Numerical Analysis Seminar

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Quadrature for Boundary Element Methods

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1. Gauss Quadrature

Reference: Davis and Rabinowitz, *Methods of Numerical Integration*, 1967.

Gauss–Legendre quadrature rule for the interval (a, b):

$$G_{n,(a,b)}f = \sum_{j=1}^{n} w_j f(x_j).$$

Error functional:

$$E_{n,(a,b)}f = \int_a^b f(x)\,dx - G_{n,(a,b)}f.$$

Notation: $\mathbb{P}_m$ denotes the set of all polynomials with degree at most m.

Rule $G_{n,(a,b)}$ characterized by property that

$$E_{n,(a,b)}f = 0 \quad \text{for } f \in \mathbb{P}_{2n-1}.$$

1.1 Theorem. If f is C^{2n} on $[a, b]$ then

$$E_{n,(a,b)}f = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi)$$

for some ξ with $a < \xi < b$.

For $\rho > 1$, let $\mathcal{E}_{\rho,(a,b)}$ denote the closed ellipse in the complex plane with foci at a and b, and with

$$\text{major axis} = (b-a)\frac{\rho^2+1}{2\rho},$$

$$\text{minor axis} = (b-a)\frac{\rho^2-1}{2\rho}.$$

1.2 Theorem. If f has an analytic continuation to $\mathcal{E}_{\rho,(a,b)}$ then

$$|E_{n,(a,b)}f| \leq C\rho^{-2n} \max_{z \in \partial\mathcal{E}_{\rho,(a,b)}} |f(z)|.$$

1.3 Example. For the integral

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0{\cdot}785398163397\ldots,$$

we can take $1 < \rho < 1 + \sqrt{2} = 2.4142\ldots$, but in fact:

Gauss-Legendre		
points	error	ρ
1	1.46E-02	
2	1.49E-03	3.13
3	-1.31E-04	3.37
4	4.81E-06	5.22
5	-3.43E-09	37.48
6	-1.06E-08	0.57
7	6.66E-10	3.99
8	-1.77E-11	6.14
9	-3.87E-13	6.75
10	6.34E-14	2.47
11	-2.89E-15	4.69
12	1.11E-16	5.10

Romberg	
points	error
2	-3.54E-02
3	-2.06E-03
5	1.31E-04
9	-1.72E-06
17	2.92E-09
33	1.21E-11
65	-1.77E-14

2. Product Rules

Consider f continuous on the closed d-dimensional interval

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d].$$

Given n_1, n_2, . . . , n_d write

$$G_i = G_{n_i,[a_i,b_i]} \text{ acting on } ith \text{ variable } x_i,$$
$$J_i f = \int_{a_i}^{b_i} f \, dx_i.$$

Product Gauss rule:

$$Gf = G_1 \cdots G_d f = \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} w_{1,j_1} \cdots w_{d,j_d} f(x_{1,j_1}, \ldots, x_{d,j_d})$$

approximates

$$Jf = J_1 \cdots J_d f = \int_R f(x) \, dx.$$

Error functional

$$E_i = J_i - G_i = E_{n_i,[a_i,b_i]} \text{ acting on } i\text{th variable } x_i.$$

If $d = 2$ then

$$J - Q = J_1 J_2 - Q_1 Q_2 = J_1(J_2 - Q_2) + (J_1 - Q_1)Q_2 = J_1 E_2 + E_2 Q_2.$$

If $d = 3$ then

$$J - Q = J_1 J_2 E_3 + J_1 E_2 Q_3 + E_1 Q_2 Q_3.$$

Putting $R_i^{\mathsf{C}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$ and

$$|R_i^{\mathsf{C}}| = \prod_{k=1, k \neq i}^{d} (b_k - a_k),$$

we have

$$|(I - Q)f| \leq \sum_{i=1}^{d} |R_i^{\mathsf{C}}| \max_{R_i^{\mathsf{C}}} |E_i f|.$$

3. Duffy's Transformation

Michael G. Duffy: Quadrature over a pyramid or cube of integrands with a singularity at a vertex *SIAM J. Numer. Anal.*, 1982, 1260–1262.

Simple example arising in boundary element collocation method for the Dirichlet problem:

$$I = \iint_T \frac{g(y_1, y_2)}{\sqrt{(y_1)^2 + (y_2)^2}} \, dy_1 \, dy_2,$$

for smooth function g and triangle

$$T : \quad 0 < y_2 < y_1 < h.$$

Putting $y_2 = y_1 \eta$ gives

$$I = \iint_{(0,h)\times(0,1)} \frac{g(y_1, y_1 \eta)}{\sqrt{1 + \eta^2}} \, dy_1 \, d\eta.$$

Transformed integrand is *smooth* (but area of integration region is larger by factor $2/h$).

4. Galerkin BEM

S. A. Sauter and C. Schwab: Quadrature for hp-Galerkin BEM in $\mathbb{R}^3$, *Numer. Math.* 78 (1997), 211–258.

Consider a 4-dimensional (principal value) integral

$$I = \int_\sigma \left(\int_\tau K(X,Y)U(Y)\,ds_Y \right) V(X)\,ds_X$$

where

σ, τ	boundary elements,
ds	element of surface area,
U, V	shape functions.

We assume the kernel has the form

$$K(X,Y) = \frac{1}{|X-Y|^2} \sum_{\nu \geq 0} K_\nu(X, Y, |X-Y|, X-Y)$$

with $K_\nu(X, Y, t, Z)$ homogeneous of degree ν in Z, and

$$K_0(X, X, 0, -Z) = -K_0(X, X, 0, Z).$$

Example:

$$K(X,Y) = \frac{e^{ik|X-Y|}}{|X-Y|}, \qquad K_\nu(X,Y,t,Z) = \begin{cases} |Z|e^{ikt}, & \nu = 1 \\ 0, & \nu \neq 1. \end{cases}$$

Example: if $n_Y =$ unit normal at Y and

$$K(X,Y) = \frac{\partial}{\partial n_Y} \frac{1}{|X-Y|} = \frac{n_Y \cdot (X-Y)}{|X-Y|^3}$$

then

$$K_\nu(X,Y,t,Z) = \begin{cases} \dfrac{n_Y \cdot Z}{|Z|}, & \nu = 0, \\ 0, & \nu \neq 0. \end{cases}$$

For a conforming and shape-regular mesh, five cases occur:

- $\sigma = \tau$;
- σ and τ meet along a common edge;
- σ and τ meet at a common vertex;
- σ and τ are close but do not touch;
- σ and τ are well separated.

5. Case $\sigma = \tau$

Parametric representation $\phi : \widehat{\sigma} \to \sigma$ for (flat) quadrilateral $\widehat{\sigma}$ with vertices a_0, a_1, a_2, a_3. Write

$$X = \phi(\widehat{x}), \qquad Y = \phi(\widehat{y}).$$

Approx side lengths of $\widehat{\sigma}$:

$$h_i = h_{\sigma,i} \simeq |a_i - a_{i-1}|.$$

Rectangular reference element

$$0 < x_1 < h_1, \qquad 0 < x_2 < h_2.$$

Bilinear mapping $x \mapsto \widehat{x}$ given by

$$\widehat{x} = a_0 + \frac{x_1}{h_1}(a_1 - a_0) + \frac{x_2}{h_2}(a_3 - a_0) + \frac{x_1 x_2}{h_1 h_2}(a_0 - a_1 + a_2 - a_3).$$

Integral over $\sigma \times \sigma$ transforms to

$$I = \int f(x, y)\, dx\, dy$$

over

$$\begin{aligned} 0 < x_1 < h_1, &\quad 0 < x_2 < h_2, \\ 0 < y_1 < h_1, &\quad 0 < y_2 < h_2. \end{aligned}$$

where

$$f(x, y) = K(X, Y)U(Y)J(y)V(X)J(x) \quad \text{and} \quad J(x) = \left| \frac{\partial X}{\partial x_1} \times \frac{\partial X}{\partial x_2} \right|.$$

Change of variable $z = x - y$ fixes singularity at $z = 0$,

$$I = \int_S f(y + z, y)\, dy\, dz,$$

but integration region becomes more complicated,

$$S : \qquad \begin{cases} 0 < y_1 < h_1, \quad 0 < y_2 < h_2, \\ \qquad -y_1 < z_1 < h_1 - y_1, \\ \qquad -y_2 < z_2 < h_2 - y_2. \end{cases}$$

5.1 Theorem. There exist transformations

$$(w, \eta) \mapsto (x^{(j)}, y^{(j)}) \quad \text{for } j = 1,\ 2,$$

such that

$$I = \sum_{j=1}^{2} \int_R \Big[f(x^{(j)}, y^{(j)}) + f(y^{(j)}, x^{(j)}) \Big] (h_1 - w_1)(h_2 - w_2)\, dw\, d\eta,$$

where the integration region is

$$R: \quad \begin{cases} 0 < w_1 < h_1, & 0 < w_2 < h_2, \\ \quad 0 < \eta_1 < 1, & 0 < \eta_2 < 1. \end{cases}$$

Moreover,

$$\Big[f(x^{(j)}, y^{(j)}) + f(y^{(j)}, x^{(j)}) \Big] = O(|w|^{-1}),$$

for $j = 1,\ 2$.

Partial Proof: $S = S_1 \cup S_2 \cup S_3 \cup S_4$ where

$$S_1: \quad \begin{cases} 0 < z_1 < h_1, \quad 0 < z_2 < h_2, \\ 0 < y_1 < h_1 - z_1, \\ 0 < y_2 < h_2 - z_2, \end{cases}$$

$$S_2: \quad \begin{cases} -h_1 < z_1 < 0, \quad -h_2 < z_2 < 0, \\ -z_1 < y_1 < h_1, \\ -z_2 < y_2 < h_2, \end{cases}$$

$$S_3: \quad \begin{cases} 0 < z_1 < h_1, \quad -h_2 < z_2 < 0, \\ 0 < y_1 < h_1 - z_1, \\ -z_2 < y_2 < h_2, \end{cases}$$

$$S_4: \quad \begin{cases} -h_1 < z_1 < 0, \quad 0 < z_2 < h_2, \\ -z_1 < y_1 < h_1, \\ 0 < y_2 < h_2 - z_2. \end{cases}$$

For S_1,

$$0 < z_1 < h_1, \quad 0 < z_2 < h_2, \quad 0 < y_1 < h_1 - z_1, \quad 0 < y_2 < h_2 - z_2.$$

Let

$$w_1 = z_1, \quad w_2 = z_2, \quad w_3 = y_1, \quad w_4 = y_2,$$

and

$$w_3 = (h_1 - w_1)\eta_1, \quad w_4 = (h_2 - w_2)\eta_2.$$

Since

$$0 < y_i < h_i - z_i \iff 0 < w_{i+2} < h_i - w_i \iff 0 < \eta_i < 1,$$

the map $(w_1, w_2, \eta_1, \eta_2) \mapsto (z, y)$ takes $R \to S_1$, and

$$\begin{aligned}
x_1^{(1)} &:= x_1 = y_1 + z_1 = w_3 + w_1 = w_1 + (h_1 - w_1)\eta_1, \\
x_2^{(1)} &:= x_2 = y_2 + z_2 = w_4 + w_2 = w_2 + (h_2 - w_2)\eta_2, \\
y_1^{(1)} &:= y_1 = w_3 = (h_1 - w_1)\eta_1, \\
y_2^{(1)} &:= y_2 = w_4 = (h_2 - w_2)\eta_2.
\end{aligned}$$

For S_2,

$$-h_1 < z_1 < 0, \quad -h_2 < z_2 < 0, \quad -z_1 < y_1 < h_1, \quad -z_2 < y_2 < h_2.$$

Let

$$w_1 = -z_1, \quad w_2 = -z_2, \quad w_3 = y_1 + z_1, \quad w_4 = y_2 + z_2,$$

and

$$w_3 = (h_1 - w_1)\eta_1, \quad w_4 = (h_2 - w_2)\eta_2.$$

Since

$$-h_i < z_i < 0 \iff 0 < w_i < h_i,$$
$$-z_i < y_i < h_i \iff 0 < w_{i+2} < h - w_i \iff 0 < \eta_i < 1,$$

the map $(w_1, w_2, \eta_1, \eta_2) \mapsto (z, y)$ takes $R \to S_2$, and

$$x_1 = y_1 + z_1 = w_3 = (h_1 - w_1)\eta_1 = y_1^{(1)},$$
$$x_2 = y_2 + z_2 = w_4 = (h_2 - w_2)\eta_2 = y_2^{(1)},$$
$$y_1 = w_3 - z_1 = w_1 + (h_1 - w_1)\eta_1 = x_1^{(1)},$$
$$y_2 = w_4 - w_2 = w_2 + (h_2 - w_2)\eta_2 = x_2^{(1)}.$$

Thus,

$$\int_{S_1 \cup S_2} f(y+z, y)\, dy\, dz$$
$$= \int_R \Big[f(x^{(1)}, y^{(1)}) + f(y^{(1)}, x^{(1)}) \Big] (h_1 - w_1)(h_2 - w_2)\, dw\, d\eta.$$

Similar arguments give

$$\int_{S_3 \cup S_4} f(y+z, y)\, dy\, dz$$
$$= \int_R \Big[f(x^{(2)}, y^{(2)}) + f(y^{(2)}, x^{(2)}) \Big] (h_1 - w_1)(h_2 - w_2)\, dw\, d\eta,$$

where

$$\begin{aligned} x_1^{(2)} &= (h_1 - w_1)\eta_1, \\ x_2^{(2)} &= w_2 + (h_2 - w_2)\eta_2, \\ y_1^{(2)} &= w_1 + (h_1 - w_1)\eta_1, \\ y_2^{(2)} &= (h_2 - w_2)\eta_2. \end{aligned}$$

Finally, note that

$$x^{(1)} - y^{(1)} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{and} \quad x^{(2)} - y^{(2)} = \begin{bmatrix} -w_1 \\ -w_2 \end{bmatrix}$$

so $|x^{(j)} - y^{(j)}| = |w|$ and thus

$$f(x^{(j)}, y^{(j)}) + f(y^{(j)}, x^{(j)}) = O(|w|^{-1}),$$

because if we write $g(X, Y) = U(Y)J(y)V(X)J(x)$ then

$$\begin{aligned} f(x, y) + f(y, x) &= K(X, Y)g(X, Y) + K(Y, X)g(Y, X) \\ &= [K(X, Y) + K(Y, X)]g(X, Y) \\ &\qquad + K(Y, X)[g(Y, X) - g(X, Y)] \\ &= O(|X - Y|^{-1}). \end{aligned}$$

5.2 Theorem. If, for simplicity, $h = h_1 = h_2$, then

$$I = \sum_{j=1}^{4} \int_0^h dw \int_0^1 d\eta_1 \int_0^1 d\eta_2 \int_0^1 d\eta_3$$

$$w\Big[f(x^{(j)}, y^{(j)}) + f(y^{(j)}, x^{(j)})\Big](h-w)(h-w\eta_3),$$

where

$$x_1^{(1)} = w + (h-w)\eta_1, \qquad x_1^{(2)} = w\eta_3 + (h-w\eta_3)\eta_1,$$
$$x_2^{(1)} = w\eta_3 + (h-w\eta_3)\eta_2, \qquad x_2^{(2)} = w + (h-w)\eta_2,$$
$$y_1^{(1)} = (h-w)\eta_1, \qquad y_1^{(2)} = (h-w\eta_3)\eta_1,$$
$$y_2^{(1)} = (h-w\eta_3)\eta_2, \qquad y_2^{(2)} = (h-w)\eta_2,$$

$$\ldots \qquad \ldots$$

Partial Proof: Write $R = R_1 \cup R_2$ where

$$R_1 : \quad \begin{cases} 0 < w_1 < h, & 0 < w_2 < w_1, \\ 0 < \eta_1 < h, & 0 < \eta_2 < h, \end{cases}$$

$$R_2 : \quad \begin{cases} 0 < w_2 < h, & 0 < w_1 < w_2, \\ 0 < \eta_1 < h, & 0 < \eta_2 < h, \end{cases}$$

and apply Duffy's transformation to (w_1, w_2):

$$\text{for } R_1\text{: } w_1 = w, \quad w_2 = w\eta_3,$$

$$\text{for } R_2\text{: } w_2 = w, \quad w_1 = w\eta_3,$$

with $0 < \eta_3 < 1$.

Since

$$\left| x^{(j)} - y^{(j)} \right| = \sqrt{(w_1)^2 + (w_2)^2} = w\sqrt{1 + (\eta_3)^2}$$

it follows that

$$w\Big[f(x^{(j)}, y^{(j)}) + f(y^{(j)}, x^{(j)}) \Big]$$

is a smooth function of $(w, \eta_1, \eta_2, \eta_3)$.