ANALYSIS OF A SUBELLIPTIC OPERATOR ON THE SPHERE IN COMPLEX N-SPACE

A thesis submitted for the degree of
Master of Science

By
Oldrich Klima
B. Sc. (Hons), Macquarie University

School of Mathematics,
The University of New South Wales.

April, 2004

Abstract

A great deal of classical harmonic analysis is concerned with the properties of the Laplacian $\mathcal{L}$ on Euclidean space $\mathbb{R}^n$ and on the sphere S^{n-1} in $\mathbb{R}^n$. One of the key questions which inspired many of the experts on the subject was the following.

Given a bounded Borel function m on $\mathbb{R}^n$, we can define a bounded operator $m(\mathcal{L})$ on $L^2(\mathbb{R}^n)$ using the functional calculus of self-adjoint operators on a Hilbert space. What conditions on m ensure that this operator extends to a bounded operator on $L^p(\mathbb{R}^n)$ when $p \neq 2$? Many of the answers to this question find their way into the theory of partial differential equations. There are versions of L. Hörmander's classic multiplier theorem [7, Theorem 2.5] which hold for the Laplace–Beltrami operator on the sphere, due to R.R. Coifman and G. Weiss [3] and A. Bonami and J.-L. Clerc [2].

In this thesis, we are concerned with a related operator on the sphere. In complex analysis, one of the directions tangent to the sphere is atypical, namely that which is i times the radial direction. Problems in complex analysis about the boundary behaviour of holomorphic functions are resolved using a modified version of the usual Laplacian, known as the Kohn–Laplacian, in which the square of the partial derivative in the 'atypical' direction is omitted from the usual Laplacian.

We study this operator using the complex analogue of the theory of spherical harmonics (see [9] and [1] for the case of the sphere in $\mathbb{R}^n$). To do so, in this thesis, we quickly summarise a few properties of harmonic functions, and then recall the theory of spherical harmonics on the 'real sphere'. We then develop an analogous theory for the 'complex sphere', which is certainly known to the experts, but we give a leisurely presentation which does not rely on previous exposure to the representation theory of the unitary group. Finally, we develop some 'weighted estimates' for spherical harmonics which enable us to prove a 'weighted Plancherel theorem' for the Kohn–Laplacian. This result, together with the general theorem of M. Cowling and A. Sikora [5], enable us to assert that an analogue of the Hörmander multiplier theorem holds for the Kohn–Laplacian, where the index is optimal.

Acknowledgements

It is my pleasure to acknowledge and thank those whose valuable support during my years at the University of New South Wales contributed greatly to the completion of this thesis.

My family and girlfriend Alanna provided great moral support and in this respect I would also like to thank my friends at UNSW: Keith, Ale, Jozef and Abdul.

Most importantly, I would like to express my sincerest gratitude for the continued inspiration, wisdom and patience of my supervisor Michael Cowling.

Finally, I am very grateful to the Australian Postgraduate Awards scheme and the School of Mathematics at the University of New South Wales for financial support.

Oldrich Klima, 30 November 2003.

Contents

Chapter 1

Harmonic Functions

The purpose of this chapter is to collect some results about harmonic functions in $\mathbb{R}^n$ that will be used repeatedly in this thesis. We assume throughout that $n \geq 2$, since the theory is essentially trivial but different when $n = 1$. In particular, we will need the maximum principle and the solution to the Dirichlet problem for the ball.

Many of the general properties of harmonic functions on $\mathbb{R}^n$ are more easily proved by using the mean-value property of harmonic functions than by using the definition of harmonicity directly. The mean-value theorem—which characterises harmonic functions—is in turn a consequence of an n-dimensional version of Green's theorem that plays a similar role in n dimensions to Cauchy's theorem in the two-dimensional case. References for the material in this chapter are [9] and [1]; the latter contains many more aspects of the theory of harmonic functions.

1.1 Basic Properties of Harmonic Functions

A complex-valued function f defined in a *domain* (that is, a connected open subset) D of $\mathbb{R}^n$ is *harmonic* on D if Δf is identically equal to 0 on D, where Δ is the Laplace operator, or Laplacian, defined by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \, .$$

Since the definition of harmonicity involves taking second order partial derivatives, we must of course impose smoothness conditions on the function f above. It will be assumed that f is in $C^2(D)$, the space of complex-valued functions on D which possess continuous second-order partial derivatives.

Although the above definition was for complex-valued functions, we shall need to work with real-valued harmonic functions as well—for example, in proving the mean-value theorem—and the rules for differentiating complex-valued functions of real variables establish the expected relationship between complex-valued and real-valued harmonic functions: namely, $f \colon D \to \mathbb{C}$ is harmonic if and only if there exist harmonic functions $g, h \colon D \to \mathbb{R}$ such that $f = g + ih$.

Small differences which occur between the real and complex-valued case are directly attributable to the absence of a linear ordering on $\mathbb{C}$.

Now we give some examples of harmonic functions. Apart from trivial examples like constant functions and linear forms (such as the coordinate functions) on $\mathbb{R}^n$, we know from complex analysis that all analytic functions on $\mathbb{C} = \mathbb{R}^2$ are harmonic, as are their real and imaginary parts.

When $n > 2$, we define $f(x) = |x|^{2-n}$. Then $\partial f/\partial x_j = (2-n)x_j\,|x|^{-n}$ and

$$\frac{\partial^2 f}{\partial x_j^2}(x) = (2-n)\,|x|^{-n} - n(2-n)x_j^2\,|x|^{-n-2}$$

when $1 \le j \le n$. By summing the last expression over all such j, we see that $\Delta f(x) = 0$ for all x except 0. Thus f is harmonic on $\mathbb{R}^n \setminus \{0\}$.

This radial function is very important in harmonic function theory—in the next section it will be used to prove the mean-value theorem—and its analogue in the case where $n = 2$ is $\log|x|$.

The set of harmonic functions on D is a vector space since it is obviously closed under addition and scalar multiplication. Further, harmonic functions are preserved by the usual groups of transformations occurring in harmonic analysis: translations, dilations and orthogonal maps.

When $f\colon D \to \mathbb{C}$ and h is a fixed element of $\mathbb{R}^n$, we define $\tau_h f$ to be the function on $D + h = \{y + h : y \in D\}$ given by $(\tau_h f)(x) = f(x-h)$. We refer to τ_h as a *translation operator*; it is easy to see that it commutes with the Laplacian: $\Delta(\tau_h f) = \tau_h \Delta f$. Hence if f is harmonic on D, then $\tau_h f$ is harmonic on $D + h$.

When $a > 0$, we define *dilation by* a to be the operator δ_a taking $f\colon D \to \mathbb{C}$ to $\delta_a f\colon (1/a)D \to \mathbb{C}$, where $(1/a)D = \{(1/a)y : y \in D\}$ and $(\delta_a f)(x) = f(x/a)$. Since $\Delta(\delta_a f) = a^{-2}\delta_a \Delta f$, dilates of harmonic functions are harmonic.

An *orthogonal map* is a linear operator on $\mathbb{R}^n$ that preserves the Euclidean inner product. The orthogonal maps on $\mathbb{R}^n$ form a compact subgroup of the set of all bounded maps on $\mathbb{R}^n$ under composition, which will be denoted by $\mathrm{O}(n)$. If f is in $C^2(D)$ and T in $\mathrm{O}(n)$ is represented by the matrix $[t_{ij}]$, then the chain rule implies that

$$\frac{\partial (f\circ T)}{\partial x_k}(x) = \sum_{j=1}^{n} t_{jk}\frac{\partial f}{\partial x_j}(Tx).$$

Another application of the chain rule gives

$$\frac{\partial^2 (f\circ T)}{\partial x_k^2}(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ik}t_{jk}\frac{\partial^2 f}{\partial x_i x_j}(Tx),$$

and summing this over all k implies that

$$\Delta(f\circ T) = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\sum_{k=1}^{n} t_{ik}t_{jk}\Big)\frac{\partial^2 f}{\partial x_i x_j}(Tx). \tag{1.1}$$

Now we use the fact that T preserves the inner product to deduce that

$$\sum_{k=1}^{n} t_{ik}t_{jk} = Te_i\cdot Te_j = e_i\cdot e_j = \begin{cases} 0 & \text{when } i \ne j \\ 1 & \text{when } i = j, \end{cases} \tag{1.2}$$

and substitute into (1.1) to deduce that

$$\Delta(f\circ T) = (\Delta f)\circ T \qquad \forall T \in \mathrm{O}(n).$$

The Laplacian therefore commutes with orthogonal changes of variables, and all compositions $f \circ T$ of harmonic functions f with orthogonal maps T are harmonic.

1.2 Deeper Properties of Harmonic Functions

As mentioned above, harmonic functions in $\mathbb{R}^2$ share many of the remarkable characteristics of analytic functions in $\mathbb{C}$.

Properties such as the mean-value theorem, the maximum modulus principle and the infinite-differentiability of two-variable harmonic functions are also true in the n-dimensional case. The key to proving them in this case is the following version of Stokes' theorem, in which $\nu(t)$ is the unit outward normal vector to ∂D at the point t, $d\sigma$ is the element of surface measure on ∂D, dx is the element of Lebesgue measure and ∇ denotes the divergence or the gradient.

Theorem 1.1 (Green's Theorem)**.** *Suppose that D is a bounded domain with C^2 boundary in $\mathbb{R}^n$, and that F is a C^2 vector field on (a neighbourhood of) $\overline{D}$. Then*

$$\int_{\partial D} F(y) \cdot \nu(y)\, d\sigma(y) = \int_D \nabla \cdot F(x)\, dx.$$

Now take f and g in $C^2(D)$, and substitute $f\nabla g$ for F in the theorem above; it follows that

$$\int_{\partial D} f\, \frac{\partial g}{\partial \nu}\, d\sigma = \int_D f\, \Delta g + \nabla f \cdot \nabla g\, dx,$$

where

$$\frac{\partial g}{\partial \nu} = \nu \cdot \nabla g,$$

i.e., $\partial g/\partial \nu$ is the directional derivative of g in the direction ν.

Reversing the role of f and g in the above equality and subtracting gives us the following version of Green's identity:

$$\int_{\partial D} f\, \frac{\partial g}{\partial \nu} - g\, \frac{\partial f}{\partial \nu}\, d\sigma = \int_D f\, \Delta g - g\, \Delta f\, dx. \tag{1.3}$$

We will use Green's identity to prove that if a function f is harmonic in an open set in $\mathbb{R}^n$, then the average of f over all balls with centre x whose closure is contained in the set is $f(x)$.

In the following theorem, $B(x, r)$ denotes the ball in $\mathbb{R}^n$ with centre x and radius r. We denote $B(0, 1)$ by B and its boundary by S. The surface measure $\sigma(S)$ of S is denoted by ω_{n-1}.

Theorem 1.2 (Mean-Value Theorem)**.** *Suppose that f is harmonic in a neighbourhood of $\overline{B}(x, r)$. Then*

$$f(x) = \frac{1}{\omega_{n-1}} \int_S f(x + ry)\, d\sigma(y). \tag{1.4}$$

Proof. The translation and dilation invariance properties of the last section imply that it is enough to prove (1.4) in the case where $x = 0$ and $r = 1$. In fact, we will prove that all harmonic functions g on $\overline{B}$ satisfy

$$g(0) = \frac{1}{\omega_{n-1}} \int_S g(y)\, d\sigma(y);$$

we replace g in this equation by the function $g_{x,r}$, defined by $g_{x,r}(y) = f(x + ry)$, to prove (1.4).

Suppose that $n > 2$ and f is harmonic on $\overline{B}$. Define $g(x) = |x|^{2-n}$ and $D_\epsilon = \{x \in \mathbb{R}^n : \epsilon < |x| < 1\}$ for small positive ϵ. Both f and g are harmonic on $\overline{D}_\epsilon$ and Green's identity (1.3) implies that

$$\begin{aligned}
0 &= \int_{\partial D_\epsilon} f\,\frac{\partial g}{\partial \nu} - g\,\frac{\partial f}{\partial \nu}\, d\sigma \\
&= \int_S \Big[f\,\frac{\partial\,|\cdot|^{2-n}}{\partial \nu} - |\cdot|^{2-n}\,\frac{\partial f}{\partial \nu}\Big]\, d\sigma - \int_{\epsilon S} \Big[f\,\frac{\partial\,|\cdot|^{n-2}}{\partial \nu} - |\cdot|^{n-2}\,\frac{\partial f}{\partial \nu}\Big]\, d\sigma \\
&= \int_S f\,\frac{\partial\,|\cdot|^{2-n}}{\partial \nu}\, d\sigma - \int_S \frac{\partial f}{\partial \nu}\, d\sigma - \int_{\epsilon S} f\,\frac{\partial\,|\cdot|^{2-n}}{\partial \nu}\, d\sigma + \epsilon^{2-n} \int_{\epsilon S} \frac{\partial f}{\partial \nu}\, d\sigma \\
&= (2-n) \int_S f\, d\sigma - (2-n)\epsilon^{1-n} \int_{\epsilon S} f\, d\sigma - \int_S \frac{\partial f}{\partial \nu}\, d\sigma + \epsilon^{2-n} \int_{\epsilon S} \frac{\partial f}{\partial \nu}\, d\sigma.
\end{aligned}$$

However the last two terms in the above expression are zero (this follows by choosing g identically equal to -1 in (1.3)), so

$$\int_S f\, d\sigma = \epsilon^{1-n} \int_{\epsilon S} f\, d\sigma.$$

The proof is completed upon noting that the right hand side of this expression approaches $\omega_{n-1} f(0)$ as ϵ tends to 0. $\square$

The case where $n = 2$ of the mean-value theorem is proved in complex function theory; it can also be derived by replacing $|x|^{2-n}$ by $\log |x|$ in the above argument.

Although we will not prove it here, it should be noted that Theorem 1.2 has a converse: all C^2 functions f defined on a domain D which satisfy (1.4) whenever $\overline{B}(x, r) \subset D$ are harmonic. This characterisation of harmonic functions in terms of the mean-value property is of fundamental importance. Using it, it is easy to establish standard theorems about harmonic functions.

For instance, it can be shown that harmonic functions are infinitely differentiable. Since the Laplacian commutes with differentiation, it follows that derivatives of harmonic functions are harmonic. For example, $x_j\,|x|^{-n}$ is harmonic on $\mathbb{R}^n \setminus \{0\}$ when $n \geq 2$, since it is (up to a constant) the j^{th} partial derivative of the harmonic function $|x|^{2-n}$ if $n > 2$ and of $\log |x|$ if $n = 2$.

Other important consequences of the mean-value property are

- Liouville's theorem: a bounded harmonic function on $\mathbb{R}^n$ is constant;
- closure under uniform limits: if a sequence $\{f_k\}$ of harmonic functions converges uniformly to f on compact subsets of an open set D in $\mathbb{R}^n$, then f is harmonic on D;
- the zeros of a real-valued harmonic function are never isolated;

- the maximum principle: a nonconstant harmonic function on a domain does not attain its maximum modulus.

Of these results we prove only the maximum principle, since this is the theorem whose variations we shall have occasion to use later.

Theorem 1.3 (Maximum Principle)**.** *If f is a real-valued harmonic function on a domain D in $\mathbb{R}^n$ which attains its supremum in D, then f is constant.*

Proof. Write M for $\sup_{x\in D}|f(x)|$. Since there is nothing to prove unless f is bounded, suppose that M is finite, and take a in D such that $f(a) = M$. If $\overline{B}(a,r) \subset D$, then Theorem 1.2 implies that

$$M = \frac{1}{\omega_{n-1}} \int_S f(a+ry)\, d\sigma(y).$$

Since the function f is continuous and bounded by M on D, this equality implies that f is identically equal to M on $\partial B(a,r)$. It follows that f is identically equal to M on a neighbourhood of a.

Define the set $f^{-1}(\{M\})$ to be $\{x \in D : f(x) = M\}$; then $f^{-1}(\{M\})$ is both open and closed in D, and the connectedness of D implies that $f^{-1}(\{M\}) = D$. □

Applying the above argument to the function $-f$ establishes a corresponding minimum principle for real-valued harmonic functions.

An appropriate version of the maximum principle will be needed when working with complex-valued harmonic functions.

Corollary 1.4. *Suppose that f is a complex-valued harmonic function on a domain D in $\mathbb{R}^n$ which attains $\sup_{x\in D}|f(x)|$ at some point in D. Then f is constant.*

Proof. Write M for $\sup_{x\in D}|f(x)|$. We may again assume that $M < \infty$, and take a in D such that $|f(a)| = M$.

Suppose that $f(a) = Me^{i\theta}$. Define $g = \mathrm{Re}\{e^{-i\theta}f\}$. Then g is a harmonic, real-valued function, and

$$g(y) = \mathrm{Re}\{e^{-i\theta}f(y)\} \le \sup_{x\in D}\left|e^{-i\theta}f(x)\right| = M = g(a)$$

for every y in D. Thus Theorem 1.3 implies that g is identically equal to M, and the inequality

$$\begin{aligned} M^2 \ge |f(x)|^2 &= \mathrm{Re}\{e^{-i\theta}f(x)\}^2 + \mathrm{Im}\{e^{-i\theta}f(x)\}^2 \\ &= M^2 + \mathrm{Im}\{e^{-i\theta}f(x)\}^2 \end{aligned}$$

shows that $\mathrm{Im}(e^{-i\theta}f)$ is identically equal to 0. Thus $e^{-i\theta}f$ is constant and f is constant. □

Another way to phrase the maximum principle is that a nonconstant harmonic function on a domain D cannot attain its maximum in D. This is obvious for unbounded functions and follows directly from Corollary 1.4 for bounded ones.

On the other hand, a continuous function on a compact set must attain its maximum somewhere on that set. Combining these remarks establishes the following useful corollary (from now on we assume that all functions are complex-valued unless otherwise specified).

Corollary 1.5. *Suppose that D is a bounded domain and f in $C^2(\overline{D})$ is harmonic in D and nonconstant. Then $|f|$ attains its maximum only on ∂D.*

We can see from Corollary 1.5 that, if f is identically equal to 0 on ∂D, then f is identically equal to 0 on $\overline{D}$, and a simple application of this observation to the difference of two functions implies that a harmonic function on a bounded open set is uniquely determined by its value on the boundary.

Corollary 1.6. *Suppose that D is a bounded open set, and that f_1 and f_2 are continuous functions on $\overline{D}$, which are harmonic in D and agree on ∂D. Then f_1 is identically equal to f_2 on $\overline{D}$.*

Corollary 1.6 is the form of the maximum principle that we shall find most useful: it will be used repeatedly in the next chapter.

1.3 The Poisson Kernel

From this point on, we will be interested in the situation where the domain D of the previous sections is the unit ball B in $\mathbb{R}^n$ (which we will sometimes write B_n to emphasize that its dimension is n). Recall that S denotes the boundary of B.

Suppose that f is a harmonic function on B which is continuous on $\overline{B}$. The maximum principle (in the form of Corollary 1.6) identifies f with its restriction $f|_S$ to S, since f is the unique continuous extension of $f|_S$ that is harmonic in B. Another way to put it is that the maximum principle provides a one-to-one correspondence between the space of continuous functions f in $C(\overline{B})$ which are harmonic on B and a subspace of $C(S)$.

The following two questions arise immediately:

- Is the above subspace $C(S)$ itself?
- If g in $C(S)$ can be extended continuously to a harmonic function f on the ball, then how do we construct f from g?

These two questions comprise the Dirichlet problem for the ball, whose solution provides an affirmative answer to the first question and says that f in the second question is related to g by integrating the product of g and the *Poisson kernel* (for the ball), defined on $S \times B$ by

$$p(\xi, x) = \frac{1}{\omega_{n-1}} \cdot \frac{1 - |x|^2}{|x - \xi|^n} \qquad \forall \xi \in S \quad \forall x \in B.$$

The precise relation of g in $C(S)$ to its harmonic extension f on $\overline{B}$ is that the latter is given by

$$f(x) = \begin{cases} P[g](x) & \text{when } x \text{ is in } B \\ g(x) & \text{when } x \text{ is in } S, \end{cases} \tag{1.5}$$

where the *Poisson integral* $P[g]$ of g is defined by

$$P[g](x) = \int_S p(\xi, x)\, g(\xi)\, d\sigma(\xi) \qquad \forall x \in B.$$

A straightforward but tedious calculation shows that $p(\xi, \cdot)$ is harmonic on B, and differentiating under the integral on the right hand side proves that the same is true of $P[g]$.

Proving that f is continuous on $\overline{B}$ (that is, showing that $P[g](x)$ tends to $g(\xi)$ as x tends to ξ in S) is slightly less straightforward and follows from certain 'approximate identity' properties of the Poisson kernel. Perhaps the most important of these is that

$$\int_S p(\xi, x)\, d\sigma(\xi) = 1 \qquad \forall x \in B. \tag{1.6}$$

To prove (1.6), first expand the dot products to show that

$$|x - \xi|^2 = \left|\xi|x| - x/|x|^2\right| \qquad \forall x \in B \setminus \{0\} \quad \forall \xi \in S.$$

It follows immediately that

$$p(\xi, x) = p(x/|x|, \xi\,|x|) \qquad \forall x \in B \setminus \{0\} \quad \forall \xi \in S.$$

We now change perspective and consider ξ as a variable in $\mathbb{R}^n$. Fix x such that $0 < |x| < 1$. Since the Poisson kernel is harmonic, $\xi \mapsto p(x/\,|x|\,, \xi\,|x|)$ is a harmonic function on $B(0, |x|^{-1})$ (and hence on B). The mean-value theorem implies that

$$\int_S p(x/|x|, \xi\,|x|)\, d\sigma(\xi) = \omega_{n-1} p(x/|x|, 0) = 1,$$

and (1.6) is proved.

We can use (1.6) to write

$$P[g](x) - g(\eta) = \int_S [g(\xi) - g(\eta)]\, p(\xi, x)\, d\sigma(\xi) \qquad \forall \eta \in S.$$

When $\epsilon > 0$, we choose δ such that $|g(\xi) - g(\eta)| < \epsilon$ for all ξ such that $|\xi - \eta| < \delta$ on S. Then we can split the above integral over S into two parts: the integral over $\{\xi \in S : |\xi - \eta| < \delta\}$ is clearly bounded by ϵ, whereas the absolute value of the integral over $\{\xi \in S : |\xi - \eta| \geq \delta\}$ is less than $2\,\|g\|_\infty\, \delta^{-n}(1 - |x|^2)$, which approaches 0 as x tends to η. We conclude that, for all η in S, $P[g](x)$ tends to $g(\eta)$ as x tends to η, and hence that f is continuous.

To summarise, we have proved the following theorem.

Theorem 1.7 (Solution of the Dirichlet Problem)**.** *Suppose that g is in $C(S)$ and f is defined as in* (1.5)*. Then f is continuous on $\overline{B}$, harmonic in B and equal to g on S.*

The maximum principle in the form of Corollary 1.6 implies that f is the unique function with the properties in the above theorem.

The Poisson kernel is intimately connected to spherical harmonics. For instance, we will establish the remarkable fact that the Poisson integral of a polynomial is also a harmonic polynomial, and then use this to prove a key theorem about spherical harmonics. It is also true that the Poisson kernel can be expressed as a series of certain spherical harmonics.

Chapter 2

Real Spherical Harmonics

In this chapter we are concerned with a certain class of complex-valued polynomials on the unit sphere in $\mathbb{R}^n$ that are the n-dimensional analogues of the functions $(x, y) \mapsto (x + iy)^k$ defined on the unit circle T. Just as all L^2 functions on T can be expanded in Fourier series, we will see that L^2 functions defined on the sphere in $\mathbb{R}^n$ are uniquely expressible in terms of these 'spherical harmonics'.

Spherical harmonics are very useful in understanding polynomials on $\mathbb{R}^n$ and it is from this point of view that their theory is presented. Theorem 2.7, which expresses the precise nature of the relationship between arbitrary polynomials and spherical harmonics, is perhaps the most crucial result about spherical harmonics and leads directly to the important L^2 decomposition mentioned above.

In the final section of this chapter, we study a special subclass of spherical harmonics that depend only upon one variable. These 'zonal spherical harmonics' have a surprising connection with the Poisson kernel for the ball, and an explicit formula for zonal spherical harmonics follows from this connection.

Many of the concepts and results in this chapter have analogues in the complex n-dimensional case, to be developed in Chapter 3. Standard references for the material in this chapter are [9] and [1].

We assume throughout this chapter that $n \geq 2$. We use multi-index notation: for an n-tuple α in $\mathbb{N}^n$ (that is, $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each α_i is a nonnegative integer), we write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. We denote by $[\mathbb{N}^n]_k$ the set of all α in $\mathbb{N}^n$ for which $|\alpha| = k$. We define the monomial x^α and the partial differential operator $\partial^\alpha / \partial x^\alpha$ by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

A polynomial in $\mathbb{R}^n$ is a (finite complex) linear combination of monomials.

We will use orthogonal changes of variables quite frequently. The equality (1.2) shows that if T is orthogonal, then T^*T is the identity. We will need several such properties of orthogonal maps, and we prove a lemma here.

Lemma 2.1. *Suppose that (u, u') and (v, v') are pairs of unit vectors in $\mathbb{R}^n$, and that $u \cdot u' = v \cdot v'$. Then there exists an orthogonal map T such that $Tu = v$ and $Tu' = v'$.*

Proof. Suppose first that $|u \cdot u'| < 1$. Take u'' to be $u' - (u \cdot u')u$; then $u \cdot u'' = 0$, i.e., u and u'' are orthogonal, and

$$|u''|^2 = (u' - (u \cdot u')u) \cdot (u' - (u \cdot u')u) = 1 - (u \cdot u')^2 \neq 0.$$

Similarly, we define v'' to be $v' - (v \cdot v')v$; then v and v'' are orthogonal, and $|u''| = |v''|$.

We apply the Gram–Schmidt process to the pairs $(u, |u''|^{-1} u'')$ and $(v, |v''|^{-1} v'')$, augmented by the standard bases, and construct orthonormal bases $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ such that $u_1 = u$ and $u_2 = |u''|^{-1} u''$, while $v_1 = v$ and $v_2 = |v''|^{-1} v''$. We define the linear map T on $\mathbb{R}^n$ by requiring that $Tu_i = v_i$ when $i = 1, \dots, n$. Then T is orthogonal, and $Tu = v$ by construction; further, $Tu'' = v''$, whence $Tu - (u \cdot u')Tu' = v - (v \cdot v')v'$, and $Tu' = v'$, as required.

If $u \cdot u' = \pm 1$, then a similar but easier proof, starting with u and v only, establishes the desired result. □

2.1 Harmonic Polynomials

The set of all monomials x^α is linearly independent: suppose that S is a finite subset of $\mathbb{N}^n$ and $\sum_{\alpha \in S} a_\alpha x^\alpha = 0$. We apply $\partial^\alpha / \partial x^\alpha$ to both sides of the above equality and take x equal to 0 to show that $a_\alpha = 0$.

It is often convenient to group together all the terms $a_\alpha x^\alpha$ for which $|\alpha| = k$. We therefore introduce the concept of homogeneous polynomials. A polynomial f is said to be *homogeneous of degree* k if $f(tx) = t^k f(x)$ for all x in $\mathbb{R}^n$ and all t in $\mathbb{R}$, or equivalently, if it consists solely of terms $a_\alpha x^\alpha$ where $|\alpha| = k$. We write $\mathcal{P}_k(\mathbb{R}^n)$ for the vector space of homogeneous polynomials of degree k on $\mathbb{R}^n$, but usually abbreviate this to $\mathcal{P}_k$. The *degree* of a polynomial $\sum_{\alpha \in S} a_\alpha x^\alpha$ is the greatest integer m for which there exists an n-tuple α in S such that $|\alpha| = m$ and $a_\alpha \neq 0$; we write $\deg(f)$ for the degree of f.

If $\deg(f) = m$ and we group all terms for which $|\alpha| = k$ in the above sum, as k varies between 0 and m, then we see that f may be expressed uniquely as a sum of homogeneous polynomials:

$$f = f_0 + f_1 + \cdots + f_m, \tag{2.1}$$

where f_k is in $\mathcal{P}_k$; the uniqueness follows from the linear independence of the monomials. The polynomial f_k is called the k^{th} homogeneous component of f.

Since the monomials $\{x^\alpha : \alpha \in [\mathbb{N}^n]_k\}$ form a basis for $\mathcal{P}_k$, we see that $\dim(\mathcal{P}_k)$ is equal to the number of n-tuples $(\alpha_1, \dots, \alpha_n)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_n = k$. The following combinatorial argument gives the precise value of $\dim(\mathcal{P}_k)$. The number in question is equal to the number of ways of distributing k identical objects into n distinct groups. Each particular way of doing this can be uniquely represented by a sequence consisting of k white balls and $n-1$ black balls, since the n regions separated by the $n-1$ black balls represent the n distinct groups. For example, if $n = 4$ and $k = 7$, then the 4-tuple (3,0,2,2) is represented by the sequence

$$\circ\circ\circ\bullet\bullet\circ\circ\bullet\circ\circ\,.$$

Since the number of ways of arranging k white balls and $n-1$ black balls in a line is precisely $(n+k-1)!/k!\,(n-1)!$, we conclude that

$$\dim(\mathcal{P}_k) = \binom{n+k-1}{k}. \tag{2.2}$$

Continuing the analysis of homogeneous polynomials, recall that a polynomial f is in $\mathcal{P}_k$ if and only if $f(tx) = t^k f(x)$ for all real t and all x in $\mathbb{R}^n$. Therefore to identify a polynomial in $\mathcal{P}_k$, it suffices to specify its values on the unit sphere S:

$$f(x) = |x|^k f\big(x/|x|\big) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Since we can multiply any polynomial by $|\cdot|^2$ without changing its value on the sphere, there is significant overlap between spaces $\mathcal{P}_k$ of different degrees; if f is in $\mathcal{P}_k$, then $f|_S$ is the restriction of infinitely many homogeneous polynomials of different degrees. Thus we cannot reconstruct f uniquely from $f|_S$ unless we know its degree.

If, however, a homogeneous polynomial with a certain restriction to S is also harmonic, then the maximum principle says that we can uniquely identify it by its restriction to the unit sphere. This means that a given polynomial on S can be the restriction of at most one harmonic, homogeneous polynomial. It will be proved below that every polynomial on S is in fact the restriction to S of a harmonic polynomial on $\mathbb{R}^n$.

We write $\mathcal{H}_k(\mathbb{R}^n)$, or just $\mathcal{H}_k$, for the set of harmonic, homogeneous polynomials of degree k on $\mathbb{R}^n$. The remarks above show that we can identify $\mathcal{H}_k$ with $\mathcal{H}_k(S)$, the space of spherical harmonics of degree k, defined to be $\{f|_S : f \in \mathcal{H}_k\}$.

By writing a polynomial f in terms of its homogeneous components as in (2.1), we see that the unique homogeneous decomposition of Δf is $g_0 + \cdots + g_{m-2}$, where $g_k = \Delta f_{k+2}$. Thus Δf is identically equal to 0 if and only if every Δf_k is identically equal to 0, which holds if and only if every f_k is in $\mathcal{H}_k$. We have proved the following result.

Proposition 2.2. *All the homogeneous components of harmonic polynomials are harmonic.*

Our remarks prior to the definition of $\mathcal{H}_k$ show that the spaces $\mathcal{H}_k$ are disjoint, and so do not suffer from the same overlap (arising from multiples of $|\cdot|^2$) as $\mathcal{P}_k$. The next section will show that harmonicity is precisely the condition that allows us to sort out this 'absolute value overlap'.

2.2 Decomposition into Spherical Harmonics

Suppose that a function f on the unit sphere is the restriction to S of a harmonic polynomial g. Then g is the Poisson integral $P[f]$ of f, by Corollary 1.6 and Theorem 1.7. Note that a polynomial is harmonic on the ball if and only if it is harmonic on all of $\mathbb{R}^n$.

As the following remarkable theorem shows, the Poisson integral of a polynomial is always a (harmonic) polynomial. This means that any polynomial f on the unit sphere S is the restriction to S of a unique harmonic polynomial, and so is a sum of spherical harmonics by Proposition 2.2.

Theorem 2.3. *Suppose that f is a polynomial in $\mathbb{R}^n$. Then its Poisson integral $P[f]$ is also a polynomial, and there exists a polynomial g, whose degree is at most* $\deg(f) - 2$, *such that*

$$P[f](x) = (1 - |x|^2)g(x) + f(x) \qquad \forall x \in \mathbb{R}^n. \tag{2.3}$$

Proof. Given f, we seek g such that $\Delta(1-|\cdot|^2)g = -\Delta f$.

Let V denote the finite-dimensional vector space of polynomials of degree at most $\deg(f)$. Then $g \mapsto \Delta(1-|\cdot|^2)g$ is a linear operator on V. This operator must be injective because if $\Delta(1-|\cdot|^2)g = 0$, then $(1-|\cdot|^2)g$ is harmonic, and since $(1-|\cdot|^2)g$ is identically 0 on S, it follows that g is identically 0 by the maximum principle.

Since a linear operator on a finite-dimensional vector space is injective if and only if it is surjective, there exists g in V satisfying (2.3). ☐

Theorem 2.3 is fundamental to the study of spherical harmonics. The following corollary was mentioned above; when combined with the Weierstrass approximation theorem, it shows that finite sums of spherical harmonics are dense in $L^2(S)$. We shall have more to say about this later in the section.

Corollary 2.4. *Every polynomial on S is a unique sum of spherical harmonics.*

Proof. On S, the polynomial f coincides with the harmonic polynomial $P[f]$. But Proposition 2.2 says that $P[f]$ is a sum of spherical harmonics. ☐

If f is a nonzero harmonic polynomial and k is in $\mathbb{N}$, then a direct consequence of Corollary 1.6 is that $|\cdot|^{2k} f$ cannot be harmonic, otherwise $f|_S$ would have two distinct harmonic extensions to B.

Theorem 2.3 implies that this is also true for all nonzero polynomials f.

Corollary 2.5. *Suppose that f is a polynomial and that $f \neq 0$. Then $|\cdot|^2 f$ is not harmonic.*

Proof. If $|\cdot|^2 f$ were harmonic, then the uniqueness of the Dirichlet problem and the fact that $|\cdot|^2 f = f$ on S would imply that $P[f] = |\cdot|^2 f$. Since $f \neq 0$, this would imply that $\deg(P[f]) > \deg(f)$, in contradiction to Theorem 2.3. We conclude that $|\cdot|^2 f$ cannot be harmonic. ☐

From Corollary 2.4, a polynomial of degree k may be written uniquely in the form

$$f = f_0 + \cdots + f_k \tag{2.4}$$

on S, where f_j is in $\mathcal{H}_j$, but such an expression cannot possibly hold everywhere on $\mathbb{R}^n$ unless f is harmonic. The question of how the spherical harmonics in (2.3) are related to values of f on the whole of $\mathbb{R}^n$ is best approached by restricting attention to homogeneous polynomials, for we saw in the previous section that at most one element of $\mathcal{P}_k$ can have a given restriction to the unit sphere. From now on, we shall therefore assume that f is in $\mathcal{P}_k$.

The next decomposition theorem clarifies the relationship between $\mathcal{P}_k$ and $\mathcal{H}_k$. It will be used to show that finding any value of f from the f_j in (2.4) amounts to nothing more than inserting appropriate multiples of $|\cdot|^2$ in that formula. Note that $\mathcal{P}_0 = \mathcal{H}_0$ and $\mathcal{P}_1 = \mathcal{H}_1$.

Theorem 2.6. *Suppose that f is in $\mathcal{P}_k$, where $k \geq 2$. Then f can be written uniquely in the form $f_k + |\cdot|^2 g_k$, where f_k is in $\mathcal{H}_k$ and g_k is in $\mathcal{P}_{k-2}$. That is,*

$$\mathcal{P}_k = \mathcal{H}_k \oplus |\cdot|^2 \mathcal{P}_{k-2}.$$

Proof. By Theorem 2.3, there exists a polynomial g of degree at most $k-2$, such that

$$f = P[f] + |\cdot|^2 g - g.$$

We can now equate the k^{th} homogeneous components of both sides above and use Proposition 2.2 to deduce that

$$f = f_k + |\cdot|^2 g_k,$$

where f_k is in $\mathcal{H}_k$ and g_k is in $\mathcal{P}_{k-2}$.

The uniqueness of this representation follows directly from Corollary 2.5: if $F_k + |\cdot|^2 G_k$ were another such representation, then it would follow that

$$|\cdot|^2 (g_k - G_k) = F_k - f_k.$$

The right hand side of this equality is harmonic by construction, so the left hand side is harmonic too; then $g_k - G_k = 0$ by Corollary 2.5, and hence $f_k - F_k = 0$ also. □

By combining (2.2) with Theorem 2.6, we find the following formula for the dimension of $\mathcal{H}_k$ when $k \geq 2$:

$$\begin{aligned}
\dim(\mathcal{H}_k) &= \dim(\mathcal{P}_k) - \dim(\mathcal{P}_{k-2}) \\
&= \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \\
&= \frac{(n+k-1)!}{k!\,(n-1)!} - \frac{(n+k-3)!}{(k-2)!\,(n-1)!} \\
&= \frac{(n+k-1)(n+k-2) - k(k-1)}{k(n-1)} \binom{n+k-3}{k-1} \\
&= \frac{(n+k-1)(n-2) + kn}{k(n-1)} \binom{n+k-3}{k-1} \\
&= \frac{n+2k-2}{k} \binom{n+k-3}{k-1}.
\end{aligned}$$

We conclude that

$$\dim(\mathcal{H}_k) = \frac{(n+2k-2)\,(n+k-3)!}{k!\,(n-2)!}. \tag{2.5}$$

If $k = 0$ or 1, then $\dim(\mathcal{H}_k) = \dim(\mathcal{P}_k)$; the formula (2.5) still holds in these cases.

Theorem 2.6 allows us to establish the most important decomposition theorem for polynomials on $\mathbb{R}^n$.

Theorem 2.7. *The following decomposition holds:*

$$\mathcal{P}_k = \mathcal{H}_k \oplus |\cdot|^2 \mathcal{H}_{k-2} \oplus \cdots \oplus |\cdot|^{k-2\lfloor k/2 \rfloor} \mathcal{H}_{k-2\lfloor k/2 \rfloor}.$$

Proof. We apply the result of Theorem 2.6 to $\mathcal{P}_{k-2}$ and continue the process. After $\lfloor k/2 \rfloor$ applications we reach the desired result—note that $k - 2\lfloor k/2 \rfloor$ is either 0 or 1 and that $\mathcal{P}_j = \mathcal{H}_j$ when $j = 0$ or 1. □

Theorem 2.7 reveals that when f is a homogeneous polynomial of degree k, then the only terms which can be nonzero in (2.4) are those f_j where the parity of j is the same as that of k. It also shows how harmonic polynomials combine with powers of $|\cdot|^2$ to generate all polynomials on $\mathbb{R}^n$.

If we restrict attention to the unit sphere in $\mathbb{R}^n$, as we do for the rest of this chapter, then, by Theorem 2.7,

$$\mathcal{P}_k(S) = \mathcal{H}_k(S) \oplus \mathcal{H}_{k-2}(S) \oplus \cdots \oplus \mathcal{H}_{k-2\lfloor k/2 \rfloor}(S).$$

It is obvious that every $\mathcal{H}_k(S)$ is a finite-dimensional subspace of the Hilbert space $L^2(S)$ with inner product given by

$$\langle f, g \rangle = \int_S f\, \bar{g}\, d\sigma.$$

Green's identity implies that spherical harmonics of different degrees are orthogonal in $L^2(S)$.

Proposition 2.8. *Suppose that* $k \neq m$. *Then*

$$\mathcal{H}_k(S) \perp \mathcal{H}_m(S)$$

in $L^2(S)$.

Proof. We substitute f in $\mathcal{H}_k$ and $\bar{g}$ in $\mathcal{H}_m$ into Green's identity (1.3) to show that

$$\int_S f \frac{\partial \bar{g}}{\partial \nu} - \bar{g} \frac{\partial f}{\partial \nu}\, d\sigma = 0. \tag{2.6}$$

Fix x in S and define $\bar{g}_x : \mathbb{R} \to \mathbb{C}$ by $\bar{g}_x(t) = \bar{g}(tx) = t^m \bar{g}(x)$; then

$$\frac{\partial \bar{g}}{\partial \nu}(x) = \bar{g}_x{}'(1) = m\bar{g}(x).$$

Similarly,

$$\frac{\partial f}{\partial \nu}(x) = kf(x).$$

Substituting into (2.6),

$$(m-k) \int_S f\, \bar{g}\, d\sigma = 0,$$

and since $k \neq m$, the inner product must be equal to zero. □

We have seen that any polynomial on S is a finite sum of spherical harmonics, and Weierstrass' approximation theorem says that any continuous function on S is a uniform limit of polynomials. These observations, combined with the standard fact that $C(S)$ is dense in $L^2(S)$, imply that the linear span of $\bigcup_{k=0}^{\infty} \mathcal{H}_k(S)$ is dense in $L^2(S)$. We choose an orthonormal basis E_k for every $\mathcal{H}_k$; then $\bigcup_{k=0}^{\infty} E_k$ is an orthonormal basis for $L^2(S)$ and it follows that every f in $L^2(S)$ can be expressed uniquely thus:

$$f = \sum_{k=0}^{\infty} f_k,$$

where f_k is in $\mathcal{H}_k(S)$ and the infinite series converges in norm in $L^2(S)$.

Theorem 2.9. *The space $L^2(S)$ is the orthogonal direct sum of the spaces $\mathcal{H}_k(S)$, i.e., the spaces $\mathcal{H}_k(S)$ are pairwise orthogonal, and*

$$L^2(S) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S).$$

Theorem 2.9 is an n-dimensional analogue of a well-known result from the theory of Fourier series: any L^2 function f on the unit circle has a Fourier series which converges to f in the L^2 norm.

2.3 Zonal Harmonics

We will now study a special class of spherical harmonics which are, in a sense, functions of only one variable.

Fix a point ξ on the unit sphere in $\mathbb{R}^n$ and consider the linear functional on $\mathcal{H}_k(S)$ taking f to $f(\xi)$. Since $\mathcal{H}_k(S)$ is a finite-dimensional subspace of $L^2(S)$, containing only continuous functions, it follows that our 'evaluation functional' is also bounded. We can now apply the Riesz representation theorem of Hilbert space theory to prove the existence of a unique element $Z_\xi^{(k)}$ in $\mathcal{H}_k(S)$ such that

$$f(\xi) = \langle f, Z_\xi^{(k)} \rangle = \int_S f\, \bar{Z}_\xi^{(k)}\, d\sigma \qquad \forall f \in \mathcal{H}_k(S). \tag{2.7}$$

The polynomial $Z_\xi^{(k)}$ is called a *zonal spherical harmonic of degree* k with *pole* ξ.

It should be noted that a variant of (2.7) holds for all f in $L^2(S)$: if f_k is the projection of f onto the closed subspace $\mathcal{H}_k(S)$ of $L^2(S)$, then the linear functional $f \mapsto f_k(\xi)$ is bounded, being a composition of two bounded operators. The Riesz representation theorem proves the existence of a unique $Y_\xi^{(k)}$ in $L^2(S)$ such that

$$f_k(\xi) = \langle f, Y_\xi^{(k)} \rangle \qquad \forall f \in L^2(S).$$

In particular this is true whenever $f = f_k$ in $\mathcal{H}_k(S)$, so comparing with (2.7), we see that $Y_\xi^{(k)} = Z_\xi^{(k)}$, by the uniqueness of the representation. Thus

$$f_k(\xi) = \langle f, Z_\xi^{(k)} \rangle \qquad \forall f \in L^2(S).$$

A direct consequence of the defining property (2.7) is that zonal harmonics are real-valued. For all ξ and x in S, the definition implies that

$$\bar{Z}_x^{(k)}(\xi) = \langle \bar{Z}_x^{(k)}, Z_\xi^{(k)} \rangle = \int_S \bar{Z}_x^{(k)}\, \bar{Z}_\xi^{(k)}\, d\sigma = \langle \bar{Z}_\xi^{(k)}, Z_x^{(k)} \rangle = \bar{Z}_\xi^{(k)}(x), \tag{2.8}$$

so that $Z_x^{(k)}(\xi) = Z_\xi^{(k)}(x)$. On the other hand,

$$Z_\xi^{(k)}(x) = \langle Z_\xi^{(k)}, Z_x^{(k)} \rangle = \langle Z_x^{(k)}, Z_\xi^{(k)} \rangle^- = \bar{Z}_x^{(k)}(\xi). \tag{2.9}$$

It is now an immediate consequence of (2.8) and (2.9) that $Z_\xi^{(k)}$ is real-valued.

To deduce another elementary consequence of the defining property, suppose that T is an orthogonal map, and recall that $|\det T| = 1$. A simple change of variable in the integral given by the second inner product below then shows that

$$\langle f, Z_{T\xi}^{(k)} \rangle = f(T\xi) = \langle f \circ T, Z_\xi^{(k)} \rangle = \langle f, Z_\xi^{(k)} \circ T^{-1} \rangle \qquad \forall f \in \mathcal{H}_k.$$

Now we apply the uniqueness of the inner product representation to conclude that $Z_{T\xi}^{(k)} = Z_\xi^{(k)} \circ T^{-1}$. We have proved the following proposition.

Proposition 2.10. *For all ξ and x in S and T in $\mathrm{O}(n)$,*

(i) $Z_\xi^{(k)}(x) = Z_x^{(k)}(\xi) \in \mathbb{R}$;

(ii) $Z_{T\xi}^{(k)} = Z_\xi^{(k)} \circ T^{-1}$.

Part (ii) of Proposition 2.10 enables us to give a nice geometric description of zonal spherical harmonics with pole ξ. First, it implies that $Z_{T\xi}^{(k)}(Tx) = Z_\xi^{(k)}(x)$ for all T in $\mathrm{O}(n)$. Hence

$$Z_\xi^{(k)}(Tx) = Z_\xi^{(k)}(x) \tag{2.10}$$

for all x in S and T in $\mathrm{O}(n)$ such that $T\xi = \xi$.

Now a hyperplane which is perpendicular to ξ intersects S if and only if its equation is $\xi \cdot x = c$ for some c in $[-1, 1]$. We will write $\xi(c)$ for the intersection $\{x \in S : \xi \cdot x = c\}$. Our geometric result asserts that the level sets of the zonal harmonics with pole ξ are the sets $\xi(c)$.

Proposition 2.11. *The function $Z_\xi^{(k)}$ is constant on every $\xi(c)$.*

Proof. Since $\xi(1) = \{\xi\}$ and $\xi(-1) = \{-\xi\}$, we may assume that $-1 < c < 1$.

An orthogonal map T which fixes ξ maps the hyperplane $\{x \in \mathbb{R}^n : \xi \cdot x = c\}$ into itself and preserves norms, hence maps $\xi(c)$ into itself. In view of (2.10), it suffices to prove that if x_1 and x_2 are in $\xi(c)$, then $x_2 = Tx_1$ for some T in $\mathrm{O}(n)$ which fixes ξ. This follows by applying Lemma 2.1 to the pairs of vectors (ξ, x_1) and (ξ, x_2). □

We have shown that Proposition 2.11 follows from (2.10). The converse follows from the observation that every rotation T which fixes ξ maps $\xi(c)$ into itself.

Since S is the union of the sets $\xi(c)$ over $c \in [-1, 1]$, Proposition 2.11 shows that the zonal spherical harmonic $Z_\xi^{(k)}$ is a function of the real variable $\xi \cdot x$ on S (values for other x are then found by homogeneity). The next theorem shows that, up to a constant multiple, $Z_\xi^{(k)}$ is the only element of $\mathcal{H}_k(S)$ with this property.

Theorem 2.12. *Fix ξ in S. Then f in $\mathcal{H}_k(S)$ is constant on every $\xi(c)$ if and only if it is a scalar multiple of $Z_\xi^{(k)}$.*

Proof. Since every $\xi(c)$ is mapped into itself by all T in $\mathrm{O}(n)$ such that $T\xi = \xi$, it is clear that $f(Tx) = f(x)$ for all x in S and all such T.

Assume, without loss of generality, that $\xi = e_1 = (1, 0, \ldots, 0)$. Write x in $\mathbb{R}^n$ as (x_1, x') in $\mathbb{R} \times \mathbb{R}^{n-1}$. Now T in $\mathrm{O}(n)$ fixes e_1 if and only if $T(x_1, x') = (x_1, T'x')$ for some $T' \in \mathrm{O}(n-1)$. By homogeneity, $f(x_1, T'x') = f(x_1, x')$ for all (x_1, x') in $\mathbb{R} \times \mathbb{R}^{n-1}$ and all T' in $\mathrm{O}(n-1)$. Fix x_1 in $\mathbb{R}$, and consider the polynomial $g(x') = f(x_1, x')$ on $\mathbb{R}^{n-1}$. Since g is even, $g(t, 0, \ldots, 0) = \sum_{j=0}^{\lfloor k/2 \rfloor} c_j t^{2j}$, and then $g(x') = \sum_{j=0}^{\lfloor k/2 \rfloor} c_j \, |x'|^{2j}$. It follows that

$$f(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} a_j x_1^{k-2j} \, |x'|^{2j} .$$

Using the fact that f is harmonic and differentiating, we see that

$$\begin{aligned} 0 &= \Delta f(x) \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor - 1} \big[(k-2j)(k-2j-1)a_j + (2n-2+4j)(j+1)a_{j+1} \big] x_1^{k-2-2j} \, |x'|^{2j} . \end{aligned}$$

Thus

$$(k-2j)(k-2j-1)a_j + (2n-2+4j)(j+1)a_{j+1} = 0$$

when $j = 0, 1, \ldots, \lfloor k/2 \rfloor - 1$, and all a_j are iteratively determined by the first nonzero a_j (which is a_1 when $n = 1$ and a_0 otherwise). It follows that two elements of $\mathcal{H}_k(S)$ that are constant on all $e_1(c)$ must be constant multiples of each other, so the fact that $Z_{e_1}^{(k)}$ is nonzero and constant on all $e_1(c)$ implies the result. $\square$

Consider a function f in $L^2(S)$ and write $f = \sum_{k=0}^{\infty} f_k$ as in Theorem 2.9. We know from the definition of zonal harmonics and the remarks following it that $f_k(x) = \langle f, Z_x^{(k)} \rangle$ for all points x on the unit sphere. Hence the above decomposition suggests that we may be able to find f on S by integrating it against an infinite series of zonal harmonics.

Consider the following formal computation, based on (2.9):

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} f_k(x) \\ &= \sum_{k=0}^{\infty} \langle f, Z_x^{(k)} \rangle \\ &= \sum_{k=0}^{\infty} \int_S f(\xi) \, \bar{Z}_x^{(k)}(\xi) \, d\sigma(\xi) \\ &= \sum_{k=0}^{\infty} \int_S f(\xi) \, Z_\xi^{(k)}(x) \, d\sigma(\xi) \\ &= \int_S \Big(\sum_{k=0}^{\infty} Z_\xi^{(k)}(x) \Big) f(\xi) \, d\sigma(\xi) \qquad \forall x \in S. \end{aligned}$$

If the infinite sum of zonal harmonics in this formal sum converged locally uniformly in x, then we could define $q(\xi, x)$ to be this infinite sum. Being a locally uniform limit of harmonic functions, $q(\xi, \cdot)$ would be harmonic in B for every ξ in S. The above computation suggests that q would have to be the Poisson kernel, in view of the uniqueness of the solution to the Dirichlet problem.

In other words, we are led to believe that

$$p(\xi, x) = \sum_{k=0}^{\infty} Z_{\xi}^{(k)}(x) \qquad \forall \xi \in S \quad \forall x \in B,$$

where the values of $Z_{\xi}^{(k)}$ on B are found from those on S by homogeneity.

To prove this formula, we must find estimates on the size of zonal spherical harmonics that establish the locally uniform convergence of the series above.

Write $d(k)$ for $\dim(\mathcal{H}_k)$ and let $\{Y_1, \ldots Y_{d(k)}\}$ be an arbitrary orthonormal basis for $\mathcal{H}_k$. From (2.7),

$$Z_{\xi}^{(k)}(x) = \sum_{j=1}^{d(k)} \big\langle Z_{\xi}^{(k)}, Y_j \big\rangle \, Y_j(x) = \sum_{j=1}^{d(k)} \big\langle Y_j, Z_{\xi}^{(k)} \big\rangle^{-} \, Y_j(x) = \sum_{j=1}^{d(k)} \bar{Y}_j(\xi) \, Y_j(x)$$

for all x and ξ in S. Replacing ξ by x in the above formula shows that

$$Z_{x}^{(k)}(x) = \sum_{j=1}^{d(k)} |Y_j(x)|^2 \,. \tag{2.11}$$

On the other hand, part (ii) of Proposition 2.10 implies that

$$Z_{Tx}^{(k)}(Tx) = Z_{x}^{(k)}(x) \qquad \forall x \in S \quad \forall T \in \mathrm{O}(n).$$

Given any two points x_1 and x_2 in S, there exists an orthogonal map T such that $Tx_1 = x_2$, by Lemma 2.1 applied to the pairs (x_1, x_1) and (x_2, x_2). Therefore both sides of (2.11) are independent of x in S, i.e., $\sum_{j=1}^{d(k)} |Y_j|^2$ is constant on S.

To find its constant value, C say, we integrate:

$$C\,\omega_{n-1} = \int_S C \, d\sigma = \sum_{j=1}^{d(k)} \int_S |Y_j(x)|^2 \, d\sigma(x) = \sum_{j=1}^{d(k)} 1 = d(k).$$

Thus for any orthonormal basis $\{Y_1, \ldots, Y_{d(k)}\}$ of $\mathcal{H}_k(S)$,

$$Z_{x}^{(k)}(x) = \sum_{j=1}^{d(k)} |Y_j(x)|^2 = d(k)\omega_{n-1}^{-1} \qquad \forall x \in S.$$

Using orthogonality and this equality shows that

$$\begin{aligned}\|Z_\xi^{(k)}\|_2^2 &= \langle Z_\xi^{(k)}, Z_\xi^{(k)}\rangle \\ &= \sum_{i=1}^{d(k)}\sum_{j=1}^{d(k)} \bar{Y}_i(\xi)\, Y_j(\xi)\, \langle Y_i, Y_j\rangle \\ &= \sum_{j=1}^{d(k)} |Y_j(\xi)|^2 \\ &= d(k)\omega_{n-1}^{-1}.\end{aligned}$$

We can use this result and Schwarz's inequality to estimate $\left|Z_\xi^{(k)}(x)\right|$:

$$\left|Z_\xi^{(k)}(x)\right| = \left|\langle Z_\xi^{(k)}, Z_x^{(k)}\rangle\right| \le \|Z_\xi^{(k)}\|_2 \|Z_x^{(k)}\|_2 = d(k)\omega_{n-1}^{-1} \qquad \forall x, \xi \in S. \tag{2.12}$$

The final step in bounding $\left|Z_\xi^{(k)}(x)\right|$ involves estimating $d(k)$.

Lemma 2.13. *The dimension $d(k)$ may be estimated as follows:*

$$d(k) \le 2(k+1)^{n-2} \qquad \forall k \in \mathbb{N}.$$

Proof. Using (2.5) and the simple fact that, for all nonnegative numbers n_1 and positive integers n_2,

$$\frac{n_1 + n_2}{n_2} \le n_1 + 1$$

we see that

$$\begin{aligned} d(k) &= \frac{(n+2k-2)\,(n+k-3)!}{k!\,(n-2)!} \\ &= \frac{n+2k-2}{n-2} \prod_{j=1}^{n-3} \frac{k+j}{j} \\ &\le (2k+1)\prod_{j=1}^{n-3}(k+1) \\ &\le 2(k+1)^{n-2},\end{aligned}$$

as claimed. □

We can now prove that the Poisson kernel is an infinite sum of spherical harmonics.

Theorem 2.14. *Suppose that ξ is in S and x is in B. Then*

$$p(\xi, x) = \sum_{k=0}^{\infty} Z_\xi^{(k)}(x). \tag{2.13}$$

Proof. If $|x| < 1$, then (2.12) and Lemma 2.13 imply that there is a constant C, depending only on n, such that

$$\left|Z_\xi^{(k)}(x)\right| = |x|^k \left|Z_\xi^{(k)}(x/|x|)\right| \le Ck^{n-2} |x|^k .$$

Thus the Weierstrass M-test for series implies that the series on the right hand side of (2.13) converges uniformly on $\overline{B}(0,r)$ provided that $r < 1$.

We now define the harmonic function $q(\xi,x)$ on $S \times B$ to be this locally uniformly convergent sum, and it follows that

$$Q[f](x) = \int_S q(\xi,x)\, f(\xi)\, d\sigma(\xi).$$

Suppose that f in $C(S)$ is a finite sum of spherical harmonics $f_1, \dots f_m$ on S, where f_k is in $\mathcal{H}_k$. Then

$$\begin{aligned}
Q[f](x) &= \int_S q(\xi,x)\, f(\xi)\, d\sigma(\xi) \\
&= \int_S \sum_{k=0}^{\infty} Z_\xi^{(k)}(x)\, f(\xi)\, d\sigma(\xi) \\
&= \sum_{k=0}^{\infty} \int_S Z_\xi^{(k)}(x)\, f(\xi)\, d\sigma(\xi) \\
&= \sum_{k=0}^{\infty} |x|^k \int_S Z_\xi^{(k)}(x/|x|)\, f(\xi)\, d\sigma(\xi) \\
&= \sum_{k=0}^{\infty} |x|^k \left\langle f, Z_{x/|x|}^{(k)} \right\rangle \\
&= \sum_{k=0}^{m} |x|^k f_k(x/|x|) \\
&= \sum_{k=0}^{m} P[f_k](x) \\
&= P[f](x)
\end{aligned}$$

for all x in B. Since

$$\int_S [q(\xi,x) - p(\xi,x)]\, f(\xi)\, d\sigma(\xi) = 0$$

for all f in a dense subset of $C(S)$ (that is, the polynomials), we conclude that

$$p(\xi,x) = q(\xi,x) = \sum_{k=0}^{\infty} Z_\xi^{(k)}(x) \qquad \forall \xi \in S \quad \forall x \in B,$$

as required. □

We can see from Theorem 2.14 that zonal harmonics generate harmonic functions on the ball in the same sense that the Poisson kernel generates such functions by the Poisson integral and Theorem 1.7.

Using the previous theorem, it becomes possible to find an explicit expression for zonal spherical harmonics. This is done by expanding the denominator of the Poisson kernel into a power series and collecting all the terms of a given degree.

Theorem 2.15. *Suppose that ξ is in S and x is in B. Then $Z_\xi^{(k)}(x)$ is equal to*

$$\frac{n+2k-2}{\omega_{n-1}} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j n(n+2)\cdots(n+2k-2j-4)}{2^j j!\,(k-2j)!} (x\cdot\xi)^{k-2j} |x|^{2j}.$$

Proof. The function $(1-\cdot)^{-n/2}$ is holomorphic in the unit disk in $\mathbb{C}$, so has a power series expansion there:

$$(1-z)^{-n/2} = \sum_{l=0}^{\infty} c_l z^l. \tag{2.14}$$

Differentiating both sides of (2.14) and evaluating when $z=0$ shows that

$$c_l = \frac{(n/2)(n/2+1)\cdots(n/2+l-1)}{l!}.$$

So writing the Poisson kernel as

$$p(\xi,x) = \omega_{n-1} \frac{1-|x|^2}{\langle x-\xi, x-\xi\rangle^{-n/2}} = \omega_{n-1} \frac{1-|x|^2}{1-[2x\cdot\xi - |x|^2]^{-n/2}}$$

and using (2.14), we see that

$$\begin{aligned} p(\xi,x) &= \omega_{n-1}^{-1}(1-|x|^2)\sum_{l=0}^{\infty} c_l (2x\cdot\xi - |x|^2)^l \\ &= \omega_{n-1}^{-1}(1-|x|^2)\sum_{l=0}^{\infty} c_l \sum_{j=0}^{l} \binom{l}{j} (-1)^j\, 2^{l-j}\, (x\cdot\xi)^{l-j}\, |x|^{2j}. \end{aligned}$$

Thus if $k \geq 2$ and we define $q_k(x)$ to be the sum of terms of degree k in the series above, then it follows that

$$Z_\xi^{(k)}(x) = \omega_{n-1}^{-1}[q_k(x) - |x|^2\, q_{k-2}(x)]; \tag{2.15}$$

this also holds when $k=0$ or 1 when we interpret q_{-2} and q_{-1} as 0.

To find $q_k(x)$, we sum the terms $c_l \binom{l}{j}(-1)^j\, 2^{l-j}\,(x\cdot\xi)^{l-j}\,|x|^{2j}$ over all j such that $0 \leq j \leq l$ and $l+j=k$. But these two conditions are compatible exactly when $k/2 \le l \le k$, i.e., when $0 \le j \le \lfloor k/2 \rfloor$. Summing therefore gives

$$\begin{aligned} q_k(x) &= \sum_{j=0}^{\lfloor k/2 \rfloor} c_{k-j} \binom{k-j}{j} (-1)^j \, 2^{k-2j} \, (x \cdot \xi)^{k-2j} \, |x|^{2j} \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j n(n+2) \cdots (n+2k-2j-2)}{2^j \, j! \, (k-2j)!} \, (x \cdot \xi)^{k-2j} \, |x|^{2j} \, . \end{aligned} \tag{2.16}$$

If $k = 0$ or 1, then (2.15) says that $Z_\xi^{(k)}$ is equal to ω_{n-1}^{-1} multiplied by the above expression. Now $\lfloor (k-2)/2 \rfloor = \lfloor k/2 \rfloor - 1$, and so (2.16) implies that

$$|x|^2 \, q_{k-2}(x) = \sum_{l=0}^{\lfloor k/2 \rfloor - 1} \frac{(-1)^l n(n+2) \cdots (n+2k-2l-6)}{2^l \, l! \, (k-2l-2)!} \, (x \cdot \xi)^{k-2l-2} \, |x|^{2l+2}$$

when $k \ge 2$. Substituting $j = l + 1$ in the above sum gives

$$|x|^2 \, q_{k-2}(x) = -\sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j n(n+2) \cdots (n+2k-2j-4)}{2^{j-1} \, (j-1)! \, (k-2j)!} \, (x \cdot \xi)^{k-2j} \, |x|^{2j} \, ,$$

where the term corresponding to $j = 0$ is taken to be zero. This expression and (2.16) now imply that

$$\begin{aligned} &q_k(x) - |x|^2 \, q_{k-2}(x) \\ &= (n+2k-2) \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j n(n+2) \cdots (n+2k-2j-4)}{2^j \, j! \, (k-2j)!} \, (x \cdot \xi)^{k-2j} \, |x|^{2j} \, , \end{aligned}$$

and the theorem is proved. □

Theorem 2.15 will be used to find an explicit formula for complex zonal harmonics in the next chapter.

Chapter 3
Complex Spherical Harmonics

In this chapter, we present a systematic treatment of spherical harmonics in $\mathbb{C}^n$, along the lines of the real case of the previous chapter, and establish some results which will be used in Chapter 4, where results on complex spherical harmonics will be applied to analyse a differential operator on the unit sphere in $\mathbb{C}^n$.

After outlining the relationship between the real and complex cases, we develop the theory of complex spherical harmonics, which parallels the real-variable theory of Chapter 2. In the last section we shed further light on zonal harmonics by showing how the theory can be used to derive precise information on behaviour of the L^2 norm of a zonal harmonic under multiplication by a coordinate function. This result will also be used to prove estimates on the operator in Chapter 4.

The main reference for the material in this chapter is a brief section in Chapter 12 of [8].

In this chapter, we assume that $n \geq 2$. The case where $n = 1$ is nontrivial but different.

3.1 The Relationship between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$

In this chapter, z will always denote an n-tuple $(z_1, \ldots, z_n)$ of complex numbers. The n-dimensional vector space of all such n-tuples is $\mathbb{C}^n$; it has an inner product defined by

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j. \tag{3.1}$$

Just as the set of complex numbers is often identified with the plane, there is a useful relationship between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, found by writing each component of z in terms of its real and imaginary parts: $z_j = x_j + iy_j$. We shall thus identify z with the element $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of $\mathbb{R}^{2n}$, and usually write this as (x, y). Since $|z_j|^2 = x_j^2 + y_j^2$ for every j, the norms of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ agree under this identification; i.e., $|z| = |(x, y)|$. Given this metric equivalence between complex n-dimensional space and real $2n$-dimensional space, sets such as the unit ball and unit sphere in $\mathbb{C}^n$ are defined exactly as the corresponding sets in $\mathbb{R}^{2n}$; the unit ball in $\mathbb{C}^n$ is denoted by B (or B_{2n}, when we wish to highlight the dimension). The volume measure on $\mathbb{C}^n$ coincides with Lebesgue measure on $\mathbb{R}^{2n}$, denoted by m (or m_{2n}), and the natural measure on the unit sphere is the standard surface measure σ.

Corresponding to the group of orthogonal maps on real Euclidean spaces, we define the *unitary group* $\mathrm{U}(n)$ to be the set of complex linear operators on $\mathbb{C}^n$ which preserve the inner product (3.1). Elements of $\mathrm{U}(n)$ are called unitary operators.

A linear operator on $\mathbb{C}^n$ is unitary if and only if it maps orthonormal bases onto orthonormal bases, and the unitary group shares many important properties with the orthogonal group. For example, we have the following lemma, whose proof is similar to that of Lemma 2.1.

Lemma 3.1. *Suppose that (u,u') and (v,v') are pairs of unit vectors in $\mathbb{C}^n$, and that $\langle u,u'\rangle = \langle v,v'\rangle$. Then there exists a unitary map U such that $Uu = v$ and $Uu' = v'$.*

The following discussion reveals the relationship between the complex matrix representation on $\mathbb{C}^n$ and the real matrix representation on $\mathbb{R}^{2n}$ of a linear map T on $\mathbb{C}^n$ under the above identification, and shows that, every unitary operator on $\mathbb{C}^n$ is in fact a rotation on $\mathbb{R}^{2n}$.

Suppose that $[t_{jk}]$ is the complex $n\times n$ matrix of T and write t_{jk} as $r_{jk}+is_{jk}$, where r_{jk} and s_{jk} are real. Since the j^{th} complex coordinate of Tz is given by

$$\sum_{k=1}^{n} t_{jk}z_k = \Big(\sum_{k=1}^{n} r_{jk}x_k - s_{jk}y_k\Big) + i\Big(\sum_{k=1}^{n} r_{jk}y_k + s_{jk}x_k\Big),$$

T acts on (x,y) in $\mathbb{R}^{2n}$ by multiplication by the $2n\times 2n$ matrix

$$\begin{pmatrix} R & -S \\ S & R \end{pmatrix},$$

where R and S are the real $n\times n$ matrices $[r_{jk}]$ and $[s_{jk}]$.

Suppose that z and w are in C^n, and that $z=(x,y)$ and $w=(u,v)$. Then the inner product (3.1) is related to the dot product of $\mathbb{R}^{2n}$ by $(x,y)\cdot(u,v) = \operatorname{Re}\langle z,w\rangle$. Hence if T is a unitary operator on $\mathbb{C}^n$, then by taking the real part of the formula $\langle Tz,Tw\rangle = \langle z,w\rangle$, we deduce that

$$T(x,y)\cdot T(u,v) = (x,y)\cdot(u,v);$$

it follows that $\mathrm{U}(n) \subseteq \mathrm{O}(2n)$. It is to see that $\mathrm{U}(n)$ is a proper subgroup of $\mathrm{O}(2n)$.

The above inclusion is especially important when considering the relationship between integration and orthogonal maps, since surface measure σ on the unit sphere S in $\mathbb{R}^{2n}$ is invariant under orthogonal maps: $\sigma(TE) = \sigma(E)$ for all Borel subsets E of S and all T in $\mathrm{O}(2n)$. Another way of writing this rotation invariance is in terms of characteristic functions:

$$\int_S \chi_E(T\xi)\,d\sigma = \int_S \chi_E(\xi)\,d\sigma.$$

This implies that, for simple functions f,

$$\int_S f(T\xi)\,d\sigma = \int_S f(\xi)\,d\sigma. \tag{3.2}$$

It is standard that (3.2) also holds for all f in $C(S)$.

We shall need two results on integration, one relating integration on S with the group of orthogonal maps and the other dealing with functions that do not depend on all n variables. Both may be found in [8, Chapter 1].

Let G be a compact subgroup of $\mathrm{O}(2n)$ and dg be the element of normalised Haar measure on G. The following result says that integrals over S are unchanged when the integrand is first 'averaged' over G.

Proposition 3.2. *For all f in $C(S)$,*

$$\int_S f(\xi)\, d\sigma(\xi) = \int_S \int_G f(g\xi)\, dg\, d\sigma(\xi).$$

Proof. The function $(g, \xi) \mapsto f(g\xi)$ is continuous (and hence integrable) on the compact space $G \times S$. The formula is then a direct consequence of Fubini's theorem and the invariance of Lebesgue measure under orthogonal maps (3.2). □

In the special case where $G = \{e^{i\theta} I : \theta \in [-\pi, \pi]\}$, we deduce the following useful corollary of Proposition 3.2.

Corollary 3.3. *For all f in $C(S)$,*

$$\int_S f(\xi)\, d\sigma(\xi) = \frac{1}{2\pi} \int_S \int_{-\pi}^{\pi} f(e^{i\theta}\xi)\, d\theta\, d\sigma(\xi).$$

The role of Corollary 3.3 in the study of complex polynomials is similar to that of the Green's theorem argument used in Proposition 2.8 to prove that spherical harmonics of different degrees are orthogonal.

Suppose that $k < n$ and let f be a continuous function on the unit sphere $S = S_{2n-1}$ in $\mathbb{C}^n$ that depends only on the variables $z_1, \ldots, z_k$. We will see below that the surface integral of f over S can be expressed in terms of an integral over the unit ball in $\mathbb{C}^k$.

In the following theorem, P is the orthogonal projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$, so if z in $\mathbb{C}^n$ corresponds to (x, y) in $\mathbb{R}^{2n}$, then Pz corresponds to the point (x', y') in $\mathbb{R}^{2k}$, where $x' = (x_1, \ldots, x_k)$ in $\mathbb{R}^k$.

Proposition 3.4. *Let P denote the orthogonal projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$. Then for all f in $L^1(B_{2k})$,*

$$\int_S (f \circ P)\, d\sigma = \omega_{2(n-k)-1} \int_{B_{2k}} (1 - |w|^2)^{n-k-1} f(w)\, dm_{2k}(w). \tag{3.3}$$

Proof. Suppose that f in $C(B_{2k})$ is supported in $r_0 B_{2k}$ for some $r_0 < 1$. Then for every $r > 0$, define

$$\begin{aligned} I(r) &= \int_{rB_{2n}} (f \circ P)\, dm_{2n} \\ &= \int_0^r \int_S (f \circ P)(t\xi)\, t^{2n-1}\, d\sigma(\xi)\, dt. \end{aligned}$$

Differentiating the above expression at $r = 1$ implies that

$$I'(1) = \int_S (f \circ P)\, d\sigma. \tag{3.4}$$

On the other hand, Fubini's theorem gives

$$\begin{aligned} I(r) &= \int_{rB_{2k}} \Big(\int_{(r^2-|w|^2)^{1/2} B_{2(n-k)}} (f \circ P)(w, w')\, dm_{2(n-k)}(w') \Big)\, dm_{2k}(w) \\ &= m_{2(n-k)}(B_{2(n-k)}) \int_{rB_{2k}} (r^2 - |w|^2)^{n-k} f(w)\, dm_{2k}(w). \end{aligned}$$

Now if $r > r_0$, then rB_{2k} can be replaced in the above integral by B_{2k}, since f vanishes outside rB_{2k}. We deduce that

$$I'(1) = 2(n-k) m_{2(n-k)}(B_{2(n-k)}) \int_{B_{2k}} (1 - |w|^2)^{n-k-1} f(w)\, dm_{2k}(w).$$

By comparing this expression with (3.4) and using the formula

$$\omega_{2(n-k)-1} = 2(n-k) m_{2(n-k)}(B_{2(n-k)}),$$

we prove (3.3) for f in $C(B_{2k})$ supported in $r_0 B_{2k}$. We now approximate by such functions and let r_0 tend to 1 to establish the result for all f in $L^1(B_{2k})$. □

The case where $k = 1$ of Proposition 3.4 will be of interest when dealing with zonal spherical harmonics (which depend on only one variable). In this case, (3.3) becomes

$$\begin{aligned} \int_S f(z_1)\, d\sigma(z) &= \omega_{2(n-1)-1} \int_{B_{2k}} (1 - |w|^2)^{n-2} f(w)\, dm_{2k}(w) \\ &= \omega_{2(n-1)-1} \int_0^1 \int_{-\pi}^{\pi} (1 - r^2)^{n-2} f(re^{i\theta})\, r\, d\theta\, dr. \end{aligned}$$

3.2 Harmonic Polynomials

For $z = (z_1, \dots, z_n)$ in $\mathbb{C}^n$, we write $\bar{z}$ for $(\bar{z}_1, \dots, \bar{z}_n)$. The identities $x = (z + \bar{z})/2$ and $y = (z + \bar{z})/2i$ show that any complex-valued polynomial of the variables (x, y) on $\mathbb{R}^{2n}$ can be written as a polynomial in $(z, \bar{z})$ and vice versa.

We refer to polynomials in the variables $(z, \bar{z})$ as *polynomials in* $\mathbb{C}^n$, or complex polynomials. In the special case where a complex polynomial depends only on the variables $z_1, \dots, z_n$, it will be called a *holomorphic polynomial* (in $\mathbb{C}^n$).

Since we shall be interested in the degrees of both z and $\bar{z}$ in complex polynomials, it is appropriate to define differential operators on $\mathbb{C}^n$ which will enable us to speak unambiguously about these 'bidegrees'. The following derivatives play the role of partial derivatives with respect to z_j and $\bar{z}_j$ when $1 \le j \le n$:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \Big) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \Big).$$

Suppose that f is in $C^1(D)$. The Cauchy–Riemann equations imply that f is holomorphic in the variable z_j if and only if $\partial f/\partial \bar{z}_j = 0$, and in that case

$$\frac{\partial f(z)}{\partial z_j} = D_j f(z)$$

in D, the right hand side being defined as the usual complex derivative of the complex-valued function $f(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)$ at z_j.

We say that f is *holomorphic* on D if it is a holomorphic function of each of the variables $z_1, \ldots, z_n$ on D; it is easy to see that holomorphic polynomials are indeed holomorphic under this definition. Similarly, a function f is said to be *conjugate holomorphic* in z_j if $\bar{f}$ is holomorphic in z_j. The identity

$$\left(\frac{\partial f}{\partial z_j}\right)^{-} = \frac{\partial \bar{f}}{\partial \bar{z}_j}$$

implies that f in $C^1(D)$ is conjugate holomorphic on D if and only if $\partial f/\partial z_j = 0$ and

$$\frac{\partial f}{\partial \bar{z}_j} = \left(D_j \bar{f}\right)^{-}$$

on D. If f is conjugate holomorphic and we define 'differentiation in the conjugate variable' by

$$\bar{D}_j f(z) = \lim_{h \to 0} \frac{f(z + he_j) - f(z)}{\bar{h}},$$

where e_j is the j^{th} standard basis vector, then $\bar{D}_j f = \partial f/\partial \bar{z}_j$. We have shown that, in some sense, $\partial f/\partial \bar{z}_j$ is the partial derivative of f with respect to the variable $\bar{z}_j$, and that a function is conjugate holomorphic in the j^{th} variable if and only if it is holomorphic in the conjugate variable.

Returning to the study of complex polynomials, any polynomial in $(z, \bar{z})$ is a linear combination of complex monomials

$$z^\alpha \bar{z}^\beta = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n},$$

where α and β are n-tuples of nonnegative integers. We can combine the product rule with the remarks above to see that $\partial z^\alpha \bar{z}^\beta/\partial z_j$ or $\partial z^\alpha \bar{z}^\beta/\partial \bar{z}_j$ are nonzero if and only if $\alpha_j \neq 0$ or $\beta_j \neq 0$, in which case the derivative is found by regarding all other variables as constant and differentiating with respect to z_j or $\bar{z}_j$.

With the aid of the derivatives defined above, it is a simple matter to prove that the monomials $z^\alpha \bar{z}^\beta$ are linearly independent over $\mathbb{C}$, and thus that every polynomial in $\mathbb{C}^n$ is uniquely expressible as a sum of these monomials. Indeed, suppose that S is a finite subset of $\mathbb{N}^n \times \mathbb{N}^n$, and that $\sum_{(\alpha,\beta) \in S} a_{\alpha,\beta} z^\alpha \bar{z}^\beta = 0$ (where each $a_{\alpha,\beta}$ is in $\mathbb{C}$); apply the differential operator $\partial^{|\alpha|+|\beta|}/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}$ to both sides of the equality to show that every $a_{\alpha,\beta}$ must be zero.

Fix nonnegative integers p and q. Then corresponding to the spaces $\mathcal{P}_k(\mathbb{R}^{2n})$ of the last chapter, we define the space $\mathcal{P}_{p,q}(\mathbb{C}^n)$ of homogeneous polynomials of bidegree (p, q) on $\mathbb{C}^n$ to be the set of complex polynomials of the form

$$\sum_{\substack{\alpha \in [\mathbb{N}^n]_p \\ \beta \in [\mathbb{N}^n]_q}} a_{\alpha,\beta} z^\alpha \bar{z}^\beta,$$

where $a_{\alpha,\beta}$ is in $\mathbb{C}$. We usually abbreviate this to $\mathcal{P}_{p,q}$. Using the linear independence of complex monomials it is easy to see that any polynomial in $\mathbb{C}^n$ decomposes uniquely as a sum of polynomials in $\mathcal{P}_{p,q}$, as p and q vary. Similarly, using the correspondence between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$, it is easy to see that spaces of homogeneous polynomials in $\mathbb{R}^{2n}$ can be decomposed into spaces $\mathcal{P}_{p,q}$:

$$\mathcal{P}_k = \bigoplus_{(p,q)\in[\mathbb{N}^2]_k} \mathcal{P}_{p,q}; \tag{3.5}$$

that is, every element on the left side of (3.5) is a unique sum of elements belonging to the sets on the right.

Using the linear independence of complex monomials, it is easy to see that a polynomial f on $\mathbb{C}^n$ lies in $\mathcal{P}_{p,q}$ if and only if

$$f(\lambda z) = \lambda^p \bar{\lambda}^q f(z) \qquad \forall \lambda \in \mathbb{C} \quad \forall z \in \mathbb{C}^n,$$

and that f in $\mathcal{P}_k$ is in $\mathcal{P}_{p,q}$ if and only if

$$f(e^{i\theta} z) = e^{i(p-q)\theta} f(z) \quad \forall \theta \in \mathbb{R} \quad \forall z \in \mathbb{C}^n. \tag{3.6}$$

The linear independence of the monomials also means that the monomials $z^\alpha \bar{z}^\beta$, where $\alpha \in [\mathbb{N}^n]_p$ and $\beta \in [\mathbb{N}^n]_q$ form a basis for $\mathcal{P}_{p,q}$. From the argument to prove (2.2), we see that

$$\dim(\mathcal{P}_{p,q}) = \binom{n+p-1}{p}\binom{n+q-1}{q}. \tag{3.7}$$

The decomposition (3.5) and the equality (3.7) can be combined to give the following combinatorial formula

$$\binom{2n+k-1}{k} = \sum_{j=0}^{k} \binom{n+j-1}{j}\binom{n+k-j-1}{k-j}.$$

As in the real case, a more detailed picture of the structure of polynomials on $\mathbb{C}^n$ is found by considering the subspace of $\mathcal{P}_{p,q}$ consisting of harmonic functions.

The definition of harmonic functions in $\mathbb{C}^n$ coincides with that in $\mathbb{R}^{2n}$, that is, a function is harmonic in (a subset of) $\mathbb{C}^n$ if it is harmonic when considered as a function of $2n$ real variables.

Note that we can express the Laplacian in terms of the derivatives defined above:

$$\Delta = \sum_{j=1}^{n} \Big(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j}\Big)^2 = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j},$$

so f is harmonic if and only if $\sum_{j=1}^n \partial^2 f/\partial z_j \partial \bar{z}_j = 0$.

We define $\mathcal{H}_{p,q}$ to be the space of harmonic, homogeneous polynomials of bidegree (p,q) in $\mathbb{C}^n$. It is clear that $\mathcal{H}_{p,q}$ is a subspace of $\mathcal{P}_{p,q}$, with equality if and only if $p=0$ or $q=0$: the 'if' part of this assertion follows from the fact that $\partial f/\partial \bar{z}_j = 0$ and $\partial \bar{f}/\partial z_j = 0$ for all holomorphic polynomials f, while the converse follows from the fact that $z_1^p \bar{z}_1^q$ is an element of $\mathcal{P}_{p,q} \setminus \mathcal{H}_{p,q}$ if $p \geq 1$ and $q \geq 1$.

The mean-value theorem again implies that a given function on the unit sphere in $\mathbb{C}^n$ can be the restriction of at most one harmonic, homogeneous polynomial; therefore any element of $\mathcal{H}_{p,q}$ can be identified with its restriction to S.

Since every polynomial in $\mathcal{H}_{p,q}(\mathbb{C}^n)$ can also be regarded as lying in $\mathcal{H}_{p+q}(\mathbb{R}^{2n})$, it follows that $\mathcal{H}_{p,q} \subseteq \mathcal{H}_k$ when $k = p+q$. We will now show that the space $\mathcal{H}_k$ is in fact an orthogonal direct sum, over all $p+q=k$, of these spaces $\mathcal{H}_{p,q}$.

We may write a polynomial f in $\mathbb{C}^n$ uniquely as follows:

$$f = \sum_{p,q=0}^{\infty} f_{p,q},$$

where $f_{p,q}$ is in $\mathcal{P}_{p,q}$ and only finitely many $f_{p,q}$ are nonzero; we call $f_{p,q}$ the homogeneous component of bidegree (p,q) of f. So when the Laplacian is applied, the unique decomposition of Δf into homogeneous components is

$$\sum_{p,q=0}^{\infty} \Delta f_{p,q},$$

where $\Delta f_{p,q}$ is in $\mathcal{P}_{p-1,q-1}$ if $p \geq 1$ and $q \geq 1$, and $\Delta f_{p,q} = 0$ otherwise. Since a complex polynomial is zero if and only if each of its homogeneous components is zero, we have proved the following result.

Proposition 3.5. *All the homogeneous components of harmonic, complex polynomials are harmonic.*

If the polynomial in this proposition is already homogeneous, then the result is a harmonic version of (3.5):

$$\mathcal{H}_k = \bigoplus_{(p,q)\in[\mathbb{N}^2]_k} \mathcal{H}_{p,q}. \tag{3.8}$$

Henceforward, we will be mainly concerned with polynomials on the unit sphere S in $\mathbb{C}^n$. Since $\mathcal{H}_{p,q}(S) \subseteq \mathcal{H}_k(S)$ when $p+q=k$, Proposition 2.8 shows that $\mathcal{H}_{p,q}(S)$ and $\mathcal{H}_{p',q'}(S)$ are orthogonal in $L^2(S)$ unless $p+q = p'+q'$. Using Corollary 3.3, we deduce that $\mathcal{H}_{p,q}(S)$ and $\mathcal{H}_{p',q'}(S)$ are also orthogonal unless $p-q = p'-q'$, and hence conclude that spherical harmonics of different bidegrees are always orthogonal.

Proposition 3.6. *Suppose that $(p,q) \neq (p',q')$. Then $\mathcal{H}_{p,q}(S) \perp \mathcal{H}_{p',q'}(S)$ in $L^2(S)$.*

Proof. The case where $p+q \neq p'+q'$ is already taken care of by the remarks above, so we may assume that $p-q \neq p'-q'$.

If f is in $\mathcal{H}_{p,q}(S)$ and g is in $\mathcal{H}_{p',q'}(S)$, then from (3.6),

$$\int_{-\pi}^{\pi} f(e^{i\theta}\xi)\,\bar{g}(e^{i\theta}\xi)\,d\theta = f(\xi)\,\bar{g}(\xi)\int_{-\pi}^{\pi} e^{i(p-q+q'-p')\theta}\,d\theta = 0 \qquad \forall \xi \in S,$$

since $p-q+q'-p' \neq 0$. Thus $\langle f, g\rangle = 0$ by Corollary 3.3. □

We can now combine Theorem 2.9, equality and Proposition 3.6 to conclude that $L^2(S)$ is the orthogonal direct sum of complex spherical harmonics, which we write as follows.

Theorem 3.7. *The various spaces $\mathcal{H}_{p,q}(S)$ are pairwise orthogonal, and*

$$L^2(S) = \bigoplus_{p,q=1}^{\infty} \mathcal{H}_{p,q}(S). \tag{3.9}$$

We have seen that homogeneous (harmonic) polynomials of degree k in $\mathbb{R}^{2n}$ split into complex homogeneous (harmonic) polynomials whose bidegree sums to k. Turning now to the question of a relationship between $\mathcal{P}_{p,q}$ and $\mathcal{H}_{p,q}$, we know that these spaces coincide when at least one of p and q is zero. So suppose that $p \geq 1$ and $q \geq 1$ and take f in $\mathcal{P}_{p,q}$. Since $\mathcal{P}_{p,q}$ may be viewed as a subspace of $\mathcal{P}_k$, Theorem 2.6 implies the existence of g in $\mathcal{H}_k$ and h in $\mathcal{P}_{k-2}$ such that

$$f = g + |\cdot|^2\, h. \tag{3.10}$$

We can write

$$g = \sum_{(r,s)\in[\mathbb{N}^2]_k} g_{r,s} \qquad \text{and} \qquad h = \sum_{(r,s)\in[\mathbb{N}^2]_{k-2}} h_{r,s},$$

as in (3.9) and (3.5) respectively. The homogeneous component of degree (r,s) on the right-hand side of (3.10) is therefore $g_{r,s} + |\cdot|^2\, h_{r-1,s-1}$, which must be zero unless $(r,s) = (p,q)$, by (3.5). Hence Corollary 2.5 implies that $g_{r,s} = h_{r-1,s-1} = 0$ if $(r,s) \neq (p,q)$, and so g is in $\mathcal{H}_{p,q}$ and h is in $\mathcal{P}_{p-1,q-1}$.

We conclude that all f in $\mathcal{P}_{p,q}$ may be written uniquely in the form $g + |\cdot|^2\, h$, where g is in $\mathcal{H}_{p,q}$ and h is in $\mathcal{P}_{p-1,q-1}$; the uniqueness follows from Corollary 2.5, as in Theorem 2.6. We have proved the following result.

Theorem 3.8. *Suppose that $p \geq 1$ and $q \geq 1$. Then*

$$\mathcal{P}_{p,q} = \mathcal{H}_{p,q} \oplus |\cdot|^2\, \mathcal{P}_{p-1,q-1}.$$

The following corollary follows immediately by induction and shows precisely how harmonic polynomials generate all complex polynomials.

Corollary 3.9. *Suppose that $p \geq 1$ and $q \geq 1$. Then*

$$\mathcal{P}_{p,q} = \mathcal{H}_{p,q} \oplus |\cdot|^2\, \mathcal{H}_{p-1,q-1} \oplus \cdots \oplus |\cdot|^{2m}\, \mathcal{H}_{p-m,q-m},$$

where $m = \min\{p,q\}$.

Corollary 3.10. *Suppose that* $f \in \mathcal{P}_{p,q}$ *and* $g \in \mathcal{H}_{r,s}$*, where* $r > p$ *or* $s > q$*. Then*

$$\langle f, g \rangle = 0$$

in $L^2(S)$*.*

We can easily compute the dimension $d(p,q)$ of the space $\mathcal{H}_{p,q}$ using Theorem 3.8 and the formula (3.7) for $\dim(\mathcal{P}_{p,q})$. If $p \geq 1$ and $q \geq 1$, then

$$\begin{aligned} d(p,q) &= \dim(\mathcal{H}_{p,q}) \\ &= \dim(\mathcal{P}_{p,q}) - \dim(\mathcal{P}_{p-1,q-1}) \\ &= \binom{n+p-1}{p}\binom{n+q-1}{q} - \binom{n+p-2}{p-1}\binom{n+q-2}{q-1} \\ &= \frac{(n-1)(n+p+q-1)}{pq}\binom{n+p-2}{p-1}\binom{n+q-2}{q-1}. \end{aligned} \tag{3.11}$$

To bound the growth of $d(p,q)$, write

$$\frac{1}{p}\binom{n+p-2}{p-1} = \frac{(p+n-2)\cdots(p+1)p}{(n-1)!\,p} = \frac{1}{n-1}\prod_{j=1}^{n-2}\Big(\frac{p+j}{j}\Big),$$

and use the fact that $(p+j)/j \leq p+1$ for every j in the product above. Doing this for both binomial coefficients in (3.11), we see that

$$\begin{aligned} d(p,q) &\leq \frac{n+p+q-1}{n-1}\,(p+1)^{n-2}\,(q+1)^{n-2} \\ &\leq (p+q+1)\,(p+1)^{n-2}\,(q+1)^{n-2}. \end{aligned} \tag{3.12}$$

It is easy to check that this estimate is also valid if $p = 0$ or $q = 0$.

3.3 Zonal Harmonics

In this section we are interested in analogues of the zonal spherical harmonics of the previous chapter. Since we shall be dealing with functions on the unit sphere S in $\mathbb{C}^n$, the homogeneous complex polynomials defined above will henceforth be understood to be restricted to S, unless stated otherwise.

The space $\mathcal{H}_{p,q}$ is a finite-dimensional polynomial subspace of $L^2(S)$, and for fixed w in S, the linear functional $f \mapsto f(w)$ is bounded on $\mathcal{H}_{p,q}$. The Riesz representation theorem of Hilbert space theory therefore implies the existence of a unique polynomial $Y_w^{(p,q)}$ in $\mathcal{H}_{p,q}$ such that

$$f(w) = \big\langle f, Y_w^{(p,q)} \big\rangle = \int_S f\, \bar{Y}_w^{(p,q)}\, d\sigma \qquad \forall f \in \mathcal{H}_{p,q}. \tag{3.13}$$

We refer to the function $Y_w^{(p,q)}$ as the zonal spherical harmonic of bidegree (p,q) with pole w.

Since the linear functional $f \mapsto f(w)$ is not bounded on $L^2(S)$, we cannot expect (3.13) to hold for an arbitrary function in $L^2(S)$. However, the fact that $\mathcal{H}_{p,q}(S)$ is a closed subspace of $L^2(S)$ means that the orthogonal projection $f \mapsto f_{p,q}$ onto

$\mathcal{H}_{p,q}$ is a bounded operator on $L^2(S)$. We can now compose this operator with the (bounded) evaluation functional on $\mathcal{H}_{p,q}$ to conclude that $f \mapsto f_{p,q}(w)$ is a bounded linear functional on $L^2(S)$. Since this latter functional agrees with (3.13) on $\mathcal{H}_{p,q}$, the Riesz representation theorem implies that

$$f_{p,q}(w) = \langle f, Y_w^{(p,q)} \rangle \qquad \forall f \in L^2(S). \tag{3.14}$$

The complex zonal harmonics defined above share many of the properties of their real counterparts (defined in Section 2.3), though not all. We now proceed to explore these properties.

Recall that if we identify points in $\mathbb{C}^n$ with those of $\mathbb{R}^{2n}$ as in Section 3.1, then the space $\mathcal{H}_k(S)$ of real spherical harmonics of degree k on $\mathbb{R}^{2n}$ is a direct sum of all the spaces $\mathcal{H}_{p,q}$ such that $p+q=k$. It is therefore natural to ask how the zonal harmonic $Z_w^{(k)}$ in $\mathcal{H}_k(S)$ is related to the complex zonal spherical harmonics $Y_w^{(p,q)}$ under this decomposition.

Suppose that f is in $\mathcal{H}_k(S)$. From (3.8), we may write f uniquely in the form $\sum_{(p,q)\in[\mathbb{N}^2]_k} f_{p,q}$, where $f_{p,q}$ is in $\mathcal{H}_{p,q}(S)$. Now (3.14) shows that

$$\Big\langle f, \sum_{(p,q)\in[\mathbb{N}^2]_k} Y_w^{(p,q)} \Big\rangle = \sum_{(p,q)\in[\mathbb{N}^2]_k} \langle f, Y_w^{(p,q)} \rangle = \sum_{(p,q)\in[\mathbb{N}^2]_k} f_{p,q}(w) = f(w).$$

By comparing this with (2.7) and using the uniqueness of the representation, we can prove the following result.

Proposition 3.11. *For all w in S and all k in $\mathbb{N}$,*

$$Z_w^{(k)} = \sum_{(p,q)\in[\mathbb{N}^2]_k} Y_w^{(p,q)}.$$

The connection between real and complex zonal harmonics in Proposition 3.11 will be used later to find an explicit expression for the complex zonal spherical harmonics.

Note that

$$Y_w^{(p,q)}(z) = \langle Y_w^{(p,q)}, Y_z^{(p,q)} \rangle = \langle Y_z^{(p,q)}, Y_w^{(p,q)} \rangle^{-} = \bar{Y}_z^{(p,q)}(w) \tag{3.15}$$

for all w and z in S, we cannot conclude that $Y_w^{(p,q)}$ is real-valued by attempting to prove an analogue of (2.8) unless $p=q$. This is because $\mathcal{P}_{p,q}$ and $\mathcal{H}_{p,q}$ are not closed under conjugation unless $p=q$: indeed the conjugate of an element f of $\mathcal{P}_{p,q}$ is in $\mathcal{P}_{q,p}$, and the intersection of the latter two spaces is $\{0\}$ when $p \neq q$. Alternatively, we can just recall from (3.6) that if f is in $\mathcal{P}_{p,q} \setminus \{0\}$, then $f(e^{i\theta}z) = e^{i(p-q)\theta} f(z)$ for all z in $\mathbb{C}^n$ and all θ in $\mathbb{R}$, so f cannot be real unless $p=q$. However, $\mathcal{H}_{p,p}$ is closed under conjugation, as we shall see.

Complex zonal harmonics are generally not real-valued, but there is nevertheless a nice symmetry between zonal harmonics of bidegree (p,q) and (q,p): namely, $Y_w^{(p,q)}$ and $Y_w^{(q,p)}$ are complex conjugates. To see this, note first that if f is in $\mathcal{H}_{q,p}$, then $\bar{f}$ is in $\mathcal{H}_{p,q}$. Therefore $\bar{f}(w) = \langle \bar{f}, Y_w^{(p,q)} \rangle$ from (3.13), and conjugating this last inequality gives

$$f(w) = \langle \bar{f}, Y_w^{(p,q)} \rangle^{-} = \langle f, \bar{Y}_w^{(p,q)} \rangle .$$

We now apply uniqueness of the representation to conclude that

$$Y_w^{(q,p)} = \bar{Y}_w^{(p,q)}. \tag{3.16}$$

Combining this fact with (3.15) implies that

$$Y_w^{(p,q)}(z) = Y_z^{(q,p)}(w) \qquad \forall w, z \in S.$$

Taking $p = q$ in the last two equations shows that every $Y_w^{(p,p)}$ is real-valued and $Y_w^{(p,p)}(z) = Y_z^{(p,p)}(w)$ for all w and z in S.

We now turn to considering the relationship between complex zonal harmonics and unitary changes of variables. As in the real case, this will lead to nice geometric information about the behaviour of zonal harmonics on the sphere, and show that $Y_w^{(p,q)}$ is essentially a function of only one variable on S.

Recall from Section 3.1 that a unitary operator U on $\mathbb{C}^n$ is an orthogonal map on $\mathbb{R}^{2n}$. Therefore $|\det U| = 1$ and we can change variables in the inner product to prove the following result.

Proposition 3.12. *For all U in $\mathrm{U}(n)$ and all w in S,*

$$Y_{Uw}^{(p,q)} = Y_w^{(p,q)} \circ U^{-1}.$$

Proof. For all f in $\mathcal{H}_{p,q}$,

$$\langle f, Y_{Uw}^{(p,q)} \rangle = f(Uw) = (f \circ U)(w) = \langle f \circ U, Y_w^{(p,q)} \rangle = \langle f, Y_w^{(p,q)} \circ U^{-1} \rangle ,$$

where we have used the observation that $\mathcal{H}_{p,q}$ is closed under composition with unitary operators. The result now follows by the uniqueness of the inner product representation. □

As an application of the uniqueness in Theorem 3.7, we can easily prove that the orthogonal projection $f \mapsto f_{p,q}$ of $L^2(S)$ onto $\mathcal{H}_{p,q}$ commutes with compositions with unitary changes of variables, in the sense that

$$(f \circ U)_{p,q} = f_{p,q} \circ U \qquad \forall f \in L^2(S) \quad \forall U \in \mathrm{U}(n). \tag{3.17}$$

To see this, note that $f = \sum_{p,q=0}^{\infty} f_{p,q}$ by Theorem 3.7. The uniqueness of the representation therefore implies that the components of bidegree (p,q) on both sides of

$$\sum_{p,q=0}^{\infty} (f_{p,q} \circ U) = f \circ U = \sum_{p,q=0}^{\infty} (f \circ U)_{p,q}$$

must be equal, and (3.17) is proved.

Equality (3.17) shows that if f in $L^2(S)$ is invariant under some subgroup of $\mathrm{U}(n)$ in the sense that $f = f \circ U$ for every U in the subgroup, then the same is true of each component $f_{p,q}$ of f.

Proposition 3.12 can be used to show that zonal harmonics with pole w are invariant under a particular subgroup of $\mathrm{U}(n)$, namely, the set of unitary operators that fix w, and this invariance characterises zonal harmonics as those depending on only one variable, in the 'direction' of the pole. To see this, note that

$$Y_{Uw}^{(p,q)}(Uz) = Y_w^{(p,q)}(z) \qquad \forall U \in \mathrm{U}(n) \quad \forall z, w \in S$$

from Proposition 3.12, so if U is a unitary operator such that $Uw = w$, then

$$Y_w^{(p,q)}(Uz) = Y_w^{(p,q)}(z) \qquad \forall w, z \in S. \tag{3.18}$$

Equality (3.18) is equivalent to the statement that $Y_w^{(p,q)}$ is constant on all sets which are intersections of S with a hyperplane orthogonal to w.

Now a hyperplane in $\mathbb{C}^n$ is orthogonal to w if and only if its equation is of the form $\langle z, w\rangle = \lambda$ for some fixed λ in $\mathbb{C}$, where $\langle \cdot, \cdot \rangle$ is the inner product (3.1) on $\mathbb{C}^n$. But since $\langle z, w\rangle = \lambda$ can be written as $\langle z - \lambda w, w\rangle = 0$, the hyperplane $\langle z, w\rangle = \lambda$ passes through the point λw, and hence it can intersect S if and only if $|\lambda| \leq 1$. A subset of S is therefore an intersection of S with a hyperplane orthogonal to w if and only if it has the form

$$w(\lambda) = \{z \in S : \langle z, w\rangle = \lambda\},$$

where $|\lambda| \leq 1$ in $\mathbb{C}$. We shall call such sets 'w-slices of S'. Using the fact that unitary operators map S into S and that

$$\langle Uz, w\rangle = \langle Uz, Uw\rangle = \langle z, w\rangle$$

for all U in $\mathrm{U}(n)$ which fix w, it is easy to see from (3.18) that every such U maps the slice $w(\lambda)$ into itself.

The following proposition is an analogue of Proposition 2.11.

Proposition 3.13. *The complex spherical harmonic $Y_w^{(p,q)}$ is constant on every w-slice $w(\lambda)$ of S, where $|\lambda| \leq 1$.*

Proof. When $\theta \in [0, 2\pi)$, it is easy to see that $w(e^{i\theta}) = \{e^{i\theta}\}$. Thus $Y_w^{(p,q)}$ is trivially constant on $w(e^{i\theta})$, and we may assume that $|\lambda| < 1$.

In view of (3.18) and the remark preceding the statement of the proposition, it suffices to show that, if z_1 and z_2 are in $w(\lambda)$, then $z_2 = Uz_1$ for some U in $\mathrm{U}(n)$ that fixes w. This follows by applying Lemma 3.1 to the pairs (w, z_1) and (w, z_2). □

In the above proof, we showed that Proposition 3.13 follows from equality (3.18); the converse follows from the fact that $U[w(\lambda)] = w(\lambda)$ for every U in $\mathrm{U}(n)$ that fixes w. Thus (3.18) and Proposition 3.13 are equivalent.

We shall now show that Proposition 3.13 characterises zonal harmonics, by proving that any element of $\mathcal{H}_{p,q}$ which is constant on all slices of S must be a constant multiple of $Y_w^{(p,q)}$.

Proposition 3.14. *Suppose that w is in S, and f in $\mathcal{H}_{p,q}$ is constant on all w-slices of S. Then there exists c in $\mathbb{C}$ such that*

$$f = cY_w^{(p,q)}.$$

Proof. Without loss of generality, assume that $w = e_1$. The remarks following Proposition 3.13 imply that $f(Uz) = f(z)$ for all z in S and all U in $\mathrm{U}(n)$ which fix e_1.

Now U in $\mathrm{U}(n)$ fixes e_1 if and only if there exists U' in $\mathrm{U}(n-1)$ such that $U(z_1, z') = (z_1, U'z')$ for all $z = (z_1, z')$ in $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$. Arguing as in the proof of Theorem 2.12, we see that

$$f(z) = \sum_{j=0}^{m} c_j z_1^{p-j} \bar{z}_1^{q-j} |z'|^{2j},$$

where $m = \min(p, q)$.

By applying the Laplacian $4\sum_{j=1}^{n} \partial^2/\partial z_j \partial \bar{z}_j$ to both sides of this equality and using Proposition 3.5, we see that

$$0 = \Delta f(z) = \sum_{j=0}^{m-1} b_j z_1^{p-1-j} \bar{z}_1^{q-1-j} |z'|^{2j},$$

where

$$b_j = 4(p-j)(q-j)c_j + 4(j+1)(j+n-1)c_{j+1}$$

when $j = 0, 1, \ldots, m$. Since every $b_j = 0$, as in Theorem 2.12 we can find all c_j from the first nonzero c_j by iteration, and any two such f in $\mathcal{H}_{p,q}$ must be constant multiples of each other.

Since $Y_w^{(p,q)}$ is a nonzero element of $\mathcal{H}_{p,q}$, the result is proved. □

The previous proposition introduces sums of the form $\sum_{j=0}^{m} c_j z_1^{p-j} \bar{z}_1^{q-j} |z'|^{2j}$, and we will use similar sums below.

Lemma 3.15. *Suppose that p and q are nonnegative integers and $m = \min(p, q)$, and that*

$$\sum_{j=0}^{m} a_j z_1^{p-j} \bar{z}_1^{q-j} |z'|^{2j} = \sum_{j=0}^{m} b_j z_1^{p-j} \bar{z}_1^{q-j} |z'|^{2j} \qquad \forall (z_1, z') \in S,$$

$$\sum_{j=0}^{m} c_j z_1^{p-j} \bar{z}_1^{q-j} |z|^{2j} = \sum_{j=0}^{m} d_j z_1^{p-j} \bar{z}_1^{q-j} |z|^{2j} \qquad \forall z \in S.$$

Then $a_j = b_j$ and $c_j = d_j$ for all j.

Proof. Taking z of the form $(z_1, z_2, 0, \ldots, 0)$, we see that

$$\sum_{j=0}^{m} a_j z_1^{p-j} \bar{z}_1^{q-j} z_2^j \bar{z}_2^j = \sum_{j=0}^{m} b_j z_1^{p-j} \bar{z}_1^{q-j} z_2^j \bar{z}_2^j \qquad \forall z \in S;$$

the linear independence of monomials implies that $a_j = b_j$ for all j. Further, we see that

$$\sum_{j=0}^{m} c_j z_1^{p-j} \bar{z}_1^{q-j} (z_1\bar{z}_1 + z_2\bar{z}_2)^j = \sum_{j=0}^{m} d_j z_1^{p-j} \bar{z}_1^{q-j} (z_1\bar{z}_1 + z_2\bar{z}_2)^j \qquad \forall z \in S;$$

expanding this out and equating coefficients recursively (starting with the highest order coefficients of $z_2\bar{z}_2$) shows that indeed $c_j = d_j$ for all j. □

We shall now prove another interesting consequence of Proposition 3.12; we evaluate the L^2 norms of zonal spherical harmonics, and relate these norms to special values of zonal harmonics and orthonormal bases of $\mathcal{H}_{p,q}$ spaces.

For all w in S and all unitary operators U, Proposition 3.12 implies that

$$Y_{Uw}^{(p,q)}(Uw) = Y_w^{(p,q)}(w).$$

Since unitary operators preserve norms, U maps S into S. Every point z on S is the image of w under some unitary operator U, by Lemma 3.1 applied to the pairs (w, w) and (z, z). Therefore there is a number c such that $Y_w^{(p,q)}(w) = c$ for all w in S. To determine the constant c, we first make a simple and useful observation relating zonal harmonics to the dimension of the space $\mathcal{H}_{p,q}$.

Recall that $d(p,q) = \dim(\mathcal{H}_{p,q})$, the precise value of which is given by (3.11). Also, let $\{u_1, \ldots, u_{d(p,q)}\}$ be an arbitrary orthonormal basis of $\mathcal{H}_{p,q}$. Then writing a zonal spherical harmonic in terms of this orthonormal basis and using (3.13) implies that

$$\begin{aligned} Y_w^{(p,q)}(z) &= \sum_{j=1}^{d(p,q)} \left\langle Y_w^{(p,q)}, u_j \right\rangle u_j(z) \\ &= \sum_{j=1}^{d(p,q)} \left\langle u_j, Y_w^{(p,q)} \right\rangle^{-} u_j(z) \\ &= \sum_{j=1}^{d(p,q)} \bar{u}_j(w)\, u_j(z) \qquad \forall w, z \in S. \end{aligned} \tag{3.19}$$

This equality is interesting since it shows that $\sum_{j=1}^{d(p,q)} \bar{u}j(w)\, uj(z)$ is independent of the orthonormal basis $\{u_1, \ldots, u_{d(p,q)}\}$ for $\mathcal{H}_{p,q}$. This provides another proof of (3.15).

Our purpose in proving (3.19) was, however, to find the constant value, c say, of $Y_w^{(p,q)}(w)$. Putting $w = z$ in (3.19) and integrating over S shows that

$$\begin{aligned} c\,\omega_{2n-1} &= \int_S Y_w^{(p,q)}(w)\, d\sigma(w) \\ &= \sum_{j=1}^{d(p,q)} \int_S |u_j(w)|^2 \, d\sigma(w) \\ &= d(p,q), \end{aligned}$$

because $\|u_j\|_2 = 1$. Since $Y_w^{(p,q)}(w) = \langle Y_w^{(p,q)}, Y_w^{(p,q)} \rangle = \|Y_w^{(p,q)}\|_2^2$, we have proved the following proposition.

Proposition 3.16. *For any orthonormal basis* $\{u_1, \ldots, u_{d(p,q)}\}$ *of* $\mathcal{H}_{p,q}$,

$$\sum_{j=1}^{d(p,q)} |u_j(w)|^2 = d(p,q)\omega_{2n-1}^{-1} \qquad \forall w \in S.$$

In particular,

$$Y_w^{(p,q)}(w) = \|Y_w^{(p,q)}\|_2^2 = d(p,q)\omega_{2n-1}^{-1} \qquad \forall w \in S.$$

Combining this with the Schwarz inequality, we derive the following interesting consequence.

Corollary 3.17. *The zonal spherical harmonic* $Y_w^{(p,q)}$ *reaches its maximum absolute value on* S *at the point* w.

Proof. For all z in S,

$$\begin{aligned} |Y_w^{(p,q)}(z)| &= |\langle Y_w^{(p,q)}, Y_z^{(p,q)} \rangle| \\ &= \Big| \int_S Y_w^{(p,q)}(\xi)\, \bar{Y}_z^{(p,q)}(\xi)\, d\sigma(\xi) \Big| \\ &\le \|Y_w^{(p,q)}\|_2 \, \|Y_z^{(p,q)}\|_2 \\ &= d(p,q)\omega_{2n-1}^{-1}. \end{aligned}$$

Proposition 3.16 now implies the result. □

We will now compute a precise expression for $Y_w^{(p,q)}$ in the case where $w = e_1$, the first standard basis vector in $\mathbb{C}^n$. This case is sufficiently general since we may then compute any complex zonal spherical harmonic by composing with an appropriate unitary operator.

We denote by z and w points in $\mathbb{C}^n$. We will write $z = (x, y)$, where $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$.

Fix w in $\mathbb{C}^n$ and let $Z_w^{(k)}$ be the zonal spherical harmonic of degree k on $\mathbb{R}^{2n}$ with pole w. Replacing n by $2n$ in Theorem 2.15 shows that $Z_w^{(k)}(z)$ is equal to

$$\begin{aligned} &Z_{(u,v)}^{(k)}(x, y) \\ &= c \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j 2n(2n+2)\cdots(2n+2k-2j-4)}{2^j\, j!\,(k-2j)!} \big((x,y)\cdot(u,v)\big)^{k-2j} |(x,y)|^{2j} \\ &= \frac{c}{2} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j 2^{k-2j}\,(n+k-j-2)!}{j!\,(k-2j)!\,(n-1)!} \big((x,y)\cdot(u,v)\big)^{k-2j} |(x,y)|^{2j}, \end{aligned}$$

where $c = {\omega_{2n-1}}^{-1}(2n+2k-2)$. To express the above in terms of z and w, note that

$$(x, y) = \Big(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\Big) \qquad \text{and} \qquad (u, v) = \Big(\frac{w+\bar{w}}{2}, \frac{w-\bar{w}}{2i}\Big),$$

and moreover,

$$(x,y)\cdot(u,v) = \frac{1}{4}\Big((z+\bar{z})\cdot(w+\bar{w}) - (z-\bar{z})\cdot(w-\bar{w})\Big) = \frac{1}{2}\left(z\cdot\bar{w} + \bar{z}\cdot w\right).$$

Therefore $Z_w^{(k)}(z)$ is equal to

$$\frac{(n+k-1)}{\omega_{2n-1}} \sum_{j=0}^{\lfloor k/2\rfloor} \frac{(-1)^j (n+k-j-2)!}{j!\,(k-2j)!\,(n-1)!} \left(z\cdot\bar{w} + \bar{z}\cdot w\right)^{k-2j} |z|^{2j}. \tag{3.20}$$

Now Proposition 3.5 and Proposition 3.11 show that, if $(p,q)\in[\mathbb{N}^2]_k$, then $Y_w^{(p,q)}$ may be found from the above expression by collecting the terms of bidegree (p,q).

We are interested in the case where w is the standard basis vector e_1 in $\mathbb{C}^n$, and henceforth we abbreviate $Z_{e_1}^{(k)}$ to $Z^{(k)}$ and $Y_{e_1}^{(p,q)}$ to $Y^{(p,q)}$. In this particular case, the expression (3.20) simplifies to

$$\begin{aligned} &Z^{(k)}(z) \\ &= \frac{(n+k-1)}{\omega_{2n-1}} \sum_{j=0}^{\lfloor k/2\rfloor} \frac{(-1)^j (n+k-j-2)!}{j!\,(k-2j)!\,(n-1)!} \left(z_1 + \bar{z}_1\right)^{k-2j} |z|^{2j} \\ &= \frac{(n+k-1)}{\omega_{2n-1}} \sum_{j=0}^{\lfloor k/2\rfloor} \sum_{l=0}^{k-2j} \frac{(-1)^j (n+k-j-2)!}{j!\,l!\,(k-2j-l)!\,(n-1)!} z_1^{k-2j-l} \bar{z}_1^{l} |z|^{2j}. \end{aligned}$$

Since $k=p+q$, it is now easy to see that $Y^{(p,q)}$ is found by restricting the sum above to all j and l such that

$$0 \le j \le \min(p,q), \qquad 0 \le l \le p+q-2j \qquad \text{and} \qquad j+l=q.$$

Since $j\le\min(p,q)$ and $l=q-j$, it follows that $0\le l = q-j \le p+q-2j$.

The following result is a consequence. Recall that we write $Y^{(p,q)}$ for $Y_{e_1}^{(p,q)}$.

Theorem 3.18. *Suppose that p and q are in $\mathbb{N}$ and that $m=\min(p,q)$. Then*

$$Y^{(p,q)}(z) = \frac{(n+p+q-1)}{\omega_{2n-1}} \sum_{j=0}^{m} \frac{(-1)^j (n+p+q-j-2)!}{(p-j)!\,(q-j)!\,j!\,(n-1)!} z_1^{p-j} \bar{z}_1^{q-j} |z|^{2j}.$$

3.4 Products of Zonal Harmonics

Again in this section, we shall write $Y^{(p,q)}$ in place of $Y_{e_1}^{(p,q)}$. We will study $z_1 Y^{(p,q)}$, $\bar{z}_1 Y^{(p,q)}$ and $|z_1|^2 Y^{(p,q)}$. We shall express these as sums of other zonal harmonics, and we shall estimate $\left\|z_1 Y^{(p,q)}\right\|_2$ in terms of $\left\|Y^{(p,q)}\right\|_2$. The fact that we are integrating over S implies that the former is less than the latter. This relationship will be used in Chapter 4.

Lemma 3.19. *There exist unique complex constants $\delta_{p,q}$ and $\epsilon_{p,q}$ such that*

$$z_1 Y^{(p,q)}(z) = \delta_{p,q} Y^{(p+1,q)}(z) + \epsilon_{p,q} |z|^2 Y^{(p,q-1)}(z) \qquad \forall z\in\mathbb{C}^n; \tag{3.21}$$

they are given by

$$\delta_{p,q} = \frac{p+1}{n+p+q}$$

and

$$\epsilon_{p,q} = \begin{cases} \dfrac{n+q-2}{n+p+q-2} & \text{when } q > 0 \\ 0 & \text{when } q = 0. \end{cases}$$

Proof. Write m for $\min\{p+1, q\}$. As $f\colon z \mapsto z_1 Y^{(p,q)}(z)$ is in $\mathcal{P}_{p+1,q}$, we see from Corollary 3.9 that there exist f_j in $\mathcal{H}_{p-j+1,q-j}$ (where $j = 0, 1, \ldots, m$) such that

$$f = \sum_{j=0}^{m} |\cdot|^{2j} f_j.$$

But since $f \circ U = f$ for every U in $\mathrm{U}(n)$ that fixes e_1, it follows from (3.17) that the same is true of each f_j. Now Proposition 3.13, the remarks following it, and Proposition 3.14 imply that there exists c_j in $\mathbb{C}$ such that $f_j = c_j Y^{(p+1-j,q-j)}$. From the uniqueness of the f_j in the decomposition above, it therefore follows that there exist unique constants c_j such that

$$f = \sum_{j=0}^{m} c_j \, |\cdot|^{2j} \, Y^{(p+1-j,q-j)}.$$

It is easy to see from the above expression and from Proposition 3.6 that

$$\begin{aligned} c_j \left\| Y^{(p+1-j,q-j)} \right\|_2^2 &= \left\langle f, Y^{(p+1-j,q-j)} \right\rangle \\ &= \int_S z_1 Y^{(p,q)}(z) \, \bar{Y}^{(p+1-j,q-j)}(z) \, d\sigma(z) \\ &= \left\langle Y^{(p,q)}, g_j \right\rangle, \end{aligned}$$

where $g_j\colon z \mapsto \bar{z}_1 Y^{(p+1-j,q-j)}(z)$. Now if $1 < j \leq m$, then $g_j \in \mathcal{P}_{p+1-j,q+1-j}$, and so $c_j \left\| Y^{(p+1-j,q-j)} \right\|_2^2 = 0$ from Corollary 3.10. Thus $c_j = 0$, since $\left\| Y^{(p+1-j,q-j)} \right\|_2^2 \neq 0$ by Proposition 3.16. We have thus proved the first part of the lemma.

To compute $\delta_{p,q}$ and $\epsilon_{p,q}$, we use Lemma 3.15 and Theorem 3.18, and equate the coefficients of $z_1^{p+1} \bar{z}_1^{q}$ in (3.21):

$$\begin{aligned} &\frac{n+p+q-1}{\omega_{2n-1}} \cdot \frac{(n+p+q-2)!}{p!\, q!\, (n-1)!} \\ &= \delta_{p,q} \frac{n+p+q}{\omega_{2n-1}} \cdot \frac{(n+p+q-1)!}{(p+1)!\, q!\, (n-1)!}, \end{aligned}$$

from which

$$\delta_{p,q} = \frac{p+1}{n+p+q}. \tag{3.22}$$

Similarly, when $q > 0$, equating the coefficients of $z_1^p \bar{z}_1^{q-1} |z|^2$ in (3.21) shows that

$$\begin{aligned}
&- \frac{n+p+q-1}{\omega_{2n-1}} \cdot \frac{(n+p+q-3)!}{(p-1)!\,(q-1)!\,(n-1)!} \\
&= \epsilon_{p,q} \frac{n+p+q-2}{\omega_{2n-1}} \cdot \frac{(n+p+q-3)!}{p!\,(q-1)!\,(n-1)!} \\
&\quad - \delta_{p,q} \frac{n+p+q}{\omega_{2n-1}} \cdot \frac{(n+p+q-2)!}{p!\,(q-1)!\,(n-1)!}
\end{aligned}$$

Multiplying the above expression by

$$\frac{\omega_{2n-1}\, p!\,(q-1)!\,(n-1)!}{(n+p+q-2)!}$$

and using (3.22) shows that

$$\epsilon_{p,q} = \frac{(p+1)(n+p+q-2) - p(n+p+q-1)}{n+p+q-2} = \frac{n+q-2}{n+p+q-2},$$

as required. □

Corollary 3.20. *There exist unique complex numbers $\delta'_{p,q}$ and $\epsilon'_{p,q}$ such that*

$$\bar{z}_1 Y^{(p,q)}(z) = \delta'_{p,q} Y^{(p,q+1)}(z) + \epsilon'_{p,q} |z|^2 Y^{(p-1,q)}(z) \qquad \forall z \in \mathbb{C}^n;$$

they are given by

$$\delta'_{p,q} = \delta_{q,p} = \frac{q+1}{n+p+q}$$

and

$$\epsilon'_{p,q} = \epsilon_{q,p} = \begin{cases} \dfrac{n+p-2}{n+p+q-2} & \text{when } p > 0 \\ 0 & \text{when } p = 0. \end{cases}$$

Proof. By using Lemma 3.19, write

$$z_1 Y^{(q,p)}(z) = \delta_{q,p} Y^{(q+1,p)}(z) + \epsilon_{q,p} |z|^2 Y^{(q,p-1)}(z) \qquad \forall z \in \mathbb{C}^n.$$

Taking complex conjugates of both sides in the above expression and using (3.16) gives

$$\begin{aligned}
\bar{z}_1 Y^{(p,q)}(z) &= \bar{\delta}_{q,p} Y^{(p,q+1)}(z) + \bar{\epsilon}_{q,p} |z|^2 Y^{(p-1,q)}(z) \\
&= \delta_{q,p} Y^{(p,q+1)}(z) + \epsilon_{q,p} |z|^2 Y^{(p-1,q)}(z) \\
&= \delta'_{p,q} Y^{(p,q+1)}(z) + \epsilon'_{p,q} |z|^2 Y^{(p-1,q)}(z) \qquad \forall z \in \mathbb{C}^n,
\end{aligned}$$

and the existence is proved.

To prove the uniqueness, conjugate and use the uniqueness in Lemma 3.19. □

Corollary 3.21. *There are unique complex constants $\alpha_{p,q}$, $\beta_{p,q}$ and $\gamma_{p,q}$ such that*

$$z_1\bar{z}_1 Y^{(p,q)}(z) = \alpha_{p,q} Y^{(p+1,q+1)}(z) + \beta_{p,q} |z|^2 Y^{(p,q)}(z) + \gamma_{p,q} |z|^4 Y^{(p-1,q-1)}(z)$$

for all z in $\mathbb{C}^n$; further, if $n = 2$ and $p = q = 0$, then $\beta_{p,q} = 1/2$, while otherwise

$$\beta_{p,q} = \frac{p^2 + q^2 + (n-1)(p+q) + n - 2}{(n+p+q-1)^2 - 1}.$$

Proof. From Corollary 3.20 and Lemma 3.19, we see that, if $p > 0$ and $q > 0$, then

$$\begin{aligned} z_1\bar{z}_1 Y^{(p,q)}(z) &= z_1\big(\delta_{q,p} Y^{(p,q+1)}(z) + \epsilon_{q,p} |z|^2 Y^{(p-1,q)}(z)\big) \\ &= \delta_{q,p}\big(\delta_{p,q+1} Y^{(p+1,q+1)}(z) + \epsilon_{p,q+1} |z|^2 Y^{(p,q)}(z)\big) \\ &\qquad + \epsilon_{q,p} |z|^2 \big(\delta_{p-1,q} Y^{(p,q)}(z) + \epsilon_{p-1,q} |z|^2 Y^{(p-1,q-1)}(z)\big) \\ &= \delta_{q,p}\delta_{p,q+1} Y^{(p+1,q+1)}(z) + |z|^2 \big(\delta_{q,p}\epsilon_{p,q+1} + \epsilon_{q,p}\delta_{p-1,q}\big) Y^{(p,q)}(z) \\ &\qquad + \epsilon_{q,p}\epsilon_{p-1,q} |z|^4 Y^{(p-1,q-1)}(z) \\ &= \alpha_{p,q} Y^{(p+1,q+1)}(z) + \beta_{p,q} |z|^2 Y^{(p,q)}(z) + \gamma_{p,q} |z|^4 Y^{(p-1,q-1)}(z), \end{aligned}$$

for all z in $\mathbb{C}^n$, say. Further,

$$\begin{aligned} \beta_{p,q} &= \delta_{q,p}\epsilon_{p,q+1} + \epsilon_{q,p}\delta_{p-1,q} \\ &= \frac{q+1}{n+p+q} \cdot \frac{n+q-1}{n+p+q-1} + \frac{n+p-2}{n+p+q-2} \cdot \frac{p}{n+p+q-1} \\ &= \frac{p^2 + q^2 + (n-1)(p+q) + n - 2}{(n+p+q-1)^2 - 1}, \end{aligned}$$

as claimed. If $p = 0$ or $q = 0$, then similar calculations give the result claimed. □

Corollary 3.22. *Suppose that p and q are nonnegative integers. Then*

$$\big\| z_1 Y^{(p,q)} \big\|_2^2 = \beta_{p,q} \big\| Y^{(p,q)} \big\|_2^2$$

where $\beta_{p,q}$ is given in Corollary 3.21.

Proof. By Corollary 3.21, if $p > 0$ and $q > 0$, then $z_1\bar{z}_1 Y^{(p,q)}(z)$ is equal to

$$\alpha_{p,q} Y^{(p+1,q+1)}(z) + \beta_{p,q} |z|^2 Y^{(p,q)}(z) + \gamma_{p,q} |z|^4 Y^{(p-1,q-1)}(z)$$

for all z in $\mathbb{C}^n$. By orthogonality and Proposition 3.16,

$$\begin{aligned} \big\| z_1 Y^{(p,q)} \big\|_2^2 &= \int_S z_1 Y^{(p,q)}(z)\, \bar{z}_1 \bar{Y}^{(p,q)}(z)\, d\sigma(z) \\ &= \big\langle z_1\bar{z}_1 Y^{(p,q)}, Y^{(p,q)} \big\rangle \\ &= \beta_{p,q} \big\| Y^{(p,q)} \big\|_2^2 . \end{aligned}$$

The obvious variant of this holds if $p = 0$ or $q = 0$. □

Chapter 4
The Kohn Sublaplacian

Properties of spherical harmonics, such as orthogonality, make them very useful in Fourier analysis and in the study of certain linear operators acting on $L^2(S)$.

An example of this for $\mathbb{R}^n$ may be found in [9], where the connection between spherical harmonics and the Fourier transform is developed. In particular, the spherical harmonics lead to a decomposition of $L^2(S)$ into spaces on which the Fourier transform is invariant and behaves in a sufficiently simple way. In addition to enabling a deeper understanding of the Fourier transform, several applications to certain convolution operators are thus found.

In this chapter we will show how complex spherical harmonics can be used to understand the action of a particular Laplace-type differential operator acting on $L^2(S)$, where S is the unit sphere in complex n-dimensional space.

The operator $\mathcal{L}$ defined below is important in the study of the 'Kohn–Laplacian' associated with the Cauchy–Riemann complex on S. An explicit formula for the Kohn–Laplacian is given in [6].

We intend to prove that $\mathcal{L}$ acts by scalar multiplication on the spaces $\mathcal{H}_{p,q}$ of Chapter 3, and hence deduce some basic operator-theoretic information about $\mathcal{L}$.

4.1 The Study of an Operator on $L^2(S)$

Consider again the differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\Big) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\Big),$$

defined on differentiable functions on $\mathbb{C}^n$.

We shall be dealing with a second-order differential operator related to the Laplacian on $\mathbb{C}^n$ and defined in terms of the operators above. The behaviour of this operator on $L^2(S)$ will be clarified by considering the spaces $\mathcal{H}_{p,q}$ of complex spherical harmonics of the previous chapter.

When $1 \le j, k \le n$, we define

$$M_{jk} = \bar{z}_j\frac{\partial}{\partial z_k} - \bar{z}_k\frac{\partial}{\partial z_j} \qquad \text{and} \qquad \bar{M}_{jk} = z_j\frac{\partial}{\partial \bar{z}_k} - z_k\frac{\partial}{\partial \bar{z}_j}.$$

It is easy to see that M_{jk} maps every $\mathcal{P}_{p,q}$ into $\mathcal{P}_{p-1,q+1}$ and $\bar{M}_{jk}$ maps every $\mathcal{P}_{p,q}$ into $\mathcal{P}_{p+1,q-1}$. We define the operator $\mathcal{L}$ by

$$\mathcal{L} = \sum_{1\le j<k\le n} M_{jk}\bar{M}_{jk} + \bar{M}_{jk}M_{jk};$$

clearly $\mathcal{L}$ maps every $\mathcal{P}_{p,q}$ into itself.

The following theorem examines the effect of $\mathcal{L}$ on each $\mathcal{H}_{p,q}$.

Theorem 4.1. *The operator $\mathcal{L}$ maps every space $\mathcal{H}_{p,q}$ into itself. Moreover,*

$$\mathcal{L}f = -\lambda_{p,q} f \qquad \forall f \in \mathcal{H}_{p,q}, \tag{4.1}$$

where $\lambda_{p,q} = 2pq + (n-1)(p+q)$.

Proof. First,

$$\begin{aligned}
\mathcal{L} &= \sum_{1\le j<k\le n} M_{jk}\bar{M}_{jk} + \bar{M}_{jk}M_{jk} \\
&= \sum_{1\le j<k\le n} \Big(\bar{z}_j \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial z_j}\Big)\Big(z_j \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial \bar{z}_j}\Big) \\
&\quad + \Big(z_j \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial \bar{z}_j}\Big)\Big(\bar{z}_j \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial z_j}\Big) \\
&= \sum_{1\le j<k\le n} 2z_j\bar{z}_j \frac{\partial}{\partial z_k}\frac{\partial}{\partial \bar{z}_k} + 2z_k\bar{z}_k \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_k} z_k \\
&\quad - \bar{z}_k \frac{\partial}{\partial \bar{z}_k}\frac{\partial}{\partial z_j} z_j - z_j \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_k}\bar{z}_k - z_k \frac{\partial}{\partial z_k}\frac{\partial}{\partial \bar{z}_j}\bar{z}_j \\
&= \sum_{\substack{1\le j,k\le n \\ j\neq k}} 2z_j\bar{z}_j \frac{\partial}{\partial z_k}\frac{\partial}{\partial \bar{z}_k} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_k} z_k - z_j \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_k}\bar{z}_k \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}\Big(2z_j\bar{z}_j \frac{\partial}{\partial z_k}\frac{\partial}{\partial \bar{z}_k} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_k} z_k - z_j \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_k}\bar{z}_k\Big) \\
&\quad - \sum_{j=1}^{n}\Big(2\,|z_j|^2 \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_j} z_j - z_j \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}\bar{z}_j\Big) \\
&= \frac{1}{2}|z|^2\Delta - \Big(\sum_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\Big)\Big(\sum_{k=1}^{n} \frac{\partial}{\partial z_k} z_k\Big) - \Big(\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\Big)\Big(\sum_{k=1}^{n} \frac{\partial}{\partial \bar{z}_k}\bar{z}_k\Big) \\
&\quad + \sum_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_j} z_j + \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}\bar{z}_j - \sum_{j=1}^{n} 2\,|z_j|^2 \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}.
\end{aligned}$$

Now if f is in $\mathcal{P}_{p,q}$, then

$$\sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j} = pf \qquad \text{and} \qquad \sum_{j=1}^{n} \bar{z}_j \frac{\partial f}{\partial \bar{z}_j} = qf.$$

Using this in the above expression for $\mathcal{L}$ shows that

$$\mathcal{L}f = -q(p+n)f - p(q+n)f + qf + pf = -[2pq + (n-1)(p+q)]f,$$

for all f in $\mathcal{H}_{p,q}$, as required. □

In view of Theorems 3.7 and 4.1, it is easy to see that $\mathcal{L}$ is a densely-defined, unbounded linear operator on the Hilbert space $L^2(S)$. Theorem 4.1 also gives complete information about the eigenvalues and eigenvectors of $\mathcal{L}$.

Corollary 4.2. *The set $\sigma_p(\mathcal{L})$ of eigenvalues of $\mathcal{L}$ is given by*

$$\sigma_p(\mathcal{L}) = \{-\lambda_{p,q} : p, q \in \mathbb{N}\},$$

where $\lambda_{p,q} = 2pq + (n-1)(p+q)$, and $\mathcal{H}_{p,q}$ is a subspace of the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue $-\lambda_{p,q}$.

Proof. This follows immediately from Theorem 4.1 and the fact that the union of orthonormal bases of the $\mathcal{H}_{p,q}$ is a basis of $L^2(S)$. □

To give a precise definition of the domain of $\mathcal{L}$, recall from Theorem 3.7 that $L^2(S)$ is the direct sum of the family of pairwise orthogonal, finite-dimensional spaces $\mathcal{H}_{p,q}$.

Choose an orthonormal basis for every $\mathcal{H}_{p,q}$ and let $\{e_j : j \in \mathbb{N}\}$ be the union of all these. Then this is an orthonormal basis for $L^2(S)$ and every f in $L^2(S)$ is given by a 'Fourier expansion'

$$f = \sum_{j=0}^{\infty} \langle f, e_j \rangle \, e_j, \tag{4.2}$$

where the series converges unconditionally in $L^2(S)$ (see [4]).

We are now in a position to define the domain of $\mathcal{L}$ precisely. Let $D(\mathcal{L})$ denote the set of elements f in $L^2(S)$ for which the series

$$\sum_{j=0}^{\infty} c_j \, \langle f, e_j \rangle \, e_j \tag{4.3}$$

converges unconditionally in $L^2(S)$, where the constant c_j is defined to be equal to the $\lambda_{p,q}$ for which e_j is in $\mathcal{H}_{p,q}$. For f in $D(\mathcal{L})$, we define $\mathcal{L}f$ by the formula

$$\mathcal{L}f = -\sum_{j=0}^{\infty} c_j \, \langle f, e_j \rangle \, e_j.$$

Theorem 3.7 immediately implies that $D(\mathcal{L})$ is a dense subset of $L^2(S)$, and it is clear that $D(\mathcal{L})$ is a linear manifold (that is, a vector subspace under the induced operations) of $L^2(S)$. The definition of $D(\mathcal{L})$ is also independent of the choice of such a basis of $L^2(S)$.

We define $\mathcal{T}$ to be $-\mathcal{L}$; then $D(\mathcal{T}) = D(\mathcal{L})$. Since $L^2(S)$ is a complex Hilbert space, the following proposition implies that $\mathcal{L}$ and $\mathcal{T}$ are also self-adjoint operators on $L^2(S)$.

Proposition 4.3. *The unbounded, linear operator $\mathcal{T}$ is positive, that is,*

$$\langle \mathcal{T}f, f \rangle \geq 0 \qquad \forall f \in D(\mathcal{T}).$$

Proof. For all f in $D(\mathcal{T})$, we use the expansions in (4.2) and (4.3), together with the pairwise orthogonality of the e_j to conclude that

$$\langle \mathcal{T}f, f\rangle = \Big\langle \sum_{i=0}^{\infty} c_i \langle f, e_i\rangle e_i, \sum_{j=0}^{\infty} \langle f, e_j\rangle e_j \Big\rangle = \sum_{j=0}^{\infty} c_j \langle f, e_j\rangle^2 \geq 0,$$

since each $\lambda_{p,q}$ is nonnegative. □

The graph $\mathcal{G}_T$ of a linear operator T with domain $D(T)$ between Banach spaces X and Y is the set $\{(x, Tx) \in X \times Y : x \in D(T)\}$. The linear operator T is said to be *closed* if its graph is a closed set in the topology of $X \times Y$. According to the closed graph theorem, if $D(T) = X$, then T is bounded if and only if $\mathcal{G}_T$ is closed in $X \times Y$.

Although the closed graph theorem does not necessarily hold in the case where X or Y is not a complete normed space (for example, if T is an unbounded, densely defined operator) the property of being closed is nevertheless important in operator theory (see [4, Chapter X]). The following result shows that $\mathcal{L}$ is a closed operator.

Proposition 4.4. *The operator $\mathcal{L}$ is closed.*

Proof. Recall that $\mathcal{T} = -\mathcal{L}$; it suffices to show that $\mathcal{T}$ is closed. Since $\mathcal{T}$ is linear, it is enough to take a sequence $\{f_k\}$ in $D(T)$ such that $f_k \to 0$ and $\mathcal{T}f_k \to f$ in $L^2(S)$, and then to show that $f = 0$.

Parseval's identity implies that, for all positive integers k and m,

$$\begin{aligned}
|\langle f, e_m\rangle| &= |\langle f - c_m f_k + c_m f_k, e_m\rangle| \\
&\leq |\langle f - c_m f_k, e_m\rangle| + |\langle c_m f_k, e_m\rangle| \\
&\leq \Big(\sum_{j=0}^{\infty} |\langle f - c_j f_k, e_j\rangle|^2\Big)^{1/2} + c_m \|f_k\|_2 \\
&= \|f - \mathcal{T}f_k\|_2 + c_m \|f_k\|_2 .
\end{aligned}$$

Since $f_k \to 0$ and $\mathcal{T}f_k \to f$ as k tends to infinity, it follows that $\langle f, e_m\rangle = 0$. We conclude that $f = 0$ since $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis of $L^2(S)$. □

The above proposition illustrates the fact that the closed graph theorem is not true when the domain of the operator is not a complete normed space: the graph of $\mathcal{L}$ is closed yet the operator is clearly unbounded.

Combining Proposition 4.4 with the closed graph theorem also proves that the domain $D(\mathcal{L})$ of $\mathcal{L}$ is a proper (dense) subset of $L^2(S)$.

Finally, we turn to computing the spectrum of $\mathcal{L}$.

Theorem 4.5. *The spectrum of the operator $\mathcal{L}$ is given by*

$$\sigma(\mathcal{L}) = \{-\lambda_{p,q} : p, q \in \mathbb{N}\},$$

where $\lambda_{p,q} = 2pq + (n-1)(p+q)$.

Proof. It is enough to prove that $\sigma(\mathcal{T}) = \{\lambda_{p,q} : p, q \in \mathbb{N}\}$. But since the $\lambda_{p,q}$ are eigenvalues of $\mathcal{T}$ it suffices to prove that

$$\sigma(\mathcal{T}) \subseteq \{\lambda_{p,q} : p, q \in \mathbb{N}\}.$$

So suppose that λ in $\mathbb{C}$ is not equal to any $\lambda_{p,q}$. Then Corollary 4.2 implies λ is not an eigenvalue of $\mathcal{T}$, and $\mathcal{T} - \lambda I$ is one to one.

To show that $\mathcal{T} - \lambda I$ maps $D(\mathcal{T})$ onto $L^2(S)$, note that $\{1/(\lambda_{p,q} - \lambda)\}$ is a bounded sequence of complex numbers that converges to zero as $p + q$ increases. Thus if f is in $L^2(S)$, then it follows that the series

$$\sum_{j=0}^{\infty} (c_j - \lambda)^{-1} \langle f, e_j \rangle e_j$$

converges unconditionally to an element g in $L^2(S)$ (where c_j is again defined to be the $\lambda_{p,q}$ such that e_j is in $\mathcal{H}_{p,q}$).

But the series

$$\sum_{j=0}^{\infty} \langle f, e_j \rangle e_j = \sum_{j=0}^{\infty} (c_j - \lambda)(c_j - \lambda)^{-1} \langle f, e_j \rangle e_j = \sum_{j=0}^{\infty} (c_j - \lambda) \langle g, e_j \rangle e_j$$

converges unconditionally to f and so g is in $D(\mathcal{T})$ and $f = (\mathcal{T} - \lambda)g$.

We have now proved that $\mathcal{T} - \lambda I : D(\mathcal{T}) \to L^2(S)$ is bijective. It follows from Proposition 4.3 and the closed graph theorem, applied to the operator $(\mathcal{T} - \lambda I)^{-1}$ on $L^2(S)$, that $(\mathcal{T} - \lambda I)^{-1}$ is bounded. Thus λ is in $\rho(\mathcal{T}) = \mathbb{C} \setminus \sigma(\mathcal{T})$ and the result is proved. □

4.2 A Multiplier Theorem

As an application of spherical harmonics, we shall now prove a 'weighted Plancherel estimate' (in the sense of [5]) for zonal spherical polynomials on S. The weight function will be $(1 - |z_1|^2)^{\alpha/2}$ where $0 < \alpha < 1/2$.

We say that f is a zonal spherical polynomial on S if

$$f = \sum_{p,q=0}^{\infty} c_{p,q} Y^{(p,q)}, \tag{4.4}$$

where only finitely many constants $c_{p,q}$ are nonzero.

The following lemma bounds $\left\|(1-|z_1|^2)^{1/2} f\right\|_2$ in terms of the $c_{p,q}$ and constants $m(p,q)$, defined by $m(p,q) = 1/2^{1/2}$ if $n = 2$ and $p = q = 0$, and

$$m(p,q) = \Big(\frac{2pq + (n-1)(p+q) + (n-1)(n-2)}{(n+p+q-1)^2 - 1}\Big)^{1/2} \tag{4.5}$$

otherwise. That is, $m(p,q)^2 = 1 - \beta_{p,q}$, where $\beta_{p,q}$ is defined as in Corollary 3.21. It is easy to see that $0 < m(p,q) < 1$ for all p and q in $\mathbb{N}$. For a zonal spherical polynomial f on S, we also define Mf by the formula

$$Mf = \sum_{p,q=0}^{\infty} m(p,q) c_{p,q} Y^{(p,q)}.$$

As the $m(p,q)$ are bounded, M extends to a bounded linear operator on $L^2(S)$.

Lemma 4.6. *Suppose that f is a zonal spherical polynomial on S. Then*

$$\left\| (1-|z_1|^2)^{1/2} f \right\|_2 \le 3 \left\| Mf \right\|_2 .$$

Proof. Write $f = \sum_{j=0}^{2} f_j$, where f_j is defined by

$$f_j = \sum_{\substack{p,q \in \mathbb{N} \\ p+q \equiv j}} c_{p,q} Y^{(p,q)}$$

(henceforth, $\equiv$ denotes congruence modulo 3). We define Mf_j in the obvious way; then $Mf = \sum_{j=0}^{2} Mf_j$, and the Mf_j are orthogonal by Theorem 3.7. Thus

$$\left\| Mf \right\|_2 = \left(\left\| Mf_0 \right\|_2^2 + \left\| Mf_1 \right\|_2^2 + \left\| Mf_2 \right\|_2^2 \right)^{1/2},$$

and so $\left\| Mf_j \right\|_2 \le \left\| Mf \right\|_2$. It follows that it is enough to prove that

$$\left\| (1-|z_1|^2)^{1/2} f_j \right\|_2 \le \left\| Mf_j \right\|_2 . \tag{4.6}$$

To prove (4.6), first note that

$$\left\| f_j \right\|_2^2 = \sum_{\substack{p,q \in \mathbb{N} \\ p+q \equiv j}} |c_{p,q}|^2 \left\| Y^{(p,q)} \right\|_2^2 .$$

A similar expression is needed for $\left\| z_1 f_j \right\|_2$. To find it, note that

$$z_1 f_j = \sum_{\substack{p,q \in \mathbb{N} \\ p+q \equiv j}} c_{p,q} z_1 Y^{(p,q)}.$$

Recall from (3.21) that $z_1 Y^{(p,q)} = \delta_{p,q} Y^{(p+1,q)} + \epsilon_{p,q} Y^{(p,q-1)}$; it follows that, when $p+q \equiv r+s \equiv j$,

$$\left\langle z_1 Y^{(p,q)}, z_1 Y^{(r,s)} \right\rangle = 0$$

unless $(p,q) = (r,s)$. It follows immediately from Corollary 3.22 that

$$\begin{aligned} \left\| z_1 f_j \right\|_2^2 &= \sum_{\substack{p,q \in \mathbb{N} \\ p+q \equiv j}} |c_{p,q}|^2 \left\| z_1 Y^{(p,q)} \right\|_2^2 \\ &= \sum_{\substack{p,q \in \mathbb{N} \\ p+q \equiv j}} |c_{p,q}|^2 \beta_{p,q} \left\| Y^{(p,q)} \right\|_2^2 , \end{aligned}$$

where $\beta_{p,q}$ is as in Corollary 3.21.

Combining the above expressions for $\|f_j\|_2^2$ and $\|z_1 f_j\|_2^2$ shows that

$$\begin{aligned}
&\left\|(1-|z_1|^2)^{1/2} f_j\right\|_2^2 \\
&= \|f_j\|_2^2 - \|z_1 f_j\|_2^2 \\
&= \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} |c_{p,q}|^2 (1-\beta_{p,q}) \left\|Y^{(p,q)}\right\|_2^2 \\
&= \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} m(p,q)^2 |c_{p,q}|^2 \left\|Y^{(p,q)}\right\|_2^2 \\
&= \|Mf_j\|_2^2,
\end{aligned}$$

and (4.6) is established. □

As a corollary to Lemma 4.6, we now prove an analogue of the 'Plancherel-type estimate' in Assumption 2.5 of [5]. Recall that the constant $\lambda_{p,q}$ is given by (4.1). For a positive integer i, we define the subset H_i of $\mathbb{N}^2$ by

$$H_i = \{(p,q) \in \mathbb{N}^2 : (i-1)^2 \le \lambda_{p,q} \le i^2\}.$$

Proposition 4.7. *Suppose that N is a positive integer and that f in $C(S)$ is given by*

$$f = \sum_{p,q=0}^{\infty} c_{p,q} Y^{(p,q)},$$

where $c_{p,q} = 0$ when $\lambda_{p,q} > N^2$. Then

$$\left\|(1-|z_1|^2)^{\alpha/2} f\right\|_2 \le C\, N^{n-\alpha+1/2} \Big(\sum_{i=1}^{N} \max\{|c_{p,q}| : (p,q) \in H_i\}^2\Big)^{1/2},$$

if $0 < \alpha < 1/2$, where the constant C depends only on n and α.

Proof. We define C_i by

$$C_i = \max\{|c_{p,q}| : p,q \in H_i\}.$$

By Lemma 4.6,

$$\left\|(1-|z_1|^2)^{1/2} f\right\|_2 \le 3\,\|Mf\|_2,$$

and $\|f\|_2 \le \|f\|_2$, trivially. We now define

$$M^\alpha f = \sum_{p,q=0}^{\infty} m(p,q)^\alpha c_{p,q} Y^{(p,q)};$$

then the M. Riesz convexity theorem implies that

$$\left\|(1-|z_1|^2)^{\alpha/2} f\right\|_2 \le 3^\alpha \|M^\alpha f\|_2 \qquad \forall \alpha \in (0,1).$$

Now the Plancherel theorem (i.e., the orthogonality relations) implies that

$$
\begin{aligned}
\|M^\alpha f\|_2 &= \Big(\sum_{p,q=0}^{\infty} m(p,q)^{2\alpha}\,|c_{p,q}|^2\,\big\|Y^{(p,q)}\big\|_2^2\Big)^{1/2} \\
&\le \Big(\sum_{i=1}^{N} C_i^2 \sum_{(p,q)\in H_i} m(p,q)^{2\alpha}\,\big\|Y^{(p,q)}\big\|_2^2\Big)^{1/2} \\
&\le \Big(\sum_{i=1}^{N} C_i^2\Big)^{1/2} \max_{i=1,\dots,N}\Big(\sum_{(p,q)\in H_i} m(p,q)^{2\alpha}\,\big\|Y^{(p,q)}\big\|_2^2\Big)^{1/2} \\
&= \Big(\sum_{i=1}^{N} C_i^2\Big)^{1/2} \max_{i=1,\dots,N}\Big(\sum_{(p,q)\in H_i} m(p,q)^{2\alpha}\omega_{2n-1}^{-1} d(p,q)\Big)^{1/2},
\end{aligned}
$$

by Proposition 3.16. Hence it is enough to prove that

$$
\sum_{(p,q)\in H_i} m(p,q)^{2\alpha} d(p,q) \le C\, N^{2(n-\alpha)+1}, \tag{4.7}
$$

where C depends only on n and α. From (4.5), if $p > 0$ and $q > 0$, then

$$
\begin{aligned}
m(p,q)^2 &= \frac{2pq + (n-1)(p+q) + (n-1)(n-2)}{(n+p+q-1)^2 - 1} \\
&= \frac{2pq}{(n+p+q-1)^2 - 1} + \frac{n-1}{n+p+q} \\
&\le \frac{3pq}{(p+q+n-1)^2} + \frac{n-1}{p+q+1},
\end{aligned}
$$

whence

$$
m(p,q)^2 \le \frac{(n+2)(p+1)(q+1)}{(p+q+1)^2};
$$

if $p = 0$ or $q = 0$ the same inequality holds by a similar but easier argument. Combining this with the estimate (3.12) for $d(p,q)$, we see that the left-hand side of (4.7) is bounded by

$$
\begin{aligned}
&(n+2) \sum_{(p,q)\in H_i} \frac{(p+1)^\alpha (q+1)^\alpha}{(p+q+1)^{2\alpha}} (p+q+1)(p+1)^{n-2}(q+1)^{n-2} \\
&= (n+2) \sum_{(p,q)\in H_i} (p+1)^{n+\alpha-2}(q+1)^{n+\alpha-2}(p+q+1)^{1-2\alpha}.
\end{aligned} \tag{4.8}
$$

Now

$$
\lambda_{p,q} = 2pq + (n-1)(p+q) = 2[(p+a)(q+a) - a^2]
$$

where $a = (n-1)/2$, so p and q in the above sum satisfy the inequalities when $0 \le p \le \lfloor i^2/2a \rfloor$ and q is in I_p, where

$$I_p = \Big\{ k \in \mathbb{N} : \frac{(i-1)^2 + 2a^2}{2(p+a)} \le k + a \le \frac{i^2 + 2a^2}{2(p+a)} \Big\}.$$

Note that

$$\frac{i^2 + 2a^2}{2(p+a)} \le \frac{i^2 + a^2}{p+a}.$$

It follows that, when $0 < \alpha < 1/2$, the sum in (4.8) is bounded by

$$\begin{aligned}
&\sum_{p=0}^{\lfloor i^2/2a \rfloor} (p+a)^{n+\alpha-2} \sum_{q \in I_p} (q+a)^{n+\alpha-2} (p+q+a)^{1-2\alpha} \\
&\le \sum_{p=0}^{\lfloor i^2/2a \rfloor} (p+a)^{n+\alpha-2} \left(\frac{2i-1}{2(p+a)} + 1 \right) \left(\frac{i^2+a^2}{p+a} \right)^{n+\alpha-2} \left(p + \frac{i^2+a^2}{p+a} \right)^{1-2\alpha} \\
&\le \sum_{p=0}^{\lfloor i^2/2a \rfloor} 2\,i \big(i^2+a^2\big)^{n+\alpha-2} \left(p + \frac{i^2+a^2}{p+a} \right)^{1-2\alpha} \\
&\le 4\,i \big(i^2+a^2\big)^{n+\alpha-2} \sum_{p=0}^{\lfloor i^2/2a \rfloor} \big(p + i^2 + a^2\big)^{1-2\alpha} \\
&\le 4\,i \big(i^2+a^2\big)^{n+\alpha-2} \int_0^{\lfloor i^2/2a \rfloor + 1} \big(x + i^2 + a^2\big)^{1-2\alpha}\, dx \\
&\le \frac{4\,i}{2-2\alpha} \big(i^2+a^2\big)^{n+\alpha-2} \Big(\frac{i^2}{2a} + 1 + i^2 + a^2 \Big)^{2-2\alpha} \\
&\le \frac{18\,i}{1-\alpha} \big(i^2+a^2\big)^{n-\alpha} \\
&\le 36\,(1+n^2)^n N^{2(n-\alpha)+1},
\end{aligned}$$

and (4.7) is proved. □

Combining this 'weighted Plancherel estimate' with the general results of Cowling and Sikora [5], we are able to deduce that the Hörmander multiplier theorem for the Kohn–Laplacian on the sphere in complex n-space holds with critical index $(2n-1)/2$. To describe this result in detail would require many more pages, and we will not do so here; we intend to publish this elsewhere.

References

[1] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory.* Springer-Verlag, New York, 2001.

[2] A. Bonami and J.-L. Clerc, Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques, *Trans. Amer. Math. Soc.* **183** (1973), 223–263.

[3] R.R. Coifman and G.L. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes.* Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin, 1971.

[4] J.B. Conway, *A Course in Functional Analysis.* Springer-Verlag, New York, 1985.

[5] M. Cowling, A. Sikora, A spectral multiplier theorem for a sublaplacian on SU(2), *Math. Z.* **238** (2001), 1–36.

[6] D. Geller, The Laplacian and the Kohn Laplacian for the sphere, *J. Differ. Geometry* **15** (1980), 417–435.

[7] L. Hörmander, Estimates for translation invariant operators in L^p spaces, *Acta Math.,* 104, 93-140, 1960.

[8] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$. Springer-Verlag, New York, 1980.

[9] E. M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press, Princeton, 1970.