Some irreducible unitary representations of $G(K)$
for a simple algebraic group G
over a number field K

by

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Abstract. Let K be an algebraic number field, and let G_K be the group of K-rational points of a simply connected, simple linear algebraic group G defined over K. We construct a new family of irreducible unitary representations of G_K, as follows. It is well known that G_K embeds diagonally as a lattice in G_A, where A is the ring of adèles of K. Let π be an irreducible unitary representation of G_A. We show that $\pi|_{G_K}$, the restriction of π to G_K, is irreducible and that π is determined by $\pi|_{G_K}$ up to unitary equivalence. Many of these restrictions are not in the support of the regular representation of G_K.

1. Introduction and statement of the results

Let G be a simple linear algebraic group defined over K. Let G_K be the group of K-points in G, equipped with the discrete topology (this is the only locally compact topology on G_K). Since G_K is not type I (see [Tho]), there is no hope for a classification of its irreducible unitary representations. The aim of the paper is to show that some natural unitary representations of G_K are irreducible. All the representations we consider are assumed to be strongly continuous and unitary; equivalence of representations, without further qualification, means unitary equivalence.

Denote by V the set of all places of K, that is, the set of all equivalence classes of valuations of K; V is the union of the set V_f of the finite places (or non-archimedean valuations), and of the finite set V_∞ of the finite places (or archimedean valuations). A statement which holds for all but finitely many places is said to hold for "almost all places".

For any valuation v, let K_v denote the corresponding completion of K. Let $\mathcal{O}$ be the ring of integers of K. For any non-archimean valuation v, let $\mathcal{O}_v = \{x \in K_v : v(x) \leq 1\}$, the closure of $\mathcal{O}$ in K_v. (For instance, if $K = \mathbf{Q}$ and v is the valuation given by a prime p, then K_v is the field of p-adic numbers and $\mathcal{O}_v$ is the ring of p-adic integers.) The group G_{K_v} of the K_v-points, equipped with the topology induced by K_v, is a simple p-adic Lie group when v is finite and is a simple real Lie group when v is infinite. Let π be an irreducible representation of G_{K_v} for some place v. Since G_K is dense in G_{K_v} and π is strongly continuous, it is obvious that the restriction $\pi|_{G_K}$ of π to G_K is irreducible. It is equally clear that π is determined by this restriction, that is, if π and π' are representations of G_{K_v} such that $\pi|_{G_K}$ and $\pi'|_{G_K}$ are equivalent, then π and π' are equivalent. This may be generalised as follows.

Take finitely many distinct places $v_1, \ldots, v_n$ of K, and let $\pi_1, \ldots, \pi_n$ be irreducible representations of $G(K_{v_1}), \ldots, G(K_{v_n})$. Then the tensor product $\pi_1 \otimes \ldots \otimes \pi_n$ is an irreducible representation of $G(K_{v_1}) \times \ldots \times G(K_{v_n})$. The group G_K embeds diagonally as a dense subgroup of $G(K_{v_1}) \times \ldots \times G(K_{v_n})$, by the Weak Approximation Theorem (see [PlR], Proposition 7.11). Hence, the (inner) tensor product

$$(\pi_1 \otimes \ldots \otimes \pi_n)|_{G_K}$$

is an irreducible representation of G_K and determines $\pi_1 \otimes \ldots \otimes \pi_n$.

A setting for unification and generalisation of these facts is given by the embedding of G_K in G_A, where A is the ring of adèles of K. Recall that the adèle group G_A is defined to be

$$\{(g_v)_{v\in V} \in \prod_{v\in V} G_{K_v} : g_v \in G_{\mathcal{O}_v} \text{ for almost all finite places } v\}.$$

The subgroup $G_{\mathcal{O}_v}$ of G_{K_v} is compact and open, and G_A is a locally compact group when endowed with the restricted product topology, for which the subgroup

$$\prod_{v\in V_\infty} G(K_v) \times \prod_{v\in V_f} G_{\mathcal{O}_v},$$

with its product topology, is open. By diagonal embedding, we identify G_K with a subgroup of G_A, which is no longer dense; indeed, it is discrete. Moreover, under our assumptions on G, it has finite covolume (see [Bor], 5.6 Theorem); that is, G_K is a lattice in G_A. For almost all valuations v, G_{K_v} is not compact (see [Spr], 4.9 Lemma). These facts will play a crucial role in this paper.

For any place v, we identify G_{K_v} with the subgroup

$$\{e\} \times \ldots \times \{e\} \times G_{K_v} \times \{e\} \times \ldots$$

of G_A. Since all G_{K_v} are of type I (see [Ber]), the irreducible representations π of G_A are the restricted infinite tensor products of the form $\otimes_{v \in V} \pi_v$, where π_v is an irreducible representation of G_{K_v} which, for almost all finite valuations v, is spherical, i.e., the Hilbert space of π_v contains a non-zero $G_{\mathcal{O}_v}$-invariant vector. Observe that for any irreducible representation of G_{K_v}, there exists at most one such vector (up to scalar multiples), since $(G_{K_v}, G_{\mathcal{O}_v})$ is a Gel'fand pair. In the rest of this paper, whenever we have an irreducible representation π of G_A we always write π_v for its local component at v. Two irreducible representations π and π' are equivalent if and only if π_v and π'_v are equivalent for every valuation v. For all of this, see [GGPS], Chap. 3, §3, No 3; see also [Gui] and [Tad].

Theorem A. *Let G be a simply connected connected, simple linear algebraic group defined over K. Let π and π' be inequivalent irreducible representations of G_A. Then*

(i) *the restriction $\pi|_{G_K}$ of π to G_K is irreducible*

(ii) *the restrictions $\pi|_{G_K}$ and $\pi'|_{G_K}$ are inequivalent.*

This theorem and its proof are in the spirit of [CoS] where similar facts are shown to be true for restrictions of representations of real semisimple Lie groups to lattices. Denote by $\widehat{G}$ the unitary dual of a group G, that is, the set of (unitary) equivalence classes of its irreducible (unitary) representations. Then our result says that $\widehat{G_K}$ contains a copy of the much better known space $\widehat{G_A}$.

Our theorem applies, for example, to SL_n and Sp_n and yields irreducible representations of $SL_n(\mathbf{Q})$ and $Sp_n(\mathbf{Q})$ for $n \geq 2$.

Part (ii) in the above theorem may fail if G is not simply connected. Indeed, for $G = PGL_2$ and $K = \mathbf{Q}$, it is easy to see that there exists a non-trivial unitary character of $PGL_2(A)$ which is trivial on $PGL_2(\mathbf{Q})$. We do not known whether the irreducibility result (i) above remains true when G is not simply connected.

It is perhaps surprising that the restrictions $\pi|_{G_K}$ are not contained in the support of the left regular representation λ_{G_K} of G_K for most irreducible representations π of G_A. In particular, they are not equivalent to representations obtained by internal constructions, such as induction from amenable parabolic subgroups (see [HoR] for the case $GL_n(\mathbf{Q})$). More precisely, the following is true and easy to prove.

Theorem B. *Let G be a semisimple linear algebraic group defined over K, and let π be an irreducible representation of G_A. The following are equivalent*

(i) $\pi|_{G_K}$ *is weakly contained in* λ_{G_K}

(ii) π_v *is weakly contained in* $\lambda_{G_{K_v}}$, *for every place* v

(iii) π *is weakly contained in* λ_{G_A}.

In particular, if π is any representation of G_{K_v} for some place p, then $\pi|_{G_K}$ is not weakly contained in λ_{G_K}.

Recall that a unitary representation π of a locally compact group G is said to be weakly contained in another unitary representation ρ if any diagonal matrix coefficient of π is the limit, uniformly on compacta, of sums of diagonal matrix coefficients of ρ (see [Dix], Chap. 18). The support of ρ is the set of all irreducible representations which are weakly contained in ρ. The representations π and ρ are weakly equivalent if each is weakly contained in the other.

The question whether $\pi|_{G_K}$ and $\pi'|_{G_K}$ might be weakly equivalent for inequivalent irreducible representations π and π' is more difficult, and we do not have a complete answer. Our result uses Kazhdan's Property (T); for an account of this, see [HaV].

Theorem C. *Let G be a simply connected, simple linear algebraic group, defined over K, and assume that G_{K_v} has Kazhdan's Property (T) for all $v \in V$. Let π and π' be inequivalent irreducible representations of G_A, not both weakly contained in the regular representation λ_{G_A}. Then $\pi|_{G_K}$ and $\pi'|_{G_K}$ are not weakly equivalent.*

The assumption that π and π' are not both weakly contained in the regular representation is necessary for the following reason. Suppose, for simplicity, that G has a trivial centre. It was shown in [BCH] that the reduced C^*-algebra of G_K is simple. Equivalently, any representation of G_K which is weakly contained in λ_{G_K} is weakly equivalent to λ_{G_K}. In particular, $\pi|_{G_K}$ and $\pi'|_{G_K}$ are always weakly equivalent if π and π' are weakly contained in λ_{G_A}.

Our results extend to S-arithmetic groups. Let S be any set of valuations containing the set V_∞. Let $\mathcal{O}_S$ be the subring of K consisting of all $x \in K$ for which $v(x) \leq 1$ for all $v \in V \setminus S$. For a semisimple algebraic group G, one may identify $G_{\mathcal{O}_S}$ with a lattice in $G_S = \prod_{v\in S} G_{K_v}$ by diagonal embedding ([Bor]).

Theorem D. *Let G be a simply connected simple, linear algebraic group. Let π and π' be inequivalent irreducible representations of $G_S = \prod_{v\in S} G_{K_v}$. Assume that not all $\pi_v, v \in S$, are square integrable. Then*

(i) *the restriction $\pi|_{G_{\mathcal{O}_S}}$ of π to $G_{\mathcal{O}_S}$ is irreducible*

(ii) *the restrictions $\pi|_{G_{\mathcal{O}_S}}$ and $\pi'|_{G_{\mathcal{O}_S}}$ are inequivalent.*

If, in addition, G_{K_v} has Property (T) for all $v \in S$, then $\pi|_{G_{\mathcal{O}_S}}$ and $\pi'|_{G_{\mathcal{O}_S}}$ are not even weakly equivalent.

Observe that if all π_v are square integrable, then $\pi|_{G_{\mathcal{O}_S}}$ is square integrable and hence cannot be irreducible. Indeed, it is well known that an infinite discrete group has no square integrable irreducible representations.

The paper is organised as follows. In Section 1, we give the proofs of Theorem A and Theorem B, and in Section 2 the proof of Theorem C. The proof of Theorem D is essentially just a combination of the proofs of Theorems A and C, with some minor changes, and we omit it.

2. Proofs of Theorems A and B

Let G be a unimodular locally compact group, and let Γ be a lattice in G. Denote by ρ the quasi-regular representation of G in $L^2(G/\Gamma)$, that is, the induced representation $\mathrm{Ind}_\Gamma^G 1_\Gamma$. Let ρ^0 denote the restriction of ρ to $L_0^2(G/\Gamma)$, the subspace orthogonal to the constant functions, i.e.,

$$\{f \in L^2(G/\Gamma) : \int_{G/\Gamma} f(x)dx = 0\}.$$

Our proof of Theorem A is based on the following elementary result, which makes up Corollaries 1.2 and 1.3 in [CoS]. For the convenience of the reader, we reproduce the proof.

Lemma 1. *Let π and π' be irreducible representations of G. Then*

(i) *if π is not a subrepresentation of $\pi \otimes \rho^0$, then $\pi|_\Gamma$ is irreducible*

(ii) *if π is not a subrepresentation of $\pi' \otimes \rho$, then $\pi|_\Gamma$ and $\pi'|_\Gamma$ are inequivalent.*

Proof. Let $\mathcal{H}_\pi$ denote the Hilbert space of π, and realise the tensor product representation $\pi \otimes \rho$ in $L^2(G/\Gamma; \mathcal{H}_\pi)$, with the action of g in G on a function f given by

$$\pi \otimes \rho(g) f(g_1\Gamma) = \pi(g) f(g^{-1}g_1\Gamma) \qquad \forall g_1\Gamma \in G/\Gamma.$$

Define a map

$$\Phi : \mathrm{Hom}_\Gamma(\mathcal{H}_\pi, \mathcal{H}_{\pi'}) \to \mathrm{Hom}_G(\mathcal{H}_\pi, L^2(G/\Gamma; \mathcal{H}_{\pi'}))$$

by

$$\Phi(T)\xi(g\Gamma) = \pi'(g) T \pi(g^{-1})\xi \qquad \forall g\Gamma \in G/\Gamma \quad \forall \xi \in \mathcal{H}_\pi.$$

It is easy to verify that Φ is injective. Note that $\pi' \otimes \rho = \pi' \oplus \pi' \otimes \rho^0$. Hence, if π is not a subrepresentation of $\pi \otimes \rho^0$, then $\dim(\mathrm{Hom}_G(\mathcal{H}_\pi), L^2(G/\Gamma; \mathcal{H}_\pi)) = 1$ and if π is not a subrepresentation of $\pi' \otimes \rho$, then $\dim(\mathrm{Hom}_G(\mathcal{H}_\pi, L^2(G/\Gamma; \mathcal{H}_{\pi'}))) = 0$. This proves the lemma. □

We need the following lemma.

Lemma 2. *Let G be a simply connected, semisimple linear algebraic group defined over K. Suppose that G_{K_v} is not compact for some place v. Then the natural action of G_{K_v} on G_A/G_K is ergodic.*

Proof. By Moore's duality theorem (see [Zim], 2.2.3 Corollary), it suffices to show that G_K acts ergodically on G_A/G_{K_v}. Now G_A/G_{K_v} is isomorphic (as a G_K-space) to the subgroup $\prod_{w\neq v} G_{K_w}$ of G_A. Since G_{K_v} is not compact and G is simply connected, G_K is a dense subgroup of $\prod_{w\neq v} G_{K_w}$, by the Strong Approximation Theorem (see [PlR], Theorem 7.12). This implies that G_K acts ergodically on $\prod_{w\neq v} G_{K_w}$ (see [Zim], 2.2.13 Lemma). □

Proof of Theorem A. Let ρ denote the natural induced representation of G_A on $L^2(G_A/G_K)$ and ρ^0 denote the restriction of ρ to the orthogonal complement of the constant functions.

(i) Let π be an irreducible representation of G_A.

For almost all places v, G_{K_v} is not compact (see [Spr], 4.9 Lemma). Further, for almost all finite places v, π_v has a non-trivial $G_{\mathcal{O}_v}$-invariant unit vector, ξ_v say. Fix a finite place v such that π_v has a non-trivial $G_{\mathcal{O}_v}$-invariant unit vector, and G_{K_v} is not compact.

Suppose, by contradiction, that $\pi|_{G_K}$ is reducible. Then, by Lemma 1, π is contained in $\pi \otimes \rho^0$. Hence, π_v is contained in $\pi_v \otimes (\rho^0|_{G_{K_v}})$.

By Lemma 2, $1_{G_{K_v}}$, the trivial one-dimensional representation of G_{K_v}, is contained only once in the restriction of ρ to G_{K_v}. That is, $1_{G_{K_v}}$ is not contained in $\rho^0|_{G_{K_v}}$. Hence, arguing as in [CoS], there exists some σ in $\widehat{G}_{K_v} \setminus \{1_{G_{K_v}}\}$, in the support of $\rho^0|_{G_{K_v}}$, such that π_v is contained in $\pi_v \otimes \sigma$. But then π_v is contained in $\pi_v \otimes \sigma \otimes \sigma$ and, by induction, we see that π_v is contained in $\pi_v \otimes \sigma^{\otimes N}$ for any n in $\mathbf{N}$.

Now the kernel of σ is finite. Indeed, since G is simply connected and simple and since G_{K_v} is non-compact, every normal subgroup of G_{K_v} is either contained in the centre of G_{K_v} or coincides with G_{K_v} (see [Mar], Chap. I, (2.3.2) Corollary).

It is known that then there is a real number r in $[2, \infty)$ such that all the matrix coefficients of σ lie in $L^r(G)$ (see [BoW], Chap. XI, 3.6 Proposition; see also [Cow] for the archimedean case). Hence, for some integer N, the N-fold tensor product $\sigma^{\otimes N}$ is contained in an infinite multiple of the regular representation $\lambda_{G_{K_v}}$. This implies that π_v is contained in $\lambda_{G_{K_v}}$, so π_v is square integrable.

However, the spherical function $g \mapsto \langle \pi_v(g)\xi_v, \xi_v \rangle$ is not square integrable. Indeed, the Plancherel measure for the positive definite spherical functions has no atoms (see [Mac], Theorem 5.2.10, for the p-adic case and [Hel], Chap. IV, Theorem 7.5, for the real case). This establishes the desired contradiction.

(ii) Let π and π' be inequivalent irreducible representations of G_A. Suppose, by contradiction, that π is contained in $\pi' \otimes \rho$ and that π' is contained in $\pi \otimes \rho$. Since $\rho = 1_{G_A} \oplus \rho^0$, this implies that π is contained in $\pi' \otimes \rho^0$ and that π' is contained in $\pi \otimes \rho^0$.

Choose a place v such that π_v is not square integrable. By Lemma 2, there exist irreducible representations σ and σ' of G_{K_v}, different from $1_{G_{K_v}}$, such that π_v is contained in $\pi'_v \otimes \sigma$ and π'_v is contained in $\pi_v \otimes \sigma$. Therefore, π_v is contained in $\pi_v \otimes \sigma' \otimes \sigma$ and hence in $\pi_v \otimes \sigma'^{\otimes N} \otimes \sigma^{\otimes N}$ for every N in $\mathbf{N}$. As in (i), this contradicts the fact that π_v is not square integrable and, in view of Lemma 1, completes the proof. □

We now proceed to the proof of Theorem B. We first observe that the equivalence of (i) and (iii) in this theorem is valid in a more general situation.

Lemma 3. *Let G be a locally compact group, and let H be a closed subgroup of G. Assume that 1_G is weakly contained in the quasi-regular representation $\rho = \mathrm{Ind}_H^G 1_H$. Let π be a representation of G. Then the restriction $\pi|_H$ is weakly contained in the regular representation λ_H of H if and only if π is weakly contained in λ_G.*

Proof. It is well known that

$$\mathrm{Ind}_H^G \pi|_H = \pi \otimes \mathrm{Ind}_H^G 1_H = \pi \otimes \rho \qquad \text{and} \qquad \mathrm{Ind}_H^G \lambda_H = \lambda_G.$$

Suppose that $\pi|_H$ is weakly contained in λ_H. Then $\pi \otimes \rho$ is weakly contained in λ_G, by the continuity of induction (see [Fel], Theorem 4.1). Since 1_G is weakly contained in ρ, this implies that π is weakly contained in λ_G.

Conversely, if π is weakly contained in λ_G, then $\pi|_H$ is weakly contained in $\lambda_G|_H$, and hence in λ_H, as $\lambda_G|_H$ is a multiple of λ_H. □

Remark. The assumption that 1_G is weakly contained in the quasi-regular representation $\rho = \mathrm{Ind}_H^G 1_H$ means that G/H is amenable in the sense of Eymard [Eym].

Proof of Theorem B. Lemma 3 implies that (i) and (iii) of Theorem B are equivalent. It remains to show that (ii) and (iii) are equivalent.

Let π be an irreducible representation of G_A. If π is weakly contained in λ_{G_A}, then π_v is weakly contained in $\lambda_{G_A}|_{G_{K_v}}$, and hence in $\lambda_{G_{K_v}}$.

Conversely, suppose that, for every place v, π_v is weakly contained in $\lambda_{G_{K_v}}$. Let ξ be a vector in the Hilbert space $\otimes_{v \in V} \mathcal{H}_{\pi_v}$ of the form $\otimes \xi_v$, where every ξ_v is a unit vector, and ξ_v is invariant under $G_{\mathcal{O}_v}$ for almost all finite places v. Define $\varphi = \langle \pi(\cdot)\xi, \xi \rangle$, the positive definite function associated with ξ. It suffices to show that φ is the limit, uniformly on compacta in G_A, of positive definite functions with compact support.

Let K be a compact subset of G_A. By the definition of the topology on G_A, there exist a finite set F of places containing V_∞ and compact subsets K_v of G_{K_v} for v in F such that K is contained in $K_F \times \prod_{v\notin F} G_{\mathcal{O}_v}$, where $K_F = \prod_{v\in F} K_v$. Now

$$\varphi = \bigotimes_{v\in F} \langle \pi_v(\cdot)\xi_v, \xi_v\rangle$$

on $K_F \times \prod_{v\notin F} G_{\mathcal{O}_v}$. Since π_v is weakly contained in $\lambda_{G_{K_v}}$, there exists, for every v in F, a sequence $\psi_n^{(v)}$ of normalised positive definite functions on G_{K_v} with compact supports converging to $\langle \pi_v(\cdot)\xi_v, \xi_v\rangle$ uniformly on K_v. Set

$$\psi_n = \bigotimes_{v\in F} \psi_n^{(v)} \otimes \bigotimes_{v\notin F} \chi_v,$$

where χ_v denotes the characteristic function of $G_{\mathcal{O}_v}$. Then ψ_n is a positive definite function on G_A with compact support and $\lim_n \psi_n = \varphi$, uniformly on K. Hence, (ii) and (iii) are equivalent.

Given a representation π of G_{K_v}, we may lift it to a representation $\tilde{\pi}$ of G_A, equal to $\pi \otimes \bigotimes_{w\neq v} 1_{G_{K_w}}$. Clearly, $\pi|_{G_K} = \tilde{\pi}|_{G_K}$. Now G_{K_w} is non-compact and hence non-amenable for almost all places w, and, for any such w, $1_{G_{K_w}}$ is not weakly contained in the regular representation. Hence $\pi|_{G_K}$ is not weakly contained in λ_{G_K}. This proves the last assertion of Theorem B. □

2. Proof of Theorem C

Lemma 4. *Let G be a semisimple linear algebraic group defined over K, and let π and π' be irreducible representations of G_A. If π is weakly contained in π', then π and π' are equivalent.*

Proof. If π is weakly contained in π', then π_v is weakly contained in π'_v for every place v.

Recall that a representation π of the group G is weakly contained in a representation ρ if and only if the C^*-kernel of ρ (this is the kernel of the extension of ρ to the C^*-algebra of G) is contained in the C^*-kernel of π.

It is known that every G_{K_v} is a CCR-group, that is, the range of the C^*-algebra of G_{K_v} under an irreducible representation consists of compact operators (see [Ber]; see also [Dix], 15.5.6 for the real case). This implies that π_v is equivalent to π'_v for all places v (see [Dix], Theorem 4.3.7). Hence, π and π' are equivalent. □

Let G be a semisimple algebraic group defined over K. As in the proof of Theorem A, let ρ and ρ^0 denote the natural representation of G_A on $L^2(G_A/G_K)$ and the restriction of ρ to the orthogonal complement of the constant functions.

Proof of Theorem C. Let π and π' be inequivalent irreducible representations of G_A. Assume that π is not weakly contained in the regular representation λ_{G_A}.

Suppose, by contradiction, that $\pi|_{G_K}$ and $\pi'|_{G_K}$ are weakly equivalent. Then

$$\mathrm{Ind}_{G_K}^{G_A}\pi|_{G_K} = \pi \otimes \mathrm{Ind}_{G_K}^{G_A} 1_{G_K} = \pi \oplus (\pi \otimes \rho^0)$$

and, by the continuity of induction, this is weakly equivalent to $\mathrm{Ind}_{G_K}^{G_A}\pi'|_{G_K}$, which in turn is equal to $\pi' \oplus (\pi' \otimes \rho^0)$. Since π is irreducible, this implies that π is either weakly contained in π' or in $\pi' \otimes \rho^0$ and that π' is either weakly contained in π or in $\pi \otimes \rho^0$. In view of Lemma 4, it follows that π is weakly contained in $\pi' \otimes \rho^0$ and that π' is weakly contained in $\pi \otimes \rho^0$.

By Theorem B, we may choose a place v such that G_{K_v} is non-compact and π_v is not weakly contained in the regular representation $\lambda_{G_{K_v}}$. Since π_v is weakly contained in $\pi'_v \otimes \rho^0|_{G_{K_v}}$ and π'_v is weakly contained in $\pi_v \otimes \rho^0|_{G_{K_v}}$, there exist irreducible representations σ and σ' of G_{K_v}, weakly contained in $\rho^0|_{G_{K_v}}$, such that π_v is weakly contained in $\pi'_v \otimes \sigma$ and π'_v is weakly contained in $\pi_v \otimes \sigma'$. Therefore, π is weakly contained in $\pi \otimes \sigma' \otimes \sigma$, and hence in $\pi \otimes (\sigma' \otimes \sigma)^{\otimes N}$ for any N in $\mathbf{N}$.

Now G_{K_v} has Kazhdan's Property (T). This implies that $1_{G_{K_v}}$ is not weakly contained in the restriction $\rho^0|_{G_{K_v}}$ of ρ^0 to G_{K_v}. Indeed otherwise, $1_{G_{K_v}}$ would be contained in $\rho^0|_{G_{K_v}}$, and the action of G_{K_v} on G_A/G_K would not be ergodic, contradicting Lemma 2. So, σ and σ' are different from the trivial representation of G_{K_v}.

We now proceed as in the proof of Theorem A. The matrix coefficients of $\sigma' \otimes \sigma$ belong to $L^r(G_{K_v})$ for some finite r. Choose N so large that $(\sigma' \otimes \sigma)^{\otimes N}$ are weakly contained in $\lambda_{G_{K_v}}$. Hence, π_v is weakly contained in $\lambda_{G_{K_v}}$. This is a contradiction. □

Note. The acute reader will have noticed that we do not require the full force of Kazhdan's Property (T), but rather that $L^2(G_A/G_K)_0|_{G_{K_v}}$ does not weakly contain the trivial representation of G_{K_v} for all valuations v. Thus in particular Theorem D also holds for the case where $G = SL_2$ and $K = \mathbf{Q}$, by results of Selberg, which imply the isolation in $\widehat{G_{\mathbf{R}}}$, and estimates for modular forms which imply the isolation in $\widehat{G_{\mathbf{Q}_p}}$. There has been much interest in this question (see, e.g., Burger, Li and Sarnak [BLS] and Burger and Sarnak [BS] for the real case), but, at the time of writing, a proof that this separation property (which is also of interest in other problems) always holds was not known to the authors.

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