and by Stirling’s formula,
\[
\lim_{n \to \infty} \frac{C_n}{n} = \lim_{n \to \infty} \frac{(n! 2^{2n-1})^{1/n}}{n} = \lim_{n \to \infty} \frac{1}{n} \left( \sqrt{2\pi n} n^{n} 2^{2n-1} e^{-n} \right)^{1/n}.
\]
Now as
\[
\frac{1}{n} \left( \sqrt{2\pi n} n^{n} 2^{2n-1} e^{-n} \right)^{1/n} = \left( \frac{n \pi}{2} \right)^{1/2n} 4\frac{1}{e},
\]
we see that
\[
\lim_{n \to \infty} \frac{C_n}{n} = \frac{4}{e}.
\]

The Chebyshev polynomials will be key to the proof of Lemma 2.1, so we recall some facts that we will need. For a more complete introduction see [19]. Consider \( T_n \) defined on \([-1, 1]\) by
\[
T_n(\cos \theta) = \cos n\theta,
\]
where \( 0 \leq \theta \leq \pi \). If we take the binomial expansion of the right hand side of
\[
\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n
\]
and equate real parts, we obtain
\[
\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( (-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} \right) (\cos \theta)^{n-2k},
\]
where \( \lfloor n/2 \rfloor \) denotes the integer part of \( n/2 \). Thus we can consider \( T_n \) to be the polynomial of degree \( n \) defined on the real line by
\[
T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( (-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} \right) x^{n-2k}.
\]
It is clear that \( |T_n| \leq 1 \) on \([-1, 1]\), and that the extrema are attained at \( \eta_j = \cos(j\pi/n) \) for \( j = 0, \ldots, n \). Finally we calculate the leading coefficient,
\[
\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} = \frac{1}{2}((1 + 1)^n + (1 - 1)^n) = 2^{n-1}.
\]

We will also require the following generalization of the classical mean value theorem. It can be found in texts on numerical analysis, for example [11, p. 189]. We include a proof for convenience.
Theorem 2.2. Suppose that $f : (a, b) \rightarrow \mathbb{R}$ is $n$ times differentiable, where $n \geq 1$, and suppose that $x_0 < x_1 < \ldots < x_n$ are distinct points in $(a, b)$. Then there exists $\zeta \in (a, b)$ such that

$$f^{(n)}(\zeta) = \sum_{j=0}^{n} c_j f(x_j),$$

where $c_j = (-1)^{j+n} n! \prod_{k:k \neq j} |x_k - x_j|^{-1}$.

Proof. Define the Lagrange interpolating polynomial $P$ by

$$P(x) = \sum_{j=0}^{n} f(x_j) L_j(x), \quad \text{where} \quad L_j(x) = \prod_{k:k \neq j} \frac{x - x_k}{x_j - x_k}.$$

By construction, $P(x_j) = f(x_j)$ for $j = 0, \ldots, n$, so that $P - f$ has at least $n+1$ zeros. Thus, by Rolle’s theorem (iterated many times), there exists $\zeta \in (a, b)$ such that

$$f^{(n)}(\zeta) - P^{(n)}(\zeta) = 0.$$

Now

$$P^{(n)}(\zeta) = \sum_{j=0}^{n} f(x_j) n! \prod_{k:k \neq j} \frac{1}{x_j - x_k},$$

so that

$$f^{(n)}(\zeta) = P^{(n)}(\zeta) = \sum_{j=0}^{n} f(x_j) (-1)^{n+j} n! \prod_{k:k \neq j} \frac{1}{|x_k - x_j|},$$

as desired. \qed

We apply this result to the Chebyshev polynomials. It is easy to see that $T_n^{(n)} = n! 2^{n-1}$, and that $T_n(\eta_j) = (-1)^{j+n}$ for the Chebyshev extrema $\eta_j = \cos(j\pi/n)$. Thus we obtain

$$\sum_{j=0}^{n} \prod_{k:k \neq j} n! |\eta_k - \eta_j|^{-1} = n! 2^{n-1},$$

so that

$$\sum_{j=0}^{n} \prod_{k:k \neq j} |\eta_k - \eta_j|^{-1} = 2^{n-1}. \quad (2.1)$$
Proof of Lemma 2.1. Suppose that $E = \{x \in (a, b) : |f(x)| \leq \alpha\}$, and that $|E| > 0$. First we map $E$ to an interval of the same measure without increasing distance. Centering at the origin and scaling by $2/|E|$ if necessary, we map the interval to $[-1, 1]$. Now by (2.1) we have $\sum_{j=0}^{n} \prod_{k:k \neq j} |\eta_k - \eta_j|^{-1} = 2^{n-1}$, where $\eta_j = \cos(j\pi/n)$. Thus, mapping back to $E$, there exist $n + 1$ points $x_0, \ldots, x_n \in E$ such that $\sum_{j=0}^{n} \prod_{k:k \neq j} |x_k - x_j|^{-1} \leq 2^{n-1} \frac{2^n}{|E|^n} = \frac{2^{2n-1}}{|E|^n}$. On the other hand, by Lemma 2.2 there exists $\zeta \in (a, b)$ such that $f^{(n)}(\zeta) = \sum_{j=0}^{n} c_j f(x_j)$, where $|c_j| = n! \prod_{k:k \neq j} |x_k - x_j|^{-1}$. Putting these together, $\lambda \leq \left| \sum_{j=0}^{n} c_j f(x_j) \right| \leq \sum_{j=0}^{n} n! \prod_{k:k \neq j} |x_k - x_j|^{-1} \alpha \leq n! \frac{2^{2n-1}}{|E|^n} \alpha$, so that $|E| \leq (n! 2^{2n-1})^{1/n} (\alpha/\lambda)^{1/n}$, as desired. □

The sharpness may be observed by considering $f(x) = T_n(x)$ on $[-1, 1]$ with $\lambda = n! 2^{n-1}$ and $\alpha = 1$.

2.2 Divided differences

Divided differences were first considered by Newton and are important in interpolation theory. For a given set of nodes $x_0 < \ldots < x_m \in [-1, 1]$, the divided differences are defined recursively by

$$f[x_0, \ldots, x_n] = \frac{f[x_0, \ldots, x_{n-1}] - f[x_1, \ldots, x_n]}{x_0 - x_n},$$
where \( f[x_0] = f(x_0) \) and \( f[x_1] = f(x_1) \). It is not difficult to calculate that

\[
f[x_0, \ldots, x_n] = \sum_{j=0}^{n} f(x_j)(-1)^{j+n} \prod_{k:k\neq j} |x_k - x_j|^{-1}.
\]

The following theorem shows that the Chebyshev extrema are optimal for the minimization of divided differences.

**Theorem 2.3.** The Chebyshev extrema, \( \eta_j = \cos(j\pi/n) \) for \( j = 0, \ldots, n \), are the unique nodes for which

\[
f[\eta_0, \ldots, \eta_n] \leq 2^{n-1} \|f\|_{\infty},
\]

for all \( f \in C[-1, 1] \).

**Proof.** That the inequality holds is clear from (2.1). To show the uniqueness, we suppose there exist \( x_0, \ldots, x_n \in [-1, 1] \) other than the Chebyshev extrema, such that

\[
f[x_0, \ldots, x_n] \leq 2^{n-1} \|f\|_{\infty}
\]

for all \( f \in C[-1, 1] \). If we consider \( f \in C[-1, 1] \), so that \( \|f\|_{\infty} = 1 \) and

\[
f(x_j) = (-1)^{j+n}
\]

for \( j = 0, \ldots, n \), we see that

\[
\sum_{j=0}^{n} \prod_{k:k\neq j} |x_k - x_j|^{-1} \leq 2^{n-1}.
\]

It is clear that \( |T_n(x_j)| < 1 \) for some \( j \), as \( x_0, \ldots, x_n \) are not the Chebyshev extrema. Thus

\[
\sum_{j=0}^{n} T(x_j)(-1)^{j+n}n! \prod_{k:k\neq j} |x_k - x_j|^{-1} < n! 2^{n-1}.
\]

On the other hand,

\[
\sum_{j=0}^{n} T(x_j)(-1)^{j+n}n! \prod_{k:k\neq j} |x_k - x_j|^{-1} = n! 2^{n-1}
\]

by Lemma 2.2, and we have a contradiction. \( \Box \)
2.3 A complex mean value theorem

The following theorem is a complex version of the second mean value theorem for integrals. It would be unwise to suggest that such a classical result is new, but it does not appear to be well known.

**Theorem 2.4.** Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is monotone, $g : [a, b] \rightarrow \mathbb{C}$ is continuous, and $I = \int_a^b f(x)g(x) \, dx = |I|e^{i\theta}$. Then there exists a point $c \in [a, b]$ such that

$$I = \left( f(a) \Re\left( e^{-i\theta} \int_a^c g(x) \, dx \right) + f(b) \Re\left( e^{-i\theta} \int_c^b g(x) \, dx \right) \right) e^{i\theta},$$

so that

$$|I| \leq \left| f(a) \int_a^c g(x) \, dx \right| + \left| f(b) \int_c^b g(x) \, dx \right|.$$

Furthermore if $f$ is of constant sign and $|f|$ is decreasing, then there exists a point $c \in [a, b]$ such that

$$|I| = f(a) \Re\left( e^{-i\theta} \int_a^c g(x) \, dx \right) \leq \left| f(a) \int_a^c g(x) \, dx \right|.$$

**Proof.** Let $g(x) = w(x) + iv(x)$, so that

$$|I| = I e^{-i\theta} = \cos(\theta) \int_a^b f(x)w(x) \, dx + \sin(\theta) \int_a^b f(x)v(x) \, dx$$

$$= \int_a^b f(x) (\cos(\theta)w(x) + \sin(\theta)v(x)) \, dx,$$

where $\theta$ is the argument of the integral $I$. Now as $f$ is monotone and

$$x \mapsto \cos(\theta)w + \sin(\theta)v$$

is continuous, there exists a point $c \in [a, b]$ such that

$$|I| = f(a) \int_a^c (\cos(\theta)w + \sin(\theta)v) \, dx + f(b) \int_c^b (\cos(\theta)w + \sin(\theta)v) \, dx$$

$$= f(a) \Re\left( e^{-i\theta} \int_a^c g(x) \, dx \right) + f(b) \Re\left( e^{-i\theta} \int_c^b g(x) \, dx \right),$$

by the second mean value theorem for integrals (see for example [3, p. 304]).

Finally if $f$ is of constant sign and $|f|$ is decreasing we define $h$ to be equal to $f$ on $[a, b)$, set $h(b) = 0$ and apply the argument with $h$ in place of $f$. \qed
It is clear that there is a corresponding result when $|f|$ is increasing.

The following corollary is arguably the cornerstone of the theory of oscillatory integrals, and is proven, using different methods, in J.G. van der Corput [7], A. Zygmund [34], and E.M. Stein [27] with constants $2\sqrt{2}$, 4, and 3 respectively. We use Theorem 2.4 to find the sharp constant.

**Corollary 2.5.** Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $(a, b)$, $f'$ is monotone, and $|f'| \geq \lambda > 0$ on $[a, b]$. Then

$$\left| \int_{a}^{b} e^{if(x)} \, dx \right| \leq \frac{2}{\lambda}.$$  

**Proof.** By replacing $f$ by $-f$ if necessary, we may assume $f' \geq \lambda > 0$. We also assume $f'$ is increasing. The proof when $f'$ is decreasing is similar.

By a change of variables,

$$I = \int_{a}^{b} e^{if(x)} \, dx = \int_{f(a)}^{f(b)} \frac{e^{iy}}{f'(f^{-1}(y))} \, dy.$$  

Now as $1/(f' \circ f^{-1})$ is positive and decreasing, there exists a point $c \in [a, b]$ such that

$$I = \frac{1}{f'(a)} \Re \left( e^{-i\theta} \int_{f(a)}^{c} e^{iy} \, dy \right) = \frac{1}{f'(a)} (\sin(c - \theta) - \sin(f(a) - \theta)),$$

by Theorem 2.4, where $\theta$ is the argument of the integral $I$. Thus

$$|I| \leq \frac{1 + \sin(\theta - f(a))}{\lambda} \leq \frac{2}{\lambda},$$  

(2.2)

as desired. \qed

We note, by (2.2), that if $f'$ is positive and increasing, and the integral is nonzero, then the argument of the integral cannot be $f(a) + 3\pi/2$. Similar results can be obtained for negative functions and decreasing functions. That the constant 2 is sharp is observed by considering $f(x) = x$ on $(0, \pi)$ with $\lambda = 1$.

We can also use these methods to prove the following sharp version of the Riemann–Lebesgue lemma for monotone functions.
Lemma 2.6. Suppose that \( f : [0, 1] \rightarrow \mathbb{R} \) is monotone. Then

\[
|\hat{f}(n)| = \left| \int_0^1 f(x)e^{-2\piinx} \, dx \right| \leq \frac{|f(1) - f(0)|}{\pi n}.
\]

Proof. We assume that \( f \) is increasing. The proof when \( f \) is decreasing is similar. By a change of variables,

\[
\hat{f}(n) = \frac{1}{2\pi n} \int_0^{2\pi n} f \left( \frac{y}{2\pi n} \right) e^{-iy} \, dy,
\]

so by Theorem 2.4, there exists a point \( c \in [0, 2\pi n] \) such that

\[
|\hat{f}(n)| = \frac{f(0)}{2\pi n} \Re \left( e^{-i\theta} \int_0^c e^{-iy} \, dy \right) + \frac{f(1)}{2\pi n} \Re \left( e^{-i\theta} \int_c^{2\pi n} e^{-iy} \, dy \right)
\]

\[
= \frac{f(0)}{2\pi n} (\sin(c + \theta) - \sin \theta) + \frac{f(1)}{2\pi n} (\sin \theta - \sin(c + \theta)),
\]

where \( \theta \) is the argument of \( \hat{f}(n) \). Thus

\[
|\hat{f}(n)| \leq \frac{f(1) - f(0)}{2\pi n} (\sin \theta + 1) \leq \frac{f(1) - f(0)}{\pi n}, \tag{2.3}
\]

as desired. \( \square \)

Again we note, by (2.3), that the Fourier transform of an increasing function on \([0, 1]\) cannot take values with argument \( 3\pi/2 \). Similarly, the Fourier transform of a decreasing function cannot take values with argument \( \pi/2 \). The sharpness may be observed by considering \( f : [0, 1] \rightarrow \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
1 & \text{when } x \in \left[ \frac{2n-1}{n}, 1 \right] \\
0 & \text{otherwise},
\end{cases}
\]

so that

\[
|\hat{f}(n)| = \left| \int_0^{2\pi n} f \left( \frac{y}{2\pi n} \right) e^{-iy} \, dy \right| = \left| \int_{(2n-1)\pi}^{2\pi n} e^{-iy} \, dy \right| = \frac{1}{\pi n},
\]

and

\[
\frac{|f(1) - f(0)|}{\pi n} = \frac{1}{\pi n}.
\]
### 2.4 Van der Corput lemmas

The following lemma, with constants taken to be

\[ C_n = 2^{5/2} \pi^{1/n} (1 - 1/n), \]

is due to G.I. Arhipov, A.A. Karacuba and V.N. Čubarikov [2]. It is a more precise formulation of what is generally known as van der Corput’s lemma. We improve their constants, so that the bound becomes sharp as \( n \) tends to infinity.

**Lemma 2.7.** Suppose that \( f : (a, b) \to \mathbb{R} \) is \( n \) times differentiable, where \( n \geq 2 \), and \( |f^{(n)}(x)| \geq \lambda > 0 \) on \((a, b)\). Then

\[
\left| \int_a^b e^{if(x)} \, dx \right| \leq C_n \frac{n}{\lambda^{1/n}},
\]

where \( C_n \leq 2^{5/3} \) for all \( n \geq 2 \) and \( C_n \to 4/e \) as \( n \to \infty \).

**Proof.** We integrate over \( E_1 \) and \( E_2 \) separately, where

\[ E_1 = \{ x \in (a, b) : |f'(x)| \leq \alpha \} \quad \text{and} \quad E_2 = \{ x \in (a, b) : |f'(x)| > \alpha \}. \]

If we consider \( g = f' \), then \( |g^{(n-1)}| \geq \lambda \), so that

\[ |E_1| \leq \left( (n - 1)! 2^{2(n-1)-1} \right)^{1/(n-1)} \left( \frac{\alpha}{\lambda} \right)^{1/(n-1)}, \]

by Lemma 2.1. So trivially

\[
\left| \int_{E_1} e^{if(x)} \, dx \right| \leq \left( (n - 1)! 2^{2n-3} \right)^{1/(n-1)} \left( \frac{\alpha}{\lambda} \right)^{1/(n-1)}.
\]

Now \( E_2 \) is made up of at most \( 2(n-1) \) intervals on each of which \( |f'| \geq \alpha \) and \( f' \) is monotone, so by Corollary 2.5 we have

\[
\left| \int_{E_2} e^{if(x)} \, dx \right| \leq 2(n - 1) \frac{2}{\alpha}.
\]

Finally we optimize with respect to \( \alpha \) and deduce that

\[
\left| \int_a^b e^{if(x)} \, dx \right| \leq \left( \frac{(n - 1)! 2^{2n-1}}{(n-1)^{n-2}} \right)^{1/n} \frac{n}{\lambda^{1/n}} = C_n \frac{n}{\lambda^{1/n}}. \quad (2.4)
\]

Now \( C_n \) tends to \( 4/e \) as \( n \) tends to infinity, by Stirling’s formula and we can check the first few terms to see it is always less than or equal \( 2^{5/3} \). \( \Box \)
To see that the bound becomes sharp as \( n \) tends to infinity, we consider \( f_n \)
defined on \([-1, 1]\) by
\[
f_n(x) = \frac{T_n(x)}{n}.
\]
When \( n \geq 2 \), we have
\[
\left| \int_{-1}^{1} e^{i f_n(x)} \, dx \right| \geq \left| \int_{-1}^{1} \cos(f_n(x)) \, dx \right| \geq \left| \int_{-1}^{1} 1 - \frac{(f_n(x))^2}{2} \, dx \right|,
\]
by Taylor’s theorem. Now as \(|f_n| \leq 1/n\), we see that
\[
\left| \int_{-1}^{1} e^{i f_n(x)} \, dx \right| \geq 2 - \frac{1}{n^2},
\]
which tends to 2 as \( n \) tends to infinity. On the other hand
\[
f_n^{(n)} = (n - 1)! \cdot 2^{n-1},
\]
so that
\[
\left| \int_{-1}^{1} e^{i f_n(x)} \, dx \right| \leq \left( \frac{(n - 1)! \cdot 2^{n-1}}{(n - 1)^{n-2}} \right)^{1/n} \cdot \frac{n}{((n - 1)! \cdot 2^{n-1})^{1/n}},
\]
by (2.4) in Lemma 2.7. We manipulate this to obtain
\[
\left| \int_{-1}^{1} e^{i f_n(x)} \, dx \right| \leq \left( \frac{2^n n^n}{(n - 1)^{n-2}} \right)^{1/n}.
\]
This bound also tends to 2 as \( n \) tends to infinity. Hence Lemma 2.7 is asymptotically sharp.

We note that as
\[
\left| \int_{-2}^{2} e^{i x^2/2} \, dx \right| = 3.33346 \ldots > \frac{4}{e} \times 2
\]
and
\[
\left| \int_{-3}^{3} e^{i (x^3/6 - x)} \, dx \right| = 4.61932 \ldots > \frac{4}{e} \times 3,
\]
the asymptote could not be approached strictly from below.

**Corollary 2.8.** Let \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \) be a real polynomial of degree \( n \geq 1 \). Then for all \( a, b \in \mathbb{R} \),
\[
\left| \int_{a}^{b} e^{i f(x)} \, dx \right| < \frac{C_n}{|a_n|^{1/n}},
\]
where \( C_n < 11/2 \) for all \( n \geq 1 \) and \( C_n \to 4 \) as \( n \to \infty \).
**Proof.** When \( n = 1 \), we obtain the result simply by integrating. For higher degrees we have \( f^{(n)} = n! a_n \), so that

\[
|I| = \left| \int_a^b e^{if(x)} \, dx \right| \leq \left( \frac{(n-1)! \, 2^{2n-1}}{(n-1)^{n-2}} \right)^{1/n} \frac{n}{|a_n|^{1/n}},
\]

by (2.4) in the proof of Lemma 2.7. We can manipulate to obtain

\[
|I| \leq \left( \frac{2^{2n-1}n^{n-1}}{(n-1)^{n-2}} \right)^{1/n} \frac{1}{|a_n|^{1/n}}.
\]

This bound tends to \( 4/|a_n|^{1/n} \) and we can check the first few terms to see it is always less than the stated bound.

The constant in (2.4) of Lemma 2.7 is unfortunately not absolutely sharp. Indeed, R. Kershner [12, 13] has shown that the absolutely sharp constant when \( n = 2 \) can be given in terms of the Fresnel integrals, and is

\[
2\sqrt{2} \max_{\theta \in [0,2\pi]} \int_0^{\pi/2-\theta} \cos(x^2 + \theta) \, dx = 3.3643 \ldots
\]

When \( n \) is greater than two, the problem seems more complex. Some tentative numerical experiments have suggested that the polynomials \( a_1 x + a_3 x^3 \) with the property \( a_3/a_1^3 = -0.3547 \ldots \) are optimal for maximizing

\[
\max_{a,b} \left| \int_a^b e^{ia_1 x + a_3 x^3} \, dx \right| a_3^{1/3}.
\]

They share the property that they have local maxima that take the values \( \pm 0.5935 \ldots \). These polynomials appear to be nonstandard, so it may be difficult to find the other absolutely sharp constants in Lemma 2.7 or Corollary 2.8. On the other hand, as the maximum in (2.5) appears to be 2.6396 \ldots < 4, it may be possible to show that the asymptote in Corollary 2.8 is approached from below.
Chapter 3

Introduction to the $p$-adic numbers

For a more complete introduction to the $p$-adic numbers, see [26] or [31]. Here we will outline what we will need. Much of what is contained in the chapter is due to M.H. Taibleson [30, 31].

Fix a prime number $p$. Any nonzero rational number $x$ can be uniquely expressed in the form $p^k m/n$, where $m$ and $n$ have no common divisors and neither is divisible by $p$. Define the $p$-adic norm on the rational numbers by $|x| = p^{-k}$ when $x \neq 0$, and $|0| = 0$. We obtain the $p$-adic numbers by completing $\mathbb{Q}$ with respect to this norm. It is not difficult to show that the norm satisfies

$$|xy| = |x||y|,$$

and the ultrametric inequality,

$$|x + y| \leq \max\{|x|, |y|\}.$$

It follows from the ultrametric inequality, that every point within a ball can be considered to be its center. Similarly, it can be shown that two balls are either disjoint or one is contained in the other.

A nonzero $p$-adic number $x$ such that $|x| = p^{-k}$ may be written in the form

$$x = \sum_{j=k}^{\infty} x_j p^j,$$
where \(0 \leq x_j \leq p - 1\) and \(x_k \neq 0\). This will be called the standard \(p\)-adic expansion and the arithmetic of these expansions is done formally, with carrying.

As \(\mathbb{Q}_p\) is a locally compact commutative group, there is a Haar measure, that necessarily satisfies \(d(ax) = |a|\,dx\), where \(dx\) denotes an element of this measure. We normalize so that \(\{x \in \mathbb{Q}_p : |x| \leq p^r\}\) has measure \(p^r\).

In Chapter 5 we will be concerned with \(n\)-dimensional vector spaces over \(\mathbb{Q}_p\). To this end let \(\mathbb{Q}^n_p\) be the \(n\)-dimensional vector space over \(\mathbb{Q}_p\), and let \(|\cdot|\) denote the standard norm on \(\mathbb{Q}^n_p\) defined by

\[
|x| = \max_{1 \leq j \leq n} |x_j|.
\]

It is easy to show that this is also an ultrametric.

Balls defined using an ultrametric have some interesting properties. As

\[
\{y \in \mathbb{Q}^n_p : |y - x| < p^r\} = \{y \in \mathbb{Q}^n_p : |y - x| \leq p^{r-1}\},
\]

we see that they are both open and closed. Define \(B_k(x)\) by

\[
B_k(x) = \{y \in \mathbb{Q}^n_p : |y - x| \leq p^r\}.
\]

Balls that contain the origin are subgroups. Every point within a ball can be considered to be its center. Also, two balls are either disjoint or one is contained in the other.

A Haar measure is given by \(dx = dx_1 \ldots dx_n\), where \(dx_j\) is the Haar measure on the \(j\)th copy of \(\mathbb{Q}_p\). Hence \(|B_k(x)| = p^{nk}|\), and

\[
|\{x : |x| = p^k\}| = |B_k(x)| - |B_{k-1}(x)| = p^{nk} - p^{n(k-1)} = p^{nk}(1 - p^{-n}),
\]

so that the shell of a ball has nonzero measure.

We assume in general that all functions are complex-valued and Borel measurable. We define the compactly supported, locally constant functions to be the compactly supported functions that are constant on the cosets of some ball, and denote the space of these functions by \(\mathcal{S}(\mathbb{Q}^n_p)\).
3.1 The Hardy–Littlewood maximal theorem

We note that the proofs of the following results are general for all ultrametrics. In Chapter 5 we will define our balls using a different ultrametric, and use the analogous version of these results without further comment.

We begin with a covering lemma in the style of N. Wiener. We note that the technicality is significantly reduced when dealing with ultrametrics.

**Lemma 3.1.** Suppose that $E \subset \mathbb{Q}_p^n$ has finite Haar measure, and is covered by balls defined with an ultrametric. Suppose that the balls have uniformly bounded measure. Then there exists a countable and disjoint subcover.

**Proof.** We choose a refinement from the original cover. We start with a ball with largest measure, and discard all the balls contained within it. Then we choose a ball with largest measure from the remaining balls, and discard the balls which are contained within it. We continue until all the balls have been chosen or discarded.

The refinement is disjoint as any two balls are disjoint or one is contained in the other. It is countable as $E$ is of finite measure, and each ball has positive measure. Finally it is a cover, as we only discarded redundant elements of the original cover. 

We will now define the Hardy–Littlewood maximal function $\mathcal{M}f$ for a function $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$. Define $\Phi_k$ by

$$\Phi_k = \frac{1}{|B_k(0)|} 1_{B_k(0)},$$

where $1_{B_k(0)}$ denotes the characteristic function of $B_k(0)$, and $\mathcal{M}f$ by

$$\mathcal{M}f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|B_k(x)|} \int_{B_k(x)} |f(y)| \, dy = \sup_{k \in \mathbb{Z}} \Phi_k * |f|.$$ 

As all points within a ball can be considered to be its center, this is the analogue of both the centered and uncentered maximal function of the classical theory.
Theorem 3.2. Suppose that $Mf$ is defined as above. Then

$$|\{x \in \mathbb{Q}^n_p : Mf(x) > \alpha\}| \leq \frac{1}{\alpha} \|f\|_{L^1(\mathbb{Q}^n_p)} ,$$

(3.1)

for all $f \in L^1(\mathbb{Q}^n_p)$, and for $q > 1$,

$$\|Mf\|_{L^q(\mathbb{Q}^n_p)} \leq 2 \left( \frac{q}{q-1} \right)^{1/q} \|f\|_{L^q(\mathbb{Q}^n_p)}$$

for all $f \in L^q(\mathbb{Q}^n_p)$.

Proof. Let $F$ be any subset of $\{x \in \mathbb{Q}^n_p : Mf(x) > \alpha\}$ with finite measure. For each $x \in F$ there exists a ball $B(x)$ so that

$$|B(x)| < \frac{1}{\alpha} \int_{B(x)} |f(x)| \, dx \leq \frac{1}{\alpha} \|f\|_{L^1} .$$

We can apply Lemma 3.1 to the cover $\{B(x)\}_{x \in F}$, to leave a countable, disjoint subcover $\{B^j\}_{j \in \mathbb{N}}$. Thus

$$|F| \leq \sum_{j=0}^{\infty} |B_j| < \sum_{j=0}^{\infty} \frac{1}{\alpha} \int_{B^j} |f(x)| \, dx \leq \frac{1}{\alpha} \int_{\mathbb{Q}^n_p} |f(x)| \, dx .$$

Now this is true for all subsets of $\{x \in \mathbb{Q}^n_p : Mf(x) > \alpha\}$ with finite measure, so that

$$|\{x \in \mathbb{Q}^n_p : Mf(x) > \alpha\}| \leq \frac{1}{\alpha} \int_{\mathbb{Q}^n_p} |f(x)| \, dx ,$$

(3.2)

as desired.

To prove the second part, we split $f$ as $f^\alpha + f_\alpha$, where

$$f^\alpha(x) = \begin{cases} f(x) & \text{when } |f(x)| > \alpha/2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_\alpha(x) = \begin{cases} f(x) & \text{when } |f(x)| \leq \alpha/2 \\ 0 & \text{otherwise.} \end{cases}$$

As $\{x \in \mathbb{Q}^n_p : Mf_\alpha > \alpha/2\} = \emptyset$, we see that

$$\{Mf > \alpha\} \subseteq \{Mf^\alpha > \alpha/2\} \cup \{Mf_\alpha > \alpha/2\} = \{Mf^\alpha > \alpha/2\} ,$$

and

$$\{Mf > \alpha\} \subseteq \{Mf^\alpha > \alpha/2\} \cup \{Mf_\alpha > \alpha/2\} = \{Mf^\alpha > \alpha/2\} ,$$

as desired.
where \( \{Mf > \alpha\} \) denotes \( \{x \in \mathbb{Q}_p^n : Mf(x) > \alpha\} \) as usual. By (3.2) we have
\[
|\{Mf^\alpha > \alpha/2\}| \leq \frac{2}{\alpha} \int_{\mathbb{Q}_p^n} |f^\alpha(x)| \, dx = \frac{2}{\alpha} \int_{\{|f| > \alpha/2\}} |f(x)| \, dx,
\]
so that
\[
\int_{\mathbb{Q}_p^n} |Mf(x)|^q \, dx = \int_0^\infty q \alpha^{q-1} |\{Mf > \alpha\}| \, d\alpha
\leq 2q \int_0^\infty \alpha^{q-2} \int_{\{|f| > \alpha/2\}} |f(x)| \, dx \, d\alpha.
\]
Thus, by changing the order of integration, we obtain
\[
\int_{\mathbb{Q}_p^n} |Mf(x)|^q \, dx \leq 2q \int_{\mathbb{Q}_p^n} |f(x)| \, \int_0^{2|f|} \alpha^{q-2} \, d\alpha \, dx = \frac{2^q q}{q-1} \int_{\mathbb{Q}_p^n} |f(x)|^q \, dx,
\]
as desired. \( \square \)

The bound in (3.1) is absolutely sharp, and this is easily observed by considering \( f = 1_{B_0(0)} \). This is in contrast to the Euclidean case, where finding the sharp constant in the centered Hardy–Littlewood maximal theorem is nontrivial. In the one-dimensional case it was recently shown by A.D. Melas [16, 17], to be the largest root of the quadratic equation \( 12x^2 - 22x + 5 = 0 \), which is approximately \( 1.5675 \ldots \). This followed much work on the subject by A. Carbery, M. De Guzman, M. Trinidad Menarguez and F. Soria, R. Dror, S. Ganguli and R.S. Strichartz, J.M. Aldaz, and the author ([4], [10], [32], [8], [1], [23]).

An important corollary of the Hardy–Littlewood maximal theorem is the following differentiation theorem.

**Corollary 3.3.** Suppose that \( f \in L^1(\mathbb{Q}_p^n) \). Then for almost every \( x \in \mathbb{Q}_p^n \),
\[
\lim_{k \to -\infty} \frac{1}{|B_k(x)|} \int_{B_k(x)} f(y) \, dy = f(x).
\]

**Proof.** It will suffice to show that \( |E_\delta| = 0 \) for all \( \delta > 0 \), where
\[
E_\delta = \left\{ x \in \mathbb{Q}_p^n : \limsup_{k \to -\infty} \left| \frac{1}{|B_k(x)|} \int_{B_k(x)} f(y) \, dy - f(x) \right| > \delta \right\}.
\]
Let \( \epsilon > 0 \) and \( g_\epsilon \in \mathcal{S}(\mathbb{Q}_p^n) \) such that \( \|f - g_\epsilon\|_{L^1} \leq \epsilon \). Now when \( p^k \) is sufficiently small,
\[
\frac{1}{|B_k(x)|} \int_{B_k(x)} g_\epsilon(y) \, dy = g_\epsilon(x).
\]
for all \( x \in \mathbb{Q}_p^n \), so that
\[
\frac{1}{|B_k(x)|} \int_{B_k(x)} f(y) \, dy - f(x) \\
= \frac{1}{|B_k(x)|} \int_{B_k(x)} (f(y) - g_\epsilon(y)) \, dy + g_\epsilon(x) - f(x).
\]
Thus, \( E_\delta \subset A_\delta \cup B_\delta \), where
\[
A_\delta = \left\{ x \in \mathbb{Q}_p^n : \limsup_{k \to -\infty} \frac{1}{|B_k(x)|} \int_{B_k(x)} (f(y) - g_\epsilon(y)) \, dy > \delta/2 \right\}
\]
and
\[
B_\delta = \{ x \in \mathbb{Q}_p^n : |g_\epsilon(x) - f(x)| > \delta/2 \}.
\]
Now by Theorem 3.2,
\[
\left| \left\{ x \in \mathbb{Q}_p^n : \limsup_{k \to -\infty} \frac{1}{|B_k(x)|} \int_{B_k(x)} (f(y) - g_\epsilon(y)) \, dy > \delta/2 \right\} \right| \leq \frac{2}{\delta} \| f - g_\epsilon \|_{L^1},
\]
and by Chebyshev’s inequality,
\[
\left| \{ x \in \mathbb{Q}_p^n : |g_\epsilon(x) - f(x)| > \delta/2 \} \right| \leq \frac{2}{\delta} \| f - g_\epsilon \|_{L^1},
\]
so that
\[
|E_\delta| \leq \frac{2}{\delta} \| f - g_\epsilon \|_{L^1} + \frac{2}{\delta} \| f - g_\epsilon \|_{L^1} = \frac{4\epsilon}{\delta}.
\]
Finally we let \( \epsilon \) tend to zero to see that \( |E_\delta| = 0 \).

\[ \Box \]

### 3.2 Fourier analysis on the \( p \)-adic numbers

In order to do Fourier analysis we will need an understanding of the characters of \( \mathbb{Q}_p \) and \( \mathbb{Q}_p^n \). Define \( \chi : \mathbb{Q}_p \to \mathbb{C} \) by
\[
\chi(x) = \begin{cases} 
\prod_{j=k}^{1} e^{2\pi i x_j/p^j} & \text{when } |x| > 1 \\
1 & \text{otherwise}.
\end{cases}
\]
The additive characters of \( \mathbb{Q}_p \) are of the form \( \chi_a : \mathbb{Q}_p \to \mathbb{C} \);
\[
\chi_a(x) = \chi(ax),
\]
where \( a \in \mathbb{Q}_p \), and the additive characters of \( \mathbb{Q}_p^n \) are of the form \( \chi_a : \mathbb{Q}_p^n \rightarrow \mathbb{C} \):

\[
\chi_a(x) = \chi(a \cdot x),
\]

where \( a \in \mathbb{Q}_p^n \) and \( a \cdot x = a_1x_1 + \ldots + a_nx_n \).

The following lemmas will be crucial to our calculations. They can be proven using the fact that balls in \( \mathbb{Q}_p \) have multiple centers.

**Lemma 3.4.** Suppose that \( a \in \mathbb{Q}_p \) and \( |a| > 1 \). Then

\[
\int_{|x| \leq 1} \chi(ax) \, dx = 0.
\]

**Proof.** First we consider the standard expansion of \( a \), so that

\[
a = \sum_{j=-k}^{\infty} a_j p^j,
\]

where \( k \geq 1 \) and \( a_{-k} \neq 0 \). Now as

\[
\{ x \in \mathbb{Q}_p : |x| \leq 1 \} = \{ x \in \mathbb{Q}_p : |x - p^{k-1}| \leq 1 \},
\]

we have

\[
I = \int_{|x| \leq 1} \chi(ax) \, dx = \int_{|x - p^{k-1}| \leq 1} \chi(ax) \, dx.
\]

If we let \( y = x - p^{k-1} \), we see that

\[
I = \int_{|y| \leq 1} \chi(a(y + p^{k-1})) \, dy = \chi(ap^{k-1}) \int_{|y| \leq 1} \chi(ay) \, dy,
\]

so that

\[
I = \chi(ap^{k-1})I.
\]

Now as \( \chi(ap^{k-1}) = e^{2\pi ia_{-k}/p} \neq 1 \), we see that \( I = 0 \).

The following multidimensional version will be convenient in the next section.

**Lemma 3.5.** Suppose that \( a \in \mathbb{Q}_p^n \). Then

\[
\int_{|x| \leq p^k} \chi(a \cdot x) \, dx = \begin{cases} p^{nk} & \text{when } |a| \leq p^{-k} \\ 0 & \text{otherwise.} \end{cases}
\]
Proof. By Fubini’s theorem (see, for example, [24, p. 164]),

\[
I = \int_{B_k(0)} \chi(a \cdot x) \, dx = \int_{|x_1| \leq p^k} \ldots \int_{|x_n| \leq p^k} \chi(a_1 x_1) \ldots \chi(a_n x_n) \, dx_1 \ldots dx_n,
\]

and by changing variables,

\[
I = p^{nk} \int_{|z_1| \leq 1} \ldots \int_{|z_n| \leq 1} \chi(a_1 p^{-k} z_1) \ldots \chi(a_n p^{-k} z_n) \, dz_1 \ldots dz_n.
\]

If \(|a| \leq p^{-k}\), then \(|p^{-k} a_j| \leq 1\) for each \(j = 1, \ldots, n\), so that

\[
\chi(p^{-k} a_j z_j) = 1
\]

for all \(|z_j| \leq 1\), as \(\chi\) is trivial on \(\{x \in \mathbb{Q}_p : |x| \leq 1\}\). Thus

\[
\int_{|z_j| \leq 1} \chi(p^{-k} a_j z_j) \, dz_j = 1
\]

for all \(j = 1, \ldots, n\), so that \(I = p^{nk}\).

If \(|a| > p^{-k}\), then \(|p^{-k} a_j| > 1\) for some \(j\). Thus, by Lemma 3.4,

\[
\int_{|z_j| \leq 1} \chi(p^{-k} a_j z_j) \, dz_j = 0
\]

for some \(j\), so that \(I = 0\). \(\square\)

We note that the integral in Lemma 3.4 is with respect to \(x \in \mathbb{Q}_p\), and that the integral in Lemma 3.5 is with respect to \(x \in \mathbb{Q}_p^n\). For the remainder of this chapter, integrals will be with respect to an \(n\)-dimensional variable as in Lemma 3.5.

Finally, we present the result in a form that will be useful for calculating the Fourier transform of radial functions.

Lemma 3.6. Suppose that \(a \in \mathbb{Q}_p^n\). Then

\[
\int_{|x|=p^k} \chi(a \cdot x) \, dx = \begin{cases} 
p^{nk}(1 - p^{-n}) & \text{when } |a| \leq p^{-k} \\
-|a|^{-n} & \text{when } |a| = p^{-k+1} \\
0 & \text{otherwise.}
\end{cases}
\]
Proof. We apply Lemma 3.5 in each of the cases. When $|a| \leq p^{-k}$,

$$\int_{|x|=p^k} \chi(a \cdot x) \, dx = \int_{|x|\leq p^k} \chi(a \cdot x) \, dx - \int_{|x|\leq p^{k-1}} \chi(a \cdot x) \, dx = p^{nk} - p^{n(k-1)} = p^{ak}(1 - p^{-n}).$$

Similarly, when $|a| = p^{-k+1}$,

$$\int_{|x|=p^k} \chi(a \cdot x) \, dx = \int_{|x|\leq p^k} \chi(a \cdot x) \, dx - \int_{|x|\leq p^{k-1}} \chi(a \cdot x) \, dx = 0 - p^{n(k-1)} = -|a|^{-n}.$$

Finally, when $|a| > p^{-k+1}$,

$$\int_{|x|=p^k} \chi(a \cdot x) \, dx = \int_{|x|\leq p^k} \chi(a \cdot x) \, dx - \int_{|x|\leq p^{k-1}} \chi(a \cdot x) \, dx = 0 - 0 = 0,$$

as desired. \qed

The Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{Q}_p^n} f(x) \overline{\chi_{\xi}(x)} \, dx = \int_{\mathbb{Q}_p^n} f(x) \chi(-\xi \cdot x) \, dx$$

for all $f \in L^1(\mathbb{Q}_p^n)$, and

$$\hat{\mu}(\xi) = \int_{\mathbb{Q}_p^n} \chi_{\xi}(x) \, d\mu(x) = \int_{\mathbb{Q}_p^n} \chi(-\xi \cdot x) \, d\mu(x)$$

for all finite Borel measures $\mu$. The Fourier transform maps functions in $\mathcal{S}(\mathbb{Q}_p^n)$ to functions in $\mathcal{S}(\mathbb{Q}_p^n)$, and $\mathcal{S}(\mathbb{Q}_p^n)$ is dense in $L^q(\mathbb{Q}_p^n)$, where $1 \leq q < \infty$. Thus, $\mathcal{S}(\mathbb{Q}_p^n)$ will take the role of the Schwartz function space.

Convolution is defined as usual by

$$f \ast g(x) = \int_{\mathbb{Q}_p^n} f(x-y)g(y) \, dy = \int_{\mathbb{Q}_p^n} f(y)g(x-y) \, dy = g \ast f(x),$$

and

$$\mu \ast f(x) = \int_{\mathbb{Q}_p^n} f(x-y) \, d\mu(y) = f \ast \mu(x),$$

for all functions $f$, $g$, and finite Borel measures $\mu$ for which the integral is defined. The following three results follow easily from the definitions and Fubini’s theorem.

23
Theorem 3.7. Suppose that $f, g \in L^1(Q^p_n)$, and $\mu$ is a finite Borel measure. Then

$$\|f * g\|_{L^1(Q^p_n)} \leq \|f\|_{L^1(Q^p_n)} \|g\|_{L^1(Q^p_n)},$$

and

$$\|f * \mu\|_{L^1(Q^p_n)} \leq \|f\|_{L^1(Q^p_n)} \|\mu\|_M,$$

where $\|\mu\|_M$ is the total variation of $\mu$.

Proof. If $f, g \in L^1(Q^p_n)$, then

$$\|f * g\|_{L^1} = \left\| \int_{Q^p_n} f(x - y)g(y) \, dy \right\|_{L^1} \leq \int_{Q^p_n} \int_{Q^p_n} |f(x - y)| \, dx |g(y)| \, dy = \|f\|_{L^1} \|g\|_{L^1},$$

by Fubini’s theorem. Similarly, when $\mu$ is a finite Borel measure

$$\|f * \mu\|_{L^1} = \left\| \int_{Q^p_n} f(x - y) \, d\mu(y) \right\|_{L^1} \leq \int_{Q^p_n} \int_{Q^p_n} |f(x - y)| \, dx \, d\mu(y) = \|f\|_{L^1} \|\mu\|_M,$$

again by Fubini’s theorem. \qed

Theorem 3.8. Suppose that $f, g \in L^1(Q^p_n)$ and $\mu$ is a finite Borel measure. Then $\hat{f} * g = \hat{f} \hat{g}$, and $\hat{\mu} * \hat{f} = \hat{\mu} \hat{f}$.

Proof. If $f, g \in L^1(Q^p_n)$, then $f * g \in L^1(Q^p_n)$ by Theorem 3.7. Now

$$\hat{f} \hat{g}(\xi) = \int_{Q^p_n} \int_{Q^p_n} f(x - y)g(y) \, dy \overline{\chi(\xi)}(x) \, dx$$

$$= \int_{Q^p_n} \int_{Q^p_n} f(x - y)g(y) \chi(-\xi \cdot (x - y)) \chi(-\xi \cdot y) \, dy \, dx$$

$$= \int_{Q^p_n} \int_{Q^p_n} f(z)g(y) \chi(-\xi \cdot z) \chi(-\xi \cdot y) \, dy \, dz,$$

by a change of variables. Thus

$$\hat{f} \hat{g}(\xi) = \int_{Q^p_n} f(z) \chi(-\xi \cdot z) \, dz \int_{Q^p_n} g(y) \chi(-\xi \cdot y) \, dy = \hat{f}(\xi) \hat{g}(\xi),$$
by Fubini’s theorem.

Similarly when \( \mu \) is a finite Borel measure,

\[
\hat{f} * \mu(\xi) = \int_{\mathbb{Q}^n_p} f(x - y) d\mu(y) \chi_x(x) dx
\]

\[
= \int_{\mathbb{Q}^n_p} \int_{\mathbb{Q}^n_p} f(x - y) \chi(-\xi \cdot (x - y)) \chi(-\xi \cdot y) d\mu(y) dx
\]

\[
= \int_{\mathbb{Q}^n_p} \int_{\mathbb{Q}^n_p} f(z) \chi(-\xi \cdot z) \chi(-\xi \cdot y) d\mu(y) dz,
\]

by a change of variables, so that

\[
\hat{f} * \mu(\xi) = \int_{\mathbb{Q}^n_p} f(z) \chi(-\xi \cdot z) dz \int_{\mathbb{Q}^n_p} \chi(-\xi \cdot y) d\mu(y) = \hat{f}(\xi)\hat{\mu}(\xi),
\]

by Fubini’s theorem. \( \square \)

**Theorem 3.9.** Suppose that \( f, g \in L^1(\mathbb{Q}^n_p) \). Then

\[
\int_{\mathbb{Q}^n_p} \hat{f}(x)g(x) dx = \int_{\mathbb{Q}^n_p} f(\xi)\hat{g}(\xi) d\xi.
\]

**Proof.** If \( f, g \in L^1(\mathbb{Q}^n_p) \), then

\[
\int_{\mathbb{Q}^n_p} \hat{f}(x)g(x) dx = \int_{\mathbb{Q}^n_p} \int_{\mathbb{Q}^n_p} f(\xi) \chi(-x \cdot \xi) d\xi g(x) dx
\]

\[
= \int_{\mathbb{Q}^n_p} f(\xi) \int_{\mathbb{Q}^n_p} g(x) \chi(-x \cdot \xi) dx d\xi = \int_{\mathbb{Q}^n_p} f(\xi)\hat{g}(\xi) d\xi,
\]

by Fubini’s theorem. \( \square \)

The following lemma will be the key to proving the Fourier inversion formula. It demonstrates that the localization in space of \( f \) corresponds to the localization in ‘frequency’ of \( \hat{f} \) and vice versa.

**Lemma 3.10.** Suppose that \( f \in L^1(\mathbb{Q}^n_p) \) and \( k \in \mathbb{Z} \). Then

\[
\int_{B_k(0)} \hat{f}(\xi) \chi_x(\xi) d\xi = p^{nk} \int_{B_{-k}(x)} f(y) dy.
\]

**Proof.** We begin by noting that

\[
\int_{B_k(0)} \hat{f}(\xi) \chi_x(\xi) d\xi = \int_{\mathbb{Q}^n_p} \hat{f}(\xi) \chi_x(\xi) \chi_{B_k(0)}(\xi) d\xi = \int_{\mathbb{Q}^n_p} f(y)(\chi_x \chi_{B_k(0)})(y) dy,
\]
by Theorem 3.9. Now
\[
(\chi x 1_{B_k(0)})(y) = \int_{B_k(0)} \chi(x \cdot z) \chi(-z \cdot y) \, dz = \int_{B_k(0)} \chi((x - y) \cdot z) \, dz,
\]
so that by Lemma 3.5,
\[
(\chi x 1_{B_k(0)})(y) = \begin{cases} 
p^n_k & \text{when } |x - y| \leq p^{-k} \\
0 & \text{otherwise} \end{cases}
\]
Thus
\[
\int_{B_k(0)} \hat{f}(y) \chi x(y) \, dy = \int_{Q^n_p} f(y) p^n_k 1_{-B_k(x)} \, dy = p^n_k \int_{-B_k(x)} f(y) \, dy,
\]
as desired. \( \square \)

We combine this and Corollary 3.3 to obtain the following version of the Fourier inversion theorem.

**Theorem 3.11.** Suppose that \( f \in L^1(\mathbb{Q}^n_p) \). Then for almost every \( x \in \mathbb{Q}^n_p \),
\[
\int_{B_k(0)} \hat{f}(y) \chi x(y) \, dy \to f(x)
\]
as \( k \to \infty \). In particular, the integral converges at each point of continuity of \( f \).

**Proof.** By Lemma 3.10,
\[
\int_{B_k(0)} \hat{f}(y) \chi x(y) \, dy = p^n_k \int_{-B_k(x)} f(y) \, dy = \frac{1}{|B_k(x)|} \int_{-B_k(x)} f(y) \, dy.
\]
Thus, by Corollary 3.3,
\[
\lim_{k \to \infty} \int_{B_k(0)} \hat{f}(y) \chi x(y) \, dy = \lim_{k \to \infty} \frac{1}{|B_k(x)|} \int_{-B_k(x)} f(y) \, dy = f(x)
\]
for almost every \( x \in \mathbb{Q}^n_p \). In particular, this is true at each point of continuity. \( \square \)

Finally, we obtain Plancherel’s theorem.

**Theorem 3.12.** Suppose that \( f \in L^1(\mathbb{Q}^n_p) \cap L^2(\mathbb{Q}^n_p) \). Then
\[
\|f\|_{L^2(\mathbb{Q}^n_p)} = \|\hat{f}\|_{L^2(\mathbb{Q}^n_p)}.
\]
Proof. Let \( g(x) = \overline{f(-x)} \), so that
\[
\int_{Q_n^p} |f(x)|^2 \, dx = \int_{Q_n^p} \overline{f(x)} f(x) \, dx = \int_{Q_n^p} g(-x) f(x) \, dx = f * g(0),
\]
and \( \hat{f} \hat{g} = |\hat{f}|^2 \geq 0 \), by Theorem 3.8. By the Cauchy–Schwarz inequality,
\[
|f * g(x+y) - f * g(x)| = \left| \int_{Q_n^p} (f(x+y-z) - f(x-z)) g(z) \, dz \right|
\leq \left( \int_{Q_n^p} |f(x+y-z) - f(x-z)|^2 \, dz \right)^{1/2} \|g\|_{L^2}
\leq \|f(\cdot + y) - f(\cdot)\|_{L^2} \|g\|_{L^2}.
\]
Let \( \epsilon > 0 \) and \( h \in S(Q_n^p) \), so that \( \|f - h\|_{L^2} \leq \epsilon \), then
\[
\|f(\cdot + y) - f(\cdot)\|_{L^2}
\leq \|f(\cdot + y) - h(\cdot + y)\|_{L^2} + \|h(\cdot + y) - h(\cdot)\|_{L^2} + \|h(\cdot) - f(\cdot)\|_{L^2}
\leq \|h(\cdot + y) - h(\cdot)\|_{L^2} + 2\epsilon.
\]
Now as \( h \in S(Q_n^p) \), we have \( h(\cdot + y) - h(\cdot) \equiv 0 \) when \( y \) is sufficiently small. Hence \( \|f(\cdot + y) - f(\cdot)\|_{L^2} \) tends to zero as \( y \) tends to zero, and \( f * g \) is continuous. Thus,
\[
f * g(0) = \lim_{k \to \infty} \int_{B_k(0)} \hat{f} * g(x) \chi_0(x) \, dx = \lim_{k \to \infty} \int_{B_k(0)} \hat{f}(x)^2 \, dx,
\]
by Theorem 3.11. Finally as \( |\hat{f}|^2 \geq 0 \),
\[
\lim_{k \to \infty} \int_{B_k(0)} \hat{f}(x)^2 \, dx = \int_{Q_n^p} |\hat{f}(x)|^2 \, dx,
\]
by the monotone convergence theorem (see, for example, [24, p. 21]), so that
\[
\int_{Q_n^p} |f(x)|^2 \, dx = f * g(0) = \int_{Q_n^p} |\hat{f}(x)|^2 \, dx,
\]
by (3.3), (3.4) and (3.5).

As \( L^1(Q_n^p) \cap L^2(Q_n^p) \) is dense in \( L^2(Q_n^p) \) we can extend the definition of the Fourier transform to the whole of \( L^2(Q_n^p) \), as usual.
3.3 Bessel potentials on the $p$-adic numbers

It will be convenient to define an $n$-dimensional gamma function $\Gamma_n$ by

$$\Gamma_n(s) = \frac{1 - p^{-n-s}}{1 - p^s},$$

where $s$ is a nonzero complex number. The following theorem demonstrates that it is reasonable to describe this as a gamma function.

**Theorem 3.13.** Suppose that $\text{Re}(s) < 0$ and $|y| = 1$. Then

$$\Gamma_n(s) = \lim_{k \to \infty} \int_{p^{-k} \leq |x| \leq p^k} |x|^{-n-s} \chi_y(x) \, dx.$$

**Proof.** As $|y| = 1$, we see by Lemma 3.6 that

$$\lim_{k \to \infty} \int_{p^{-k} \leq |x| \leq p^k} |x|^{-n-s} \chi_y(-y \cdot x) \, dx = -p^{-n-s} + \sum_{k=0}^{\infty} p^{(n+s)k} p^{-nk} (1 - p^{-n}).$$

Now as $\text{Re}(s) < 0$,

$$\sum_{k=0}^{\infty} p^{(n+s)k} p^{-nk} (1 - p^{-n}) = (1 - p^{-n}) \sum_{k=0}^{\infty} p^{sk} = \frac{1 - p^{-n}}{1 - p^s}.$$

Thus

$$\lim_{k \to \infty} \int_{p^{-k} \leq |x| \leq p^k} |x|^{-n-s} \chi_y(x) \, dx = -p^{-n-s} + \frac{1 - p^{-n}}{1 - p^s} = \frac{1 - p^{-n-s}}{1 - p^s} = \Gamma_n(s),$$

as desired. \qed

We will require a $p$-adic analogue of the Bessel potential $(1 + |x|^2)^{s/2}$. Note that

$$(1 + |x|^2)^{s/2} = ((1 + |x|^2)^{1/2})^s = \|(1, x)\|^s,$$

where $\|(1, x)\|$ is the $(n + 1)$-dimensional Euclidean norm of $(1, x) \in \mathbb{R}^{n+1}$. Now the $(n + 1)$-dimensional $p$-adic norm of $(1, x) \in \mathbb{Q}_p^{n+1}$ is $\max\{1, |x|\}$, so, following Taibleson [30, 31], we define the $n$-dimensional $p$-adic Bessel potential $J^s : \mathbb{Q}_p^n \to \mathbb{C}$ by

$$J^s(x) = \max\{1, |x|\}^s,$$  \hspace{1cm} (3.6)
where $s \in \mathbb{C}$. We also define $K_s : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ by

$$K_s(x) = \begin{cases} \frac{|x|^{-n-s} - p^{-n-s}}{\Gamma_n(s)} & \text{when } |x| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

when $s \neq -n$, and

$$K_{-n}(x) = \begin{cases} (1 - p^{-n}) \log_p (p/|x|) & \text{when } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 3.14.** Suppose that $\Re(s) < 0$ and $K_s$ is defined as above. Then

$$\int_{\mathbb{Q}_p^n} K_s(x) \, dx = 1.$$

**Proof.** When $s \neq -n$,

$$\int_{\mathbb{Q}_p^n} K_s(x) \, dx = \int_{|x| \leq 1} \frac{|x|^{-n-s} - p^{-n-s}}{\Gamma_n(s)} \, dx = \frac{1}{\Gamma_n(s)} \left( \int_{|x| \leq 1} |x|^{-n-s} \, dx - p^{-n-s} \right).$$

Now as $|\{x \in \mathbb{Q}_p^n : |x| = p^k\}| = p^{nk} (1 - p^{-n})$,

$$\int_{|x| \leq 1} |x|^{-n-s} \, dx = \sum_{k=0}^{\infty} p^{(n+s)k} p^{-nk} (1 - p^{-n}) = (1 - p^{-n}) \sum_{k=0}^{\infty} p^{sk} = \frac{1}{1 - p^{-n}},$$

so that

$$\int_{\mathbb{Q}_p^n} K_s(x) \, dx = \frac{1}{\Gamma_n(s)} \left( -p^{-n-s} + \frac{1 - p^{-n}}{1 - p^{-n}} \right) = 1.$$

Similarly, when $s = -n$,

$$\int_{\mathbb{Q}_p^n} K_{-n}(x) \, dx = \int_{|x| \leq 1} (1 - p^{-n}) \log_p (p/|x|) \, dx,$$

so that

$$\int_{\mathbb{Q}_p^n} K_{-n}(x) \, dx = (1 - p^{-n}) \sum_{k=0}^{\infty} \log_p (p^{k+1}) p^{-nk} (1 - p^{-n}) = (1 - p^{-n})^2 \sum_{k=0}^{\infty} (k + 1) p^{-nk} = 1,$$

as desired. \qed
In Chapter 5 we will require the following results.

**Proposition 3.15.** Suppose that \( \text{Re}(s) < 0 \), and \( J^* \) and \( K_s \) are defined as above. Then \( \hat{J}^*(\xi) = K_s(\xi) \).

**Proof.** It is clear, from the proof of Lemma 3.14, that \( K_s \in L^1(Q^n) \) when \( \text{Re}(s) < 0 \). Thus, by Theorem 3.11, it remains to show that \( \hat{K}_s = J^* \).

Suppose that \( |\xi| \leq 1 \). We have \( \chi(\xi \cdot x) = 1 \) for all \( |x| \leq 1 \), so that

\[
\hat{K}_s(\xi) = \int_{Q^n} K_s(x) \overline{\chi(x)} \, dx = \int_{Q^n} K_s(x) \, dx = 1
\]

for all \( s \in \mathbb{C} \) with negative real part, by Lemma 3.14.

Suppose that \( |\xi| > 1 \), and that \( |\xi| = p^l \), where \( l \in \mathbb{N} \). When \( s \neq -n \),

\[
\hat{K}_s(\xi) = \frac{1}{\Gamma_n(s)} \int_{|x| \leq 1} |x|^{-n-s} \overline{\chi(x)} \, dx - \frac{p^{-n-s}}{\Gamma_n(s)} \int_{|x| \leq 1} \overline{\chi(x)} \, dx
\]

\[
= \frac{1}{\Gamma(s)} \int_{|x| \leq 1} |x|^{-n-s} \chi(-\xi \cdot x) \, dx,
\]

as the second integral is zero by Lemma 3.5. Now

\[
\int_{|x| \leq 1} |x|^{-n-s} \chi(-\xi \cdot x) \, dx = \int_{|x| \leq |\xi|^{-1}} |x|^{-n-s} \, dx - (p|\xi|^{-1})^{-n-s} |\xi|^{-n}
\]

by Lemma 3.6, and

\[
\int_{|x| \leq |\xi|^{-1}} |x|^{-n-s} \, dx = \sum_{k=l}^{\infty} p^{(k+s)} p^{-k n} (1 - p^{-n})
\]

\[
= (1 - p^{-n}) \sum_{k=0}^{\infty} p^{(k+l)s} = |\xi|^s \frac{1 - p^{-n}}{1 - p^s}.
\]

Thus

\[
\hat{K}_s(\xi) = \frac{|\xi|^s}{\Gamma_n(s)} \left( \frac{1 - p^{-n}}{1 - p^s} - p^{-n-s} \right) = |\xi|^s.
\]

When \( s = -n \),

\[
\hat{K}_{-n}(\xi) = (1 - p^{-n}) \int_{|x| \leq |\xi|^{-1}} \log_p (p/|x|) \, dx - (1 - p^{-n}) \log_p (|\xi|) |\xi|^{-n}
\]

\[
= (1 - p^{-n}) \left( \sum_{k=l}^{\infty} \log_p (p^{k+1}) p^{-n k} (1 - p^{-n}) - \log_p (p^l) p^{-n l} \right),
\]
by Lemma 3.6. Thus

\[
\hat{K}_n(\xi) = (1 - p^{-n}) \left( (1 - p^{-n}) \sum_{k=0}^{\infty} (k + 1 + l) p^{-n(k+l)} - lp^{-nl} \right)
\]

\[
= (1 - p^{-n}) p^{-nl} \left( (1 - p^{-n}) \sum_{k=0}^{\infty} (k + 1) p^{-nk} + l(1 - p^{-n}) \sum_{k=0}^{\infty} p^{-nk} - l \right)
\]

\[
= (1 - p^{-n}) p^{-nl} \left( \frac{1 - p^{-n}}{(1 - p^{-n})^2} + l \frac{1 - p^{-n}}{1 - p^{-n}} - l \right) = |\xi|^{-n},
\]

and we are done. \qed

Finally we prove the following simple lemma. We restrict ourselves to the generality that we will require in the next chapter.

**Lemma 3.16.** Suppose that $J^*$ is defined as in (3.6) and that $\text{Re}(s) < 0$. Then there is a constant $C_s$ so that

\[
\|\hat{J}^*(\cdot + y) - \hat{J}^*(\cdot)\|_{L^1(\mathbb{Q}_p^n)} \leq C_s |y|^{-\text{Re}(s)}
\]

for all $y \in \mathbb{Q}_p^n$.

**Proof.** Suppose first that $|y| \geq 1$. When $\text{Re}(s) < 0$, we have $\hat{J}^* \in L^1(\mathbb{Q}_p^n)$, so that

\[
\|\hat{J}^*(\cdot + y) - \hat{J}^*(\cdot)\|_{L^1} \leq 2\|\hat{J}^*\|_{L^1} \leq 2\|\hat{J}^*\|_{L^1} |y|^{-\text{Re}(s)}.
\]

Now suppose that $|y| \leq 1$, and that $|y| = p^{-l}$, where $l \in \mathbb{N}$. As $|x + y| = |y|$ when $|y| > |x|$, and $\hat{J}^*$ is radial,

\[
\|\hat{J}^*(\cdot + y) - \hat{J}^*(\cdot)\|_{L^1} = \int_{|x| \leq |y|} |\hat{J}^*(x + y) - \hat{J}^*(x)| \, dx.
\]

Now

\[
\int_{|x| \leq |y|} |\hat{J}^*(x + y) - \hat{J}^*(x)| \, dx = \int_{|x| < |y|} |\hat{J}^*(x + y) - \hat{J}^*(x)| \, dx
\]

\[
+ \int_{|x+y| < |y|} |\hat{J}^*(x + y) - \hat{J}^*(x)| \, dx
\]

\[
+ \int_{|x+y| = |y| = |x|} |\hat{J}^*(x + y) - \hat{J}^*(x)| \, dx,
\]

31
and the third integral is zero, as $\hat{J}^s$ is radial. By changes of variables,

$$
\int_{|x+y|<|y|} |\hat{J}^s(x+y) - \hat{J}^s(x)| \, dx = \int_{|z|<|y|} |\hat{J}^s(z) - \hat{J}^s(z-y)| \, dz
$$

$$
= \int_{|z|<|y|} |\hat{J}^s(z+y) - \hat{J}^s(z)| \, dz,
$$

so that the first two integrals are equal. Thus

$$
\|\hat{J}^s(\cdot + y) - \hat{J}^s(\cdot)\|_{L^1} = 2 \int_{|x|<|y|} |\hat{J}^s(x+y) - \hat{J}^s(x)| \, dx.
$$

When $s \neq -n$,

$$
\|\hat{J}^s(\cdot + y) - \hat{J}^s(\cdot)\|_{L^1} = \frac{2}{\Gamma_n(s)} \int_{|x|<|y|} \left| x + y \right|^{-n-s} - |x|^{-n-s} \, dx
$$

$$
\leq \frac{2}{\Gamma_n(s)} \left( \int_{|x|<|y|} |y|^{-n-\text{Re}(s)} \, dx + \int_{|x|<|y|} |x|^{-n-\text{Re}(s)} \, dx \right).
$$

Now

$$
\int_{|x|<|y|} |y|^{-n-\text{Re}(s)} \, dx = \left( \frac{|y|}{p} \right)^n |y|^{-n-\text{Re}(s)} = \frac{1}{p^n} |y|^{-\text{Re}(s)},
$$

and

$$
\int_{|x|<|y|} |x|^{-n-\text{Re}(s)} \, dx \leq \int_{|x|\leq|y|} |x|^{-n-\text{Re}(s)} \, dx = \sum_{k=l}^{\infty} p^{k(n+\text{Re}(s))} p^{-kn}(1 - p^{-n})
$$

$$
= \sum_{k=0}^{\infty} p^{(k+l)\text{Re}(s)}(1 - p^{-n})
$$

$$
= \frac{1 - p^{-n}}{1 - p^{\text{Re}(s)}} |y|^{-\text{Re}(s)},
$$

which completes the proof when $s \neq -n$.

When $s = -n$, let $|x| = p^{-k}$, where $k \in \mathbb{N}$. As $|x| < |y|$, we have $k > l$. Now

$$
\hat{J}^{-n}(x+y) = \hat{J}^{-n}(y) = (1 - p^{-n}) \log_p(p/|y|) = (1 - p^{-n})(l + 1),
$$

and

$$
\hat{J}^{-n}(x) = (1 - p^{-n}) \log_p(p/|x|) = (1 - p^{-n})(k + 1).
$$

Thus

$$
|\hat{J}^{-n}(x+y) - \hat{J}^{-n}(x)| = (1 - p^{-n})(k - l),
$$
so that

\[
\int_{|x|<|y|} |\hat{J}^{-n}(x + y) - \hat{J}^{-n}(x)| \, dx = (1 - p^{-n})^2 \sum_{k=l+1}^{\infty} (k - l)p^{-nk}
\]

\[
= (1 - p^{-n})^2 p^{-n(l+1)} \sum_{k=l+1}^{\infty} (k - l)p^{-n(k-l-1)}.
\]

Finally,

\[
\|\hat{J}^{-n}(\cdot + y) - \hat{J}^{-n}(\cdot)\|_{L^1} = 2(1 - p^{-n})^2 p^{-n(l+1)} \sum_{k=0}^{\infty} (k + 1)p^{-nk} = 2p^{-n}|y|^n,
\]

which completes the proof. \qed
Chapter 4

A van der Corput lemma for the $p$-adic numbers

We will prove a $p$-adic van der Corput lemma for polynomials, opening the way for the study of oscillatory integrals on the $p$-adics. This problem was first considered by J. Wright [33], who proved lemmas for polynomials of degree two and monomials of degree three. The result for all polynomials will be obtained as a corollary from the following lemma.

**Lemma 4.1.** Suppose that $a_1, \ldots, a_n \in \mathbb{Q}_p$. Then

$$\left| \int_{|x| \leq 1} \chi(a_1 x + \cdots + a_n x^n) \, dx \right| \leq \frac{p^m}{|ma_m|^{1/m}},$$

where $m = \max\{l : |la_l| \geq |ja_j| \text{ for all } j \neq l\}$.

We note that the integral in Lemma 4.1 is with respect to $x \in \mathbb{Q}_p$. All integrals in this chapter will be with respect to a scalar variable. Before proving Lemma 4.1 we note some corollaries.

**Corollary 4.2.** Suppose that $a_1, \ldots, a_n \in \mathbb{Q}_p$. Then

$$\left| \int_{|x| \leq 1} \chi(a_1 x + \cdots + a_n x^n) \, dx \right| \leq \frac{2p^n}{\lambda^{1/n}},$$

where $\lambda = \max_{1 \leq j \leq n} |a_j|$.  

35
Proof. Suppose that $|a_k| = \max_{1 \leq j \leq n} |a_j| = \lambda$. By Lemma 4.1, we have

$$|I| = \left| \int_{|x| \leq 1} \chi(a_1 x + \cdots + a_n x^n) \, dx \right| \leq \frac{p^m}{|ma_m|^{1/m}},$$

where $m = \max \{ l : |la_l| \geq |ja_j| \text{ for all } j \neq l \}$. Now as $|ma_m| \geq |ka_k|$, we have

$$|I| \leq \frac{p^m}{|ka_k|^{1/m}} \leq \frac{k^{1/m} p^m}{|a_k|^{1/m}} \leq \frac{n^{1/m} p^m}{\lambda^{1/n}}.$$

Finally, it is easy to calculate that $n^{1/m} p^m \leq 2p^n$ for all $m = 1, \ldots, n$, so that

$$|I| \leq \frac{n^{1/m} p^m}{\lambda^{1/n}} \leq \frac{2p^n}{\lambda^{1/n}},$$

as desired. \hfill \Box

We will need some notation. Let

$$f(y) = a_0 + a_1 y + \cdots + a_n y^n,$$

and for all $j = 1, \ldots, n - 1$, define $b_j$ by

$$b_j(y) = \frac{f^{(j)}(y)}{j!} = a_j + \binom{j+1}{j} a_{j+1} y + \cdots + \binom{n}{j} a_n y^{n-j}. \ (4.1)$$

Then,

$$a_0 + a_1 (x + y) + \cdots + a_n (x + y)^n$$

$$= f(y) + f'(y) x + \frac{f''(y)}{2!} x^2 + \cdots + \frac{f^{(n)}(y)}{n!} x^n$$

$$= a_0 + \cdots + a_n y^n + b_1(y) x + \cdots + b_{n-1}(y)x^{n-1} + a_n x^n.$$

The next corollary is the main result of the chapter. It holds uniformly over all balls, and is the $p$-adic equivalent of Corollary 2.8.

Corollary 4.3. Suppose that $x_0, a_0, \ldots, a_n \in \mathbb{Q}_p$, where $n \geq 1$, and suppose that $r \in \mathbb{Z}$. Then

$$\left| \int_{|x-x_0| \leq p^r} \chi(a_0 + a_1 x + \cdots + a_n x^n) \, dx \right| \leq \frac{2p^n}{|a_n|^{1/n}}.$$
Proof. Let $y = p^r(x - x_0)$, so that

$$I = \int_{|x-x_0| \leq p^r} \chi(a_0 + a_1 x + \cdots + a_n x^n) \, dx$$

$$= \int_{|y| \leq 1} \chi \left( a_0 + a_1 \left( \frac{y}{p^r} + x_0 \right) + \cdots + a_n \left( \frac{y}{p^r} + x_0 \right)^n \right) \, dy$$

$$= p^r \int_{|y| \leq 1} \chi \left( a_0 + \cdots + a_n y^n + \frac{b_1(x_0)y}{p^r} + \cdots + \frac{b_{n-1}(x_0)y^{n-1}}{p(n-1)r} + \frac{a_n y^n}{p^r} \right) \, dy$$

$$= p^r I_1,$$

say, where the $b_j$ are defined as in (4.1). We note that

$$|I_1| = \left| \chi(a_0 + \cdots + a_n y^n) \int_{|y| \leq 1} \chi \left( \frac{b_1(x_0)y}{p^r} + \cdots + \frac{b_{n-1}(x_0)y^{n-1}}{p(n-1)r} + \frac{a_n y^n}{p^r} \right) \, dy \right|$$

$$= \int_{|y| \leq 1} \chi \left( \frac{b_1(x_0)y}{p^r} + \cdots + \frac{b_{n-1}(x_0)y^{n-1}}{p(n-1)r} + \frac{a_n y^n}{p^r} \right) \, dy$$

$$= |I_2|,$$

say. Thus,

$$|I| = p^r |I_2| \leq p^r |b_m(y)| \leq \frac{2p^n}{|p^{-nr}a_n|^{1/n}} = 2p^n |a_n|^{1/n},$$

by Corollary 4.2.

\[ \square \]

4.1 Preliminary lemmas

We will require the following lemmas.

Lemma 4.4. Suppose that $|ma_m| > |ja_j|$ for all $j > m$, and $|y| \leq 1$. Then

$$|mb_m(y)| = |ma_m| > |jb_j(y)|$$

for all $j > m$, where the $b_j$ are given by (4.1).

Proof. Suppose that $|ma_m| > |ja_j|$ for all $j > m$. Then

$$|ma_m| > \left| \binom{j-1}{m-1} |ja_j|, \right.$$ 

so that

$$|a_m| > \left| \binom{j}{m} a_j \right|$$
for all \( j > m \). Thus

\[
|mb_m(y)| = |m| a_m + \binom{m+1}{m} a_{m+1} y + \cdots + \binom{n}{m} a_n y^{n-m} = |ma_m|
\]

for all \(|y| \leq 1\). Similarly, if \( k > j > m \), then

\[
|ma_m| > \left| \binom{k-1}{j-1} ka_k \right|
\]

so that

\[
|ma_m| > \left| j \binom{k}{j} a_k \right|
\]

Putting these together,

\[
|mb_m(y)| = |ma_m| > \left| ja_j + j \binom{j+1}{j} a_{j+1} y + \cdots + j \binom{n}{j} a_n y^{n-j} \right| = |jb_j(y)|
\]

for all \(|y| \leq 1\).

\[\square\]

**Lemma 4.5.** Suppose that \(|a_1| > p\) and \(|a_1| > |ja_j|\) for \( j > 1 \). Then

\[
\int_{|x| \leq 1} \chi(a_1 x + \cdots + a_n x^n) \, dx = 0.
\]

**Proof.** Let \(|a_1| = p^{k+1}\) where \( k \geq 1 \). We split the integral into \( p^k \) pieces:

\[
I = \int_{|x| \leq 1} \chi(a_1 x + \cdots + a_n x^n) \, dx = \sum_{y=0}^{p^k-1} \int_{|h| \leq p^{-k}} \chi(a_1 (y+h) + \cdots + a_n (y+h)^n) \, dh.
\]

Now

\[
a_1(y+h) + \cdots + a_n(y+h)^n = a_1 y + \cdots + a_n y^n + b_1(y) h + \cdots + b_{n-1}(y) h^{n-1} + a_n h^n,
\]

where \( b_j \) is defined as in (4.1). Thus

\[
I = \sum_{y=0}^{p^k-1} \chi(a_1 y + \cdots + a_n y^n) I_1(y),
\]

38
where

\[ I_1(y) = \int_{|h| \leq p^{-k}} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_nh^n) \, dh \]

\[ = \frac{1}{p^k} \int_{|x| \leq 1} \chi(b_1(y)p^kx + \cdots + b_{n-1}(y)p^{(n-1)k}x^{n-1} + a_np^{nk}x^n) \, dx. \]

When \(|y| \leq 1|b_1(y)| = |a_1| > |jb_j(y)|

for all \(j > 1|b_1(y)| = |a_1| = p,

and

\[ |jb_j(y)p^{jk}| = p^k \] for all \(|y| < 1\), by Lemma 4.4. Hence

So if \(j > 1\), then

\[ |b_1(y)p^k| = \frac{|a_1|}{p^k} = p, \] for all \(|y| \leq 1\), and we are done.

**Lemma 4.6.** Suppose that \(|ma_m| > p^2|ma_m| > |ja_j| for all \(j \neq m\). Then

\[ \int_{|x| \leq 1} \chi(a_1x + \cdots + a_nx^n) \, dx = \frac{1}{p} \int_{|x| \leq 1} \chi(a_1px + \cdots + a_np^n x^n) \, dx. \]

**Proof.** We split the integral into \(p\) pieces:

\[ I = \int_{|x| \leq 1} \chi(a_1x + \cdots + a_nx^n) \, dx = \sum_{y=0}^{p-1} \chi(a_1y + \cdots + a_ny^n)I_1(y), \]

where

\[ I_1(y) = \int_{|h| \leq 1/p} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_nh^n) \, dh \]

\[ = \frac{1}{p} \int_{|x| \leq 1} \chi(b_1(y)p^kx + \cdots + b_{n-1}(y)p^{(n-1)k}x^{n-1} + a_np^{nk}x^n) \, dx, \]
and \(b_j\) is given by (4.1).

We aim to apply Lemma 4.5. When \(y \neq 0\),

\[
|b_1(y)p| = |a_1 + 2a_2y + \cdots + na_ny^{n-1}|/p = |ma_m|/p > p.
\]

Now if \(k > j \geq 2\), then

\[
|ma_m| \geq \left| \left( \frac{k-1}{j-1} \right) |ka_k| = \left| j \left( \frac{k}{j} \right) a_k \right|
\]

so that

\[
|ma_m| \geq |ja_j + j \left( \frac{j+1}{j} \right) a_{j+1}y + \cdots + j \left( \frac{n}{j} \right) a_ny^{n-j}| = |jb_j(y)|.
\]

Hence if \(j \geq 2\), then

\[
|jb_j(y)p^j| \leq \frac{|ma_m|}{p^j} = \frac{|b_1(y)p|}{p^{j-1}} < |b_1(y)p|.
\]

Thus by Lemma 4.5, we have \(I_1(y) = 0\) for all \(y \neq 0\), so that \(I = I_1(0)\). \(\square\)

### 4.2 Proof of Lemma 4.1

We use double induction on

\[
m = \max\{l : |la_l| \geq |ja_j| \text{ for all } j \neq l\},
\]

and

\[
r = \max_{1 \leq j \leq n} \log_p |ja_j|.
\]

First we note that

\[
|I| = \left| \int_{|x| \leq 1} \chi(a_1x + \cdots + a_nx^n) \, dx \right| \leq \int_{|x| \leq 1} |\chi(a_1x + \cdots + a_nx^n)| \, dx = 1.
\]

Suppose that \(m = 1\). When \(r \leq 1\), we have

\[
\frac{p^m}{|ma_m|^{1/m}} = \frac{p}{|a_1|} \geq \frac{p}{p} = 1,
\]

so that

\[
|I| \leq 1 \leq \frac{p^m}{|ma_m|^{1/m}}.
\]
and we are done. When \( r > 1 \), we have \(|a_1| > p\), and as \( m = 1 \), we have \(|a_1| > |ja_j|\) for all \( j > 1 \). Thus, when \( m = 1 \) and \( r > 1 \), we obtain the result by Lemma 4.5.

Now suppose that \( m > 1 \) and \( r \leq 2 \). Again we are done, as

\[
\frac{p^{m}}{|ma_{m}|^{1/m}} \geq \frac{p^{2}}{p^{2/2}} \geq 1 \geq |I|.
\]

So when \( m = 1 \) or \( r \leq 2 \), we have the result.

Suppose the result holds when \( m \leq k - 1 \) and \( r \leq s - 1 \), and suppose that \( m = k \) and \( r = s \). When \(|y| \leq 1\), we have

\[
\{x \in \mathbb{Q}_p : |x| \leq 1\} = \{x \in \mathbb{Q}_p : |x - y| \leq 1\}
\]

so that by a change of variables,

\[
|I| = \left| \int_{|x - y| \leq 1} \chi(a_1 x + \cdots + a_n x^n) \, dx \right|
\]

\[
= \left| \int_{|h| \leq 1} \chi(a_1(h + y) + \cdots + a_n(h + y)^n) \, dh \right|
\]

\[
= \left| \int_{|h| \leq 1} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_n h^n) \, dh \right|
\]

for all \(|y| \leq 1\), where the \( b_j \) are given by (4.1).

As \( m = k \), we have \(|ka_k| > |ja_j|\) for all \( j > k \). Thus when \(|y| \leq 1\), we have \(|kb_k(y)| > |jb_j(y)|\) for all \( j > k \), by Lemma 4.4. We choose \( y = y_1 \), so that

\[
\max_{1 \leq j < k} |jb_j(y_1)| = \min_{|y| \leq 1} \max_{1 \leq j < k} |jb_j(y)|.
\]

Either \( \max_{1 \leq j < k} |jb_j(y_1)| < |kb_k(y_1)| \) or \( \max_{1 \leq j < k} |jb_j(y_1)| \geq |kb_k(y_1)| \).

When \( \max_{1 \leq j < k} |jb_j(y_1)| < |kb_k(y_1)| \), we have

\[
|kb_k(y_1)| > |jb_j(y_1)|
\]

for all \( j \neq k \), so we can apply Lemma 4.6 to obtain

\[
|I| = \frac{1}{p} \left| \int_{|h| \leq 1} \chi(b_1(y_1)ph + \cdots + b_{n-1}(y_1)p^{n-1}h^{n-1} + a_n p^n h^n) \, dh \right|.
\]
Now as \( \max_{1 \leq j \leq n} |jb_j(y_1)p^j| \leq p^{s-1} \), we have
\[
r = \max_{1 \leq j \leq n} \log_p |jb_j(y_1)p^j| \leq s - 1.
\]
Since \( |kb_k(y_1)| > |jb_j(y_1)| \) for all \( j > k \),
\[
m = \max \{ l : |lb_l(y_1)p^l| \geq |jb_j(y_1)p^j| \text{ for all } j \neq l \} = k_1 \leq k.
\]
Hence
\[
|I| \leq \frac{1}{p} \frac{p^{k_1}}{|k_1b_{k_1}(y_1)p^{k_1}|^{1/k_1}} = \frac{p^{k_1}}{|k_1b_{k_1}(y_1)|^{1/k_1}} \leq \frac{p^k}{|kb_k(y_1)|^{1/k}},
\]
by induction. Thus
\[
|I| \leq \frac{p^k}{|ka_k|^{1/k}},
\]
as \( |kb_k(y_1)| = |ka_k| \) by Lemma 4.4.

When \( \max_{1 \leq j < k} |jb_j(y_1)| \geq |kb_k(y_1)| \),
\[
\max_{1 \leq j < k} |jb_j(y)| \geq |kb_k(y)|,
\]
for all \( |y| \leq 1 \), as
\[
\max_{1 \leq j < k} |jb_j(y)| \geq \min_{|y| \leq 1} \max_{1 \leq j < k} |jb_j(y)| = \max_{1 \leq j < k} |jb_j(y_1)|,
\]
by definition, and
\[
|kb_k(y)| = |kb_k(y_1)|
\]
by Lemma 4.4. We split the integral into \( p \) pieces, so that
\[
I = \int_{|x| \leq 1} \chi(a_1x + \cdots + a_nx^n) \, dx = \sum_{y=0}^{p-1} \chi(a_1y + \cdots + a_ny^n) I_1(y),
\]
where
\[
I_1(y) = \int_{|h| \leq 1/p} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_nh^n) \, dh
\]
\[
= \frac{1}{p} \int_{|x| \leq 1} \chi(b_1(y)p^x + \cdots + b_{n-1}(y)p^{n-1}x^{n-1} + a_np^nx^n) \, dx,
\]
and \( b_j \) is given by (4.1). Now by Lemma 4.4, we have
\[
|kb_k(y)| = |ka_k| > |lb_l(y)|
\]
for all $l > k$, so that
\[
\max_{1 \leq j < k} |jb_j(y)| \geq |kb_k(y_1)| = |ka_k| \geq |lb_l(y)|
\]
for all $l \geq k$. Thus for $y = 0, \ldots, p - 1$, there exists $k_1 < k$ (where $k_1$ depends on $y$) such that
\[
|k_1b_{k_1}(y)| \geq |jb_j(y)|,
\]
and
\[
|k_1b_{k_1}(y)p^{k_1}| > |jb_j(y)p^j|
\]
for all $j > k_1$. Hence for each $y = 0, \ldots, p - 1$ and $I_1(y)$ defined by
\[
I_1(y) = \frac{1}{p} \int_{|x| \leq 1} \chi(b_1(y)p^jx + \cdots + b_{n-1}(y)p^{n-1}x^{n-1} + a_np^nx^n) \, dx,
\]
we have
\[
m = \max\{l : |lb_l(y)p^l| \geq |jb_j(y)p^j| \text{ for all } j \neq l\} = k_1 < k.
\]
Thus
\[
|I_1(y)| \leq \frac{1}{p} \frac{p^{k_1}}{|k_1b_{k_1}(y)p^{k_1}|^{1/k_1}} = \frac{p^{k_1}}{|k_1b_{k_1}(y)|^{1/k_1}},
\]
by induction. Finally, by Lemma 4.4, so that
\[
|I| \leq \sum_{y=0}^{p-1} |I_1(y)| \leq p \frac{p^{k-1}}{|ka_k|^{1/k}} = \frac{p^k}{|ka_k|^{1/k}},
\]
and we are done.
Chapter 5

Maximal averages along $p$-adic curves

Let $P_1, \ldots, P_n$ be $p$-adic polynomials of one $p$-adic variable and define the curve $\gamma : \mathbb{Q}_p \rightarrow \mathbb{Q}_p^n$ by $\gamma(t) = (P_1(t), \ldots, P_n(t))$. We will consider the averages

$$\frac{1}{p^k} \int_{|t| \leq p^k} f(x - \gamma(t)) \, dt,$$

and the maximal average $M_\gamma f(x)$ defined by

$$M_\gamma f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{p^k} \int_{|t| \leq p^k} f(x - \gamma(t)) \, dt.$$

We will prove the following $p$-adic version of a theorem due to E.M. Stein and S. Wainger [27, 28].

**Theorem 5.1.** Suppose that $1 < q < \infty$, and that $\gamma$ and $M_\gamma$ are defined as above. Then there is a constant $C_q$ so that

$$\|M_\gamma f\|_{L^q(\mathbb{Q}_p^n)} \leq C_q \|f\|_{L^q(\mathbb{Q}_p^n)},$$

for all $f \in L^q(\mathbb{Q}_p^n)$.

The proof will largely follow the Euclidean arguments. The difficulty lies in bounding oscillatory integrals on the $p$-adics, which we are now able to do.

We will ‘lift’ to a higher dimensional situation and prove a similar theorem where the curve is defined by monomials. Then we ‘descend’ back to the original
dimension and curve. This technique goes back to K. de Leeuw [15]. In order to do this we let \( R : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^n \) be a linear map so that
\[
R(t, t^2, \ldots, t^N) = (P_1(t), \ldots P_n(t)),
\]
where \( N \) is the maximal degree of the polynomials. We will initially consider the curve \( \tilde{\gamma}(t) = (t, t^2, \ldots, t^N) \) and the maximal function
\[
M_{\tilde{\gamma}} f(x) = \sup \frac{1}{p^k} \int_{|t| \leq p^k} f(x - \tilde{\gamma}(t)) \, dt.
\]
We will make use of non-isotropic dilations \( \rho_k : \mathbb{Q}_p^N \rightarrow \mathbb{Q}_p^N \) defined by
\[
\rho_k(x) = (p^k x_1, p^{2k} x_2, \ldots, p^{Nk} x_N),
\]
and the norm \( d : \mathbb{Q}_p^N \rightarrow \mathbb{R} \) defined by
\[
d(x) = \max \{|x_1|, |x_2|^{1/2}, \ldots, |x_N|^{1/N}\}.
\]

Unless specified otherwise, balls will be defined using \( d \), so that
\[
B(x, p^k) = \{ y : d(x - y) \leq p^k \} = \{ y : |y_j - x_j| \leq p^{jk} \text{ for all } j = 1, \ldots, N \},
\]
and \(|B(x, p^k)| = p^{Mk}\), where \( M = 1 + 2 + \ldots + N = N(N + 1)/2 \). Balls denoted by \( B^j \) for some \( j \in \mathbb{N} \), will also be defined using \( d \), but will have no specified radius or position. It is not difficult to show that \( d \) is an ultrametric so that the results of Section 3.1 apply. In particular, all points within a ball can be considered to be its center. Similarly, two balls are disjoint or one is contained in the other.

In the first section we will obtain a Calderón–Zygmund type decomposition. In the next two sections we will prove \( L^2(\mathbb{Q}_p^N) \) and \( L^q(\mathbb{Q}_p^N) \) bounds for \( M_{\tilde{\gamma}} \). In the final two sections we will finish the proof of Theorem 5.1 and consider the differentiation of integrals along curves.

### 5.1 The Calderón–Zygmund decomposition

The following proposition splits an integrable function into a large and a small part. The small part will naturally be easy to bound, and the large part will have some redeeming qualities.
Proposition 5.2. Suppose that $f \in L^1(Q^N_p)$ and that $\alpha > 0$. Then

$$f = g + \sum_{j=0}^{\infty} b_j,$$

where $g, b_j \in L^1(Q^N_p)$ and each $b_j$ is supported in a ball $B^j$, so that

(i) $|g(x)| \leq \alpha$ for almost every $x$,

(ii) $\|b_j\|_{L^1(Q^N_p)} \leq 2\alpha |B^j|$

(iii) $\int_{Q^N_p} b_j(x) \, dx = 0$

(iv) $\sum_j |B^j| \leq \frac{p^M}{\alpha} \|f\|_{L^1(Q^N_p)}$

Proof. Define $E = \{x \in Q^N_p : \mathcal{M}f(x) > \alpha\}$, where $\mathcal{M}f$ is the Hardy–Littlewood maximal function defined by

$$\mathcal{M}f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|B(x, p^k)|} \int_{B(x, p^k)} |f(y)| \, dy.$$

For $x \in Q^N_p$ let $\delta(x)$ be the minimal distance such that $B(x, \delta(x)) \cap E^c \neq \emptyset$. The set $E$ is of finite measure, by Theorem 3.2, and as $B(x, \delta(x)/p) \subset E$ we see that the balls are of uniformly bounded measure. Thus we can apply Lemma 3.1 to $\{B(x, \delta(x))\}_{x \in E}$, to obtain a countable and disjoint subcover of $E$, which we denote by $\{B^j\}$.

Define $g$ by

$$g(x) = \begin{cases} f(x) & \text{when } x \notin E \\ \frac{1}{|B^j|} \int_{B^j} f(y) \, dy & \text{when } x \in B^j, \end{cases}$$

and $b_j$ by

$$b_j(x) = 1_{B^j}(x) \left( f(x) - \frac{1}{|B^j|} \int_{B^j} f(y) \, dy \right),$$

so that $f = g + \sum_j b_j$.

When $x \notin E$, we see by Corollary 3.3 that $g(x) \leq \alpha$ for almost every $x$.

When $x \in E$,

$$g(x) = \frac{1}{|B^j|} \int_{B^j} f(y) \, dy \leq \alpha,$$
as $B_j \cap E^c \neq \emptyset$, so we have proven the first part. Similarly as $B_j \cap E^c \neq \emptyset$,

$$
\int_{Q_p^N} |b_j(x)| \, dx \leq 2 \int_{B_j} |f(x)| \, dx \\
= 2|B_j| \frac{1}{|B_j|} \int_{B_j} |f(x)| \, dx \leq 2|B_j| |\alpha|,
$$

so the second part holds. The third part is clear by the definition of $b_j$. Finally we consider the balls $B'_j$, where $B(x, p^k) = B(x, p^{k-1})$. It is clear that the $B'_j$ are disjoint, and that $\cup B'_j \subset E$. Thus

$$
\sum_j |B_j| = p^M \sum_j |B'_j| \leq p^M |E| \leq \frac{p^M}{\alpha} \|f\|_{L^1},
$$

by Theorem 3.2, and we are done.

5.2 $L^2$ boundedness

It will be useful to consider $\mathcal{M}_{\tilde{\gamma}}$ as a maximal convolution of measures. To this end we let $\mu$ be the measure defined by

$$
\int_{Q_p^N} f(x) \, d\mu(x) = \int_{|t| \leq 1} f(\tilde{\gamma}(t)) \, dt,
$$

where $\tilde{\gamma}(t) = (t, t^2, \ldots, t^N)$, and let $\mu_k$ be the measure defined by

$$
\int_{Q_p^N} f(x) \, d\mu_k(x) = \int_{Q_p^N} f(x) \, d\mu(\rho_k(x)) = \frac{1}{p^k} \int_{|t| \leq p^k} f(\tilde{\gamma}(t)) \, dt,
$$

where $\rho_k(x) = (p^k x_1, p^{2k} x_2, \ldots, p^{Nk} x_N)$. Then $\mathcal{M}_{\tilde{\gamma}} f$ is defined by

$$
\mathcal{M}_{\tilde{\gamma}} f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{p^k} \int_{|t| \leq p^k} f(x - \tilde{\gamma}(t)) \, dt = \sup_{k \in \mathbb{Z}} \mu_k \ast f.
$$

The use of square functions will be key to the proof of the following proposition. This idea was developed by Stein and Wainger [27, 28].

**Proposition 5.3.** Suppose that $\mathcal{M}_{\tilde{\gamma}}$ is defined as above. Then there is a constant $C_N$ so that

$$
\|\mathcal{M}_{\tilde{\gamma}} f\|_{L^2(Q_p^N)} \leq C_N \|f\|_{L^2(Q_p^N)}
$$

for all $f \in \mathcal{S}(Q_p^N)$.  

48
Proof. Define the square function $Gf$ by

$$Gf(x) = \left( \sum_{k \in \mathbb{Z}} |\mu_k \ast f - \Phi_k \ast f|^2 \right)^{1/2},$$

where

$$\Phi_k(\xi) = \frac{1}{|B(0,p^k)|} 1_{B(0,p^k)},$$

so that

$$Gf(x) = \left( \sum_{k \in \mathbb{Z}} \left| \frac{1}{p^k} \int_{|t| \leq p^k} |f(x - \tilde{\gamma}(t))| dt - \frac{1}{p^{Mk}} \int_{B(x,p^k)} |f(x - s)| ds \right|^2 \right)^{1/2},$$

where $M = N(N + 1)/2$. We note that $x$ and $s$ are vector-valued, and $t$ is scalar-valued. In this chapter $t$ will always be scalar-valued. Now

$$\mathcal{M}_\gamma f(x) \leq Gf(x) + \mathcal{M}f(x),$$

and $\mathcal{M}$ is bounded by Theorem 3.2. Thus, it remains to show that $G$ is bounded.

If we assume for the moment that $Gf \in L^2(\mathbb{Q}_p^N)$, then

$$\|Gf\|_{L^2}^2 = \sum_{k \in \mathbb{Z}} \|\mu_k \ast f - \Phi_k \ast f\|_{L^2}^2,$$

so that by Plancherel’s theorem,

$$\|Gf\|_{L^2}^2 = \int_{\mathbb{Q}_p^N} |\hat{f}(\xi)|^2 \sum_{k \in \mathbb{Z}} |m_k(\xi)|^2 d\xi,$$

where $m_k(\xi) = \hat{\mu}_k(\xi) - \hat{\Phi}_k(\xi)$. Thus

$$m_k(\xi) = \int_{|t| \leq 1} \chi \left( \frac{\xi_1}{p^k} t + \frac{\xi_2}{p^{2k}} t^2 + \cdots + \frac{\xi_N}{p^{Nk}} t^N \right) dt - \prod_{j=1}^N \int_{|t_j| \leq 1} \chi \left( \frac{\xi_j}{p^{jk}} t_j \right) dt_j.$$ 

When $|\xi_j| p^{jk} \leq 1$ for all $j$, we have

$$\int_{|t| \leq 1} \chi \left( \frac{\xi_1}{p^k} t + \frac{\xi_2}{p^{2k}} t^2 + \cdots + \frac{\xi_N}{p^{Nk}} t^N \right) dt = 1,$$

as $\chi$ is trivial on $\{ x \in \mathbb{Q}_p : |x| \leq 1 \}$. Similarly

$$\prod_{j=1}^N \int_{|t_j| \leq 1} \chi \left( \frac{\xi_j}{p^{jk}} t_j \right) dt_j = 1,$$
so that \( m_k(\xi) = 0 \). When \(|\xi_j|^p > 1\) for some \( j \), we have

\[
\prod_{j=1}^N \int_{|t_j| \leq 1} \chi \left( \frac{\xi_j t_j}{p^j} \right) dt_j = 0,
\]

by Lemma 3.4, so that

\[
|m_k(\xi)| = \left| \int_{|t| \leq 1} \chi \left( \frac{\xi_1 t + \cdots + \xi_N t_N}{p^N} \right) dt \right| \leq \frac{p^N}{\max_j \{|\xi_j|^p\}^{1/N}},
\]

by Lemma 4.2. Thus

\[
\sum_{k \in \mathbb{Z}} |m_k(\xi)|^2 \leq \sum_{k \in \mathbb{Z}} \left( \frac{p^N}{\max_j \{|\xi_j|^p\}^{1/N}} \right)^2 \leq C_N
\]

for some constant \( C_N \). Thus

\[
\|Gf\|_{L^2}^2 = \int_{\mathbb{Q}_p^N} |\hat{f}(\xi)|^2 \sum_{k \in \mathbb{Z}} |m_k(\xi)|^2 d\xi \leq C_N \|\hat{f}\|_{L^2}^2 = C_N \|f\|_{L^2}^2,
\]

by Plancherel’s theorem, and we are done. \( \square \)

In order to take advantage of the ‘slack’ in the above argument we introduce the measures \( \nu^s \) defined by

\[
\hat{\nu}^s(x) = \hat{\mu}(x) \max\{1, |x|\}^s,
\]

and \( \nu^s_k \) defined by

\[
\hat{\nu}^s_k(x) = \hat{\nu}^s(\rho^{-k}(x)) = \hat{\mu}(\rho^{-k}(x)) \max\{1, |\rho^{-k}(x)|\}^s,
\]

(5.2)

where \( s \in \mathbb{C} \). By a change of variables,

\[
\hat{\nu}^s_k(x) = \hat{\mu}_k(x) \max\{1, |\rho^{-k}(x)|\}^s,
\]

so by the proof of Proposition 5.3, we see that \( \mathcal{N}_s \) defined by

\[
\mathcal{N}_s f = \sup_{k \in \mathbb{Z}} \nu^s_k * f
\]

is also \( L^2(\mathbb{Q}_p^N) \) bounded when \( \text{Re}(s) < 1/N \). We state this formally for future reference.

**Proposition 5.4.** Suppose that \( \mathcal{N}_s \) is defined as above. Then for each \( s \in \mathbb{C} \) such that \( \text{Re}(s) < 1/N \), there is a constant \( C_s \) so that

\[
\|\mathcal{N}_s f\|_{L^2(\mathbb{Q}_p^N)} \leq C_s \|f\|_{L^2(\mathbb{Q}_p^N)}
\]

for all \( f \in \mathcal{S}(\mathbb{Q}_p^N) \).
5.3 \(L^q\) boundedness

We aim to bound \(N_s\) when \(\text{Re}(s) < 0\). This will enable us to use complex interpolation to obtain a bound for \(\mathcal{M}_\tilde{\gamma}\).

From (5.2) we can calculate, using Theorem 3.8, that

\[
\nu^s_k(x) = \frac{1}{pMk} \mu * \hat{J}^s(\rho_k(x)),
\]

where \(J^s\) is the Bessel potential defined as in (3.6). Thus \(\nu^s_k\) is an \(L^1(\mathbb{Q}_p^N)\) function when \(\text{Re}(s) < 0\), by Proposition 3.15 and Theorem 3.7, and by a change of variables \(\|\nu^s_k\|_{L^1} \leq \|\hat{J}^s\|_{L^1}\) for all \(k \in \mathbb{Z}\).

One of the reasons we have been considering the norm \(d\) is so that we can obtain the following version of Hörmander’s condition. When \(d(x) > d(y)\), we have \(d(\rho_k(x)) > d(\rho_k(y))\) for all \(k \in \mathbb{Z}\). It is easy to see that \(|x| \geq d(x)\) when \(d(x) \geq 1\). We also note that as \(d\) is an ultrametric, \(d(x + y) = d(x)\) when \(d(x) > d(y)\).

**Proposition 5.5.** Suppose that \(\nu^s_k\) is defined as above, and that \(\text{Re}(s) < 0\). Then there is a constant \(C_s\) so that

\[
\int_{d(x) > d(y)} \sup_k |\nu^s_k(x) - \nu^s_k(x)| \, dx \leq C_s
\]

for all \(y \in \mathbb{Q}_p^N\).

**Proof.** First we note that

\[
I = \int_{d(x) > d(y)} \sup_k |\nu^s_k(x) - \nu^s_k(x)| \, dx \\
\leq \sum_k \int_{d(x) > d(y)} |\nu^s_k(x) - \nu^s_k(x)| \, dx \\
= \sum_k \int_{d(x) > d(y)} \frac{1}{pMk} \left| \int_{|t| \leq 1} \hat{J}^s(\rho_k(x - y) - \tilde{\gamma}(t)) - \hat{J}^s(\rho_k(x) - \tilde{\gamma}(t)) \, dt \right| \, dx.
\]

When \(|\rho_k(y)| \geq 1\), we have \(d(\rho_k(y)) \geq 1\), so that

\[
|\rho_k(x - y)| \geq d(\rho_k(x - y)) > d(\rho_k(y)) \geq 1,
\]

51
as \(d(x - y) = d(x) > d(y)\). Hence, as \(\hat{J}^s\) is supported in the unit ball,

\[
\hat{J}^s(\rho_k(x - y) - \tilde{\gamma}(t)) = 0
\]

for all \(|t| \leq 1\). Similarly when \(|\rho_k(y)| \geq 1\),

\[
|\rho_k(x)| \geq d(\rho_k(x)) > d(\rho_k(y)) \geq 1,
\]

so that

\[
\hat{J}^s(\rho_k(x) - \tilde{\gamma}(t)) = 0
\]

for all \(|t| \leq 1\). Thus, by Fubini’s theorem,

\[
I \leq \sum_{k: |\rho_k(y)| < 1} \int_{|t| \leq 1} \int_{\mathbb{Q}_p^N} \left| \hat{J}^s(\rho_k(x - y) - \tilde{\gamma}(t)) - \hat{J}^s(\rho_k(x) - \tilde{\gamma}(t)) \right| \, dx \, dt,
\]

so by the change of variables \(z = \rho_k(x) - \tilde{\gamma}(t)\),

\[
I \leq \sum_{k: |\rho_k(y)| < 1} \int_{|t| \leq 1} \int_{\mathbb{Q}_p^N} \left| \hat{J}^s(z - \rho_k(y)) - \hat{J}^s(z) \right| \, dz \, dt.
\]

Finally, by Lemma 3.16,

\[
I \leq \sum_{k: |\rho_k(y)| < 1} \int_{|t| \leq 1} C_s |\rho_k(y)|^{-\text{Re}(s)} \, dt = C_s \sum_{k: |\rho_k(y)| < 1} |\rho_k(y)|^{-\text{Re}(s)} \leq C_s',
\]

as desired. \(\Box\)

We use this and our Calderón–Zygmund decomposition to prove the following proposition.

**Proposition 5.6.** Suppose that \(1 < q \leq \infty\) and that \(N^s\) is defined as above. Then for each \(s \in \mathbb{C}\) such that \(\text{Re}(s) < 0\), there is a constant \(C_{q,s}\) so that

\[
\|N_s f\|_{L^q(\mathbb{Q}_p^N)} \leq C_{q,s} \|f\|_{L^q(\mathbb{Q}_p^N)}
\]

for all \(f \in \mathcal{S}(\mathbb{Q}_p^N)\).

**Proof.** When \(\text{Re}(s) < 0\), we have

\[
\sup_{x \in \mathbb{Q}_p^N} |\nu_k^s * f(x)| \leq \|\nu_k^s\|_{L^1} \|f\|_{L^\infty} \leq \|\hat{J}^s\|_{L^1} \|f\|_{L^\infty}
\]
for all $k \in \mathbb{Z}$. Hence there is a constant $A_s = \|\hat{J}^s\|_{L^1}$ so that

$$\sup_{x \in \mathbb{Q}^N} |\mathcal{N}_s f(x)| \leq A_s \|f\|_{L^\infty}, \quad (5.3)$$

for all $f \in \mathcal{S}(\mathbb{Q}^N)$. We will show that $\mathcal{N}_s$ is weak type $(1,1)$ and then interpolate using (5.3) to obtain the result.

We split $f$ into two parts $g$ and $b = \sum_j b_j$ as in Proposition 5.2, so that

$$\{x \in \mathbb{Q}^N : \mathcal{N}_s g(x) > A_s \alpha\} = \emptyset.$$

As

$$\{x : \mathcal{N}_s f(x) > 2A_s \alpha\} \subset \{x : \mathcal{N}_s g(x) > A_s \alpha\} \cup \{x : \mathcal{N}_s b(x) > A_s \alpha\},$$

it remains to bound $|\{x \in \mathbb{Q}^N : \mathcal{N}_s b(x) > A_s \alpha\}|$. Let $b_j$ be supported on $B^j = B(x_j, \delta_j)$, say, then

$$I = \int_{\cup(B^j)^c} |\mathcal{N}_s b(x)| \, dx \leq \int_{\cup(B^j)^c} \sup_k |\nu^s_k \ast b(x)| \, dx$$

$$= \int_{\cup(B^j)^c} \sup_k \left| \sum_j \int_{B^j} b_j(y) \nu^s_k(x - y) \, dy \right| \, dx.$$

By (iii) in Proposition 5.2, we have $\int_{B^j} b_j(y) \nu^s_k(x - x_j) \, dy = 0$, so that

$$I \leq \int_{\cup(B^j)^c} \sup_k \left| \sum_j \int_{B^j} b_j(y) (\nu^s_k(x - y) - \nu^s_k(x - x_j)) \, dy \right| \, dx$$

$$\leq \sum_j \int_{(B^j)^c} \sup_k |\nu^s_k(x - y) - \nu^s_k(x - x_j)| \, dx \int_{B^j} |b_j(y)| \, dy.$$

Now if we make the change of variables $z = x - x_j$, then we see that

$$I \leq \sum_j \int_{B(0, \delta_j)^c} \sup_k |\nu^s_k(z + x_j - y) - \nu^s_k(z)| \, dz \int_{B^j} |b_j(y)| \, dy.$$

Now $d(z) > \delta_j$, as $x \in B(x_j, \delta_j)^c$, and $d(y - x_j) \leq \delta_j$, as $y \in B(x_j, \delta_j)$. Thus

$$\int_{B(0, \delta_j)^c} \sup_k |\nu^s_k(z - (y - x_j)) - \nu^s_k(z)| \, dz \leq C_s$$

by Proposition 5.5, so that

$$I \leq \sum_j C_s \int_{B^j} |b_j(y)| \, dy.$$
By (ii) and (iv) of Proposition 5.5,
\[ I = \int_{(\cup B)^c} |\mathcal{N}_s b(x)| \, dx \leq C_s \sum_j 2\alpha |B^j| \leq 2C_s p^M \|f\|_{L^1}, \]
so that
\[ |\{x \in \mathbb{Q}_p^N : \mathcal{N}_s b(x) > A_s \alpha\}| \leq \frac{2C_s p^M}{A_s \alpha} \|f\|_{L^1} + |\cup B^j|. \]
Thus by (iv) in Proposition 5.5 again, we see that
\[ |\{x \in \mathbb{Q}_p^N : \mathcal{N}_s b(x) > A_s \alpha\}| \leq \frac{(2C_s/A_s + 1)p^M}{\alpha} \|f\|_{L^1}. \]
Finally, as \( \{x \in \mathbb{Q}_p^N : \mathcal{N}_s f(x) > 2A_s \alpha\} \subset \{x \in \mathbb{Q}_p^N : \mathcal{N}_s b(x) > A_s \alpha\}, \)
\[ |\{x \in \mathbb{Q}_p^N : \mathcal{N}_s f(x) > \alpha\}| \leq \frac{2(2C_s + A_s)p^M}{\alpha} \|f\|_{L^1}. \]
We interpolate between this bound and the bound in (5.3) using the argument in the proof of Theorem 3.2, and we are done.

Finally we require the following complex interpolation theorem due to Stein [29]. We reformulate it in the generality that we require. Let
\[ D = \{z \in \mathbb{C} : a \leq \text{Re}(z) \leq b\}, \]
and call a family of operators \( \{T_z\}_{z \in \mathbb{C}} \) admissible if, for \( f, g \in \mathcal{S}(\mathbb{Q}_p^N) \), the mapping
\[ z \mapsto \int_{\mathbb{Q}_p^N} (T_z f(x))g(x) \, dx \]
is analytic in the interior of \( D \), continuous on \( D \), and uniformly bounded on \( D \).

**Theorem 5.7.** Suppose that \( \{T_z\}_{z \in \mathbb{C}} \) is an admissible family of operators satisfying
\[ \|T_{a+iy}\|_{L^{q_a}(\mathbb{Q}_p^N)} \leq M_a \|f\|_{L^{q_a}(\mathbb{Q}_p^N)} \]
and
\[ \|T_{b+iy}\|_{L^{q_b}(\mathbb{Q}_p^N)} \leq M_b \|f\|_{L^{q_b}(\mathbb{Q}_p^N)} \]
for all \( f \in \mathcal{S}(\mathbb{Q}_p^N) \), where \( 1 \leq q_a, q_b \leq \infty \) and \( M_a, M_b \) are constants. Then
\[ \|T_{(b-a)\theta + a}\|_{L^{q_\theta}(\mathbb{Q}_p^N)} \leq M_a^{1-\theta} M_b^{\theta} \|f\|_{L^{q_\theta}(\mathbb{Q}_p^N)} \]
for all \( f \in \mathcal{S}(\mathbb{Q}_p^N) \), where \( 1/q_\theta = (1-\theta)/q_a + \theta/q_b \) and \( 0 \leq \theta \leq 1 \).
We ‘linearize’ $\mathcal{N}_s$ in order to apply this theorem. Let $x \mapsto k(x)$ be an arbitrary integer-valued function on $\mathbb{Q}_p^N$. With it we define the admissible family of operators $\mathcal{U}_s$ by $\mathcal{U}_s f(x) = \nu_{k(x)}^s * f(x)$. As

$$\mathcal{U}_0 f(x) = \nu_0^0 * f(x) = \mu_{k(x)} * f(x),$$

we let $b = 1/2N$ and take $\theta$ and $a$ so that $(1/2N - a)\theta + a = 0$. It is not hard to show that $\theta$ and $a$ can be chosen so that $q_0$ can take all the values in the range $1 < q_0 < \infty$. Hence we can interpolate between the bounds in Proposition 5.6 and Proposition 5.4 so that

$$\|\mathcal{U}_0 f\|_{L^q(\mathbb{Q}_p^N)} \leq C_q \|f\|_{L^q(\mathbb{Q}_p^N)}$$

for all $f \in \mathcal{S}(\mathbb{Q}_p^N)$. Now as $x \mapsto k(x)$ is arbitrary, and

$$\mathcal{N}_0 f(x) = \sup_{k \in \mathbb{Z}} \nu_k^0 * f(x) = \mathcal{M}_{\tilde{\gamma}} f(x),$$

we obtain the following proposition.

**Theorem 5.8.** Suppose that $1 < q < \infty$ and that $\mathcal{M}_{\tilde{\gamma}}$ is defined as above. Then there is a constant $C_q$ so that

$$\|\mathcal{M}_{\tilde{\gamma}} f\|_{L^q(\mathbb{Q}_p^N)} \leq C_q \|f\|_{L^q(\mathbb{Q}_p^N)}$$

for all $f \in \mathcal{S}(\mathbb{Q}_p^N)$.

### 5.4 The method of descent

We will now return to our original curve and dimension. This technique has its origins in a paper of de Leeuw [15].

**Theorem 5.9.** Suppose that $1 < q < \infty$ and that $\mathcal{M}_{\gamma}$ is defined as above. Then there is a constant $C_q$ so that

$$\|\mathcal{M}_{\gamma} f\|_{L^q(\mathbb{Q}_p^N)} \leq C_q \|f\|_{L^q(\mathbb{Q}_p^N)}$$

for all $f \in \mathcal{S}(\mathbb{Q}_p^N)$. 
Proof. Let \( K \) be a positive integer, and define \( \mathcal{M}^K_\gamma \) by

\[
\mathcal{M}^K_\gamma f(x) = \sup_{|k| \leq K} \frac{1}{p^k} \int_{|t| \leq p^k} f(x - \gamma(t)) \, dt,
\]

where \( k \) takes integer values, and \( \mathcal{M}^K_\tilde{\gamma} \) by

\[
\mathcal{M}^K_\tilde{\gamma} f(x) = \sup_{|k| \leq K} \frac{1}{p^k} \int_{|t| \leq p^k} f(x - \tilde{\gamma}(t)) \, dt.
\]

Recall that \( R : \mathbb{Q}^N_p \rightarrow \mathbb{Q}^n_p \) is a linear map so that

\[
R(t, t^2, \ldots, t^N) = (P_1(t), \ldots, P_n(t)),
\]

and

\[
\mathcal{M}^K_\gamma f(x) = \frac{1}{p^N r} \int_{B_r(0)} \mathcal{M}^K_\gamma f \| q \, dy,
\]

where \( B_r(0) = \{ y \in \mathbb{Q}_p^N : |y| \leq p^r \} \). At this point \( K \) is fixed, so we may choose \( r \) so that \( r > NK \). If \( y \in B_r(0) \) and \( |t| \leq p^k \), then \( |y - \tilde{\gamma}(t)| \leq p^r \). Hence, when \( y \in B_r(0) \),

\[
T_y \mathcal{M}^K_\gamma f(x) = \sup_{|k| \leq K} \frac{1}{p^k} \int_{|t| \leq p^k} f(x + R(y) - \tilde{\gamma}(t)) \, dt
\]

\[
= \sup_{|k| \leq K} \frac{1}{p^k} \int_{|t| \leq p^k} f(x + R(y) - \tilde{\gamma}(t)) \mathbf{1}_r(y - \tilde{\gamma}(t)) \, dt,
\]

where \( \mathbf{1}_r \) denotes the characteristic function of

\[
B_r(0) = \{ y \in \mathbb{Q}_p^N : |y| \leq p^r \}.
\]

Thus,

\[
T_y \mathcal{M}^K_\gamma f(x) = \sup_{|k| \leq K} \frac{1}{p^k} \int_{|t| \leq p^k} F_x(y - \tilde{\gamma}(t)) \, dt = \mathcal{M}^K_\gamma F_x(y),
\]

where \( F_x(z) = f(x + R(z)) \mathbf{1}_r(z) \).

We have

\[
\int_{B_r(0)} |\mathcal{M}^K_\gamma F_x(y)|^q \, dy \leq C_q \int_{\mathbb{Q}_p^N} |F_x(y)|^q \, dy,
\]

where

\[
C_q = \sum_{k=0}^N \frac{1}{p^{kN}} \left( \int_{\mathbb{Q}_p^N} |F_x(y)|^q \, dy \right)^{\frac{1}{q}}.
\]

56
by Theorem 5.8, and by Fubini’s theorem,
\[
\int_{\mathbb{Q}_p^n} \int_{\mathbb{Q}_p^N} |F_x(y)|^q \, dy \, dx = p^{N r} \| f \|_{L^q(\mathbb{Q}_p^n)}^q.
\]
Hence we can integrate (5.6) to obtain
\[
\int_{\mathbb{Q}_p^n} \int_{B_r(0)} |M^K_\gamma F_x(y)|^q \, dy \, dx \leq p^{N r} C^q_q \| f \|_{L^q(\mathbb{Q}_p^n)}^q. \tag{5.7}
\]
Now we combine (5.4), (5.5) and (5.7) so that
\[
\| M^K_\gamma f \|_{L^q(\mathbb{Q}_p^n)}^q \leq \frac{p^{N r}}{p^{N r} C^q_q} \| f \|_{L^q(\mathbb{Q}_p^n)}^q = C^q_q \| f \|_{L^q(\mathbb{Q}_p^n)}^q,
\]
and let $K$ tend to infinity to obtain the result. \[\square\]

5.5 Conclusion

As the space of compactly supported, locally constant functions $S(\mathbb{Q}_p^n)$ is dense in $L^q(\mathbb{Q}_p^n)$, when $1 \leq q < \infty$, it is clear that Theorem 5.1 can be obtained from Theorem 5.9 by a simple limiting argument. We note the following corollary, which is a version of the fundamental theorem of calculus along $p$-adic curves. The proof is the same as that for Corollary 3.3.

**Corollary 5.10.** Suppose that $f \in L^q(\mathbb{Q}_p^n)$, where $1 < q < \infty$. Suppose that $\gamma$ is defined as above, with $\gamma(0) = 0$. Then for almost every $x \in \mathbb{Q}_p^n$:
\[
\lim_{k \to -\infty} \frac{1}{p^k} \int_{|t| \leq p^k} f(x - \gamma(t)) \, dt = f(x).
\]

It is an open question, in both the Euclidean and $p$-adic cases, as to whether there is a weak type (1,1) version of Theorem 5.1. Similarly, it is not known whether there is an $L^1$ version of Corollary 5.10.
Bibliography


[33] J. Wright, $p$-Adic van der Corput lemmas, unpublished manuscript.