Solutions in Hölder Spaces
of Singular Integral Equations
on Lipschitz Contours

William McLean

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Abstract. Some well known results for singular integral equations over smooth or piecewise smooth contours are generalized to the case of Lip-
schitz contours. The theory is developed using Hölder spaces and, in contrast to the $L_p$ theory, requires only elementary methods.

INTRODUCTION

Perhaps the best modern treatment of the theory of one-dimensional, singular integral equations is to be found in the monograph of Gohberg and Krupnik [GK]; essentially the same treatment is available in English in the early chapters of the recent book by Mihklin and Prößdorf [MP]. These authors discuss in great detail the solvability of singular integral equations in $L_p$ spaces (with weights), under the assumption that the curve of integration $\Gamma$ is piecewise Lyapunov. In fact, many of the results of Gohberg and Krupnik are now known to be valid under the much weaker assumption that $\Gamma$ is Lipschitz (see §1) because in 1982 Coifman, McIntosh and Meyer [CMM] proved, for this wider class of contours, that the Cauchy integral is a bounded linear operator on $L_2(\Gamma)$.

The aim of this paper is to develop the theory of singular integral equations on a Lipschitz contour, using Hölder spaces instead of $L_p$ spaces. Several difficulties are thereby avoided; in particular, it is quite easy to prove that the Cauchy integral defines a bounded linear operator. Of course, the case of smooth (or even piecewise Lyapunov) contours is covered in the classic text of Muskhelishvili [M]; see also Prößdorf [P, pp.97–107] for a brief discussion from a modern viewpoint. In some respects, the present work is intended to be expository, giving a concise but self-contained account of the topic.

The paper is organized as follows. In §1, Lipschitz contours are introduced, along with the concept of nontangential limits. The basic theorem on the boundedness of the Cauchy integral operator in Hölder spaces is proved in §2, while in §3 we prove the Sokhotski-Plemelj formulae and the Hölder continuity of the Cauchy integral up to the boundary. Next, in §4, the usual algebra of singular integral operators is defined,

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following [GK]. The solution structure of the singular integral equations is analyzed in §4 and §5. Regularization is used to establish the Fredholm/Noether property, and here some difficulties arise because it is desirable to avoid dealing with the duals of the Hölder spaces — in this respect, the Hölder theory is more complicated than the $L_p$ theory. A derivation of the index formula is included for the sake of completeness: our treatment is very similar to that of [GK] and [MP], but is much briefer because, by assuming the coefficients are Hölder continuous, we avoid a lengthy theoretical discussion of factorization. Finally, there is an appendix in which the Poincaré-Bertrand formula is proved — this relies on some rather technical estimates which generalize those of §2 and §3.

There are at least two directions in which our results might profitably be generalized. The first ought to be straightforward: to treat systems of one-dimensional singular integral equations by allowing matrix-valued coefficients. For this, one needs to factorize Hölder continuous, matrix-valued functions on Lipschitz contours. The second generalization would be to allow piecewise Hölder continuous coefficients, which in turn would make possible the treatment of singular integral equations on Lipschitz arcs. (Duduchava [D] has done this for piecewise Lyapunov contours, by introducing Hölder spaces with weights — a weight function is needed because of the singular behaviour of solutions at points where the coefficients are discontinuous.)

1. Lipschitz Contours

For notational convenience, the field of complex numbers $\mathbb{C}$ will be freely identified with the vector space $\mathbb{R}^2$. Thus, if $y \in \mathbb{C}$, then the real part of $y$ is denoted by $y_1 = \Re y$, the imaginary part of $y$ is denoted by $y_2 = \Im y$, and we write $y = y_1 + iy_2 = (y_1, y_2)$.

Suppose $\Gamma$ is a (closed) Jordan curve, then the complex plane decomposes into the disjoint union

$$\mathbb{C} = \Omega_+ \cup \Gamma \cup \Omega_-,$$

where $\Omega_+$ is a bounded, simply-connected open set, and $\Omega_-$ is an unbounded, connected open set. The components $\Omega_+$ and $\Omega_-$ are uniquely determined by $\Gamma$, and their boundaries and closures are given by

$$\partial \Omega_+ = \Gamma = \partial \Omega_-, \quad \overline{\Omega}_+ = \Omega_+ \cup \Gamma, \quad \overline{\Omega}_- = \Omega_- \cup \Gamma.$$

These topological facts are discussed in Hille [H, p.34]. (The theory which follows could be generalized to allow the case where $\Gamma$ consists
of several nonintersecting Jordan curves, as in [M, p.86]; see also [MP, pp.43-44].)

In the context of singular integral equations, a simple closed curve is usually called a contour, whereas a simple, open-ended curve is usually called an arc. Thus, if $\Omega_+$ is a Lipschitz domain, then we say that $\Gamma$ is a Lipschitz contour. The definition of a Lipschitz domain can be found in many texts on partial differential equations and function spaces, nevertheless, we repeat it here because the notation is needed later.

For $x \in \mathbb{C}$ and $\theta \in \mathbb{R}$, let $A_{x,\theta} : \mathbb{C} \to \mathbb{C}$ be the affine transformation which first translates by $-x$, and then rotates by $-\theta$, i.e.,

$$A_{x,\theta}(y) = (y - x)e^{-i\theta}, \quad y \in \mathbb{C}.$$  

Given $x \in \Gamma$, the triple $(a, \phi, \theta)$ is said to be a Lipschitz representation of $\Gamma$ at $x$ if $a > 0$, $\theta \in \mathbb{R}$, and $\phi : [-a, a] \to [-a, a]$ is a Lipschitz function satisfying $\phi(0) = 0$ and

$$A_{x,\theta}(\Gamma) \cap (-a, a)^2 = \{(y_1, y_2) : -a < y_1 < a \text{ and } y_2 = \phi(y_1)\}$$

$$A_{x,\theta}(\Omega_+) \cap (-a, a)^2 = \{(y_1, y_2) : -a < y_1 < a \text{ and } \phi(y_1) < y_2 < a\}$$

$$A_{x,\theta}(\Omega_-) \cap (-a, a)^2 = \{(y_1, y_2) : -a < y_1 < a \text{ and } -a < y_2 < \phi(y_1)\}.$$  

In other words, $A_{x,\theta}(\Gamma) \cap [-a, a]^2$ is the graph of the Lipschitz function $\phi$. The contour $\Gamma$ is said to be Lipschitz if a Lipschitz representation exists at every point of $\Gamma$. Note that a Lipschitz contour may have (infinitely many) corners, but may not possess cusps.

Henceforth, it is always assumed that $\Gamma$ is a Lipschitz contour, and that $\Gamma$ has a positive (i.e., counterclockwise) orientation.

The derivative of a Lipschitz function exists almost everywhere, and belongs to $L_{\infty}$, therefore $\Gamma$ is rectifiable. For points $x$ and $y$ lying on $\Gamma$, let $|x, y|$ denote the minimum of the lengths of the two sub-arcs of $\Gamma$ having $x$ and $y$ as their end points. Thus, if $\gamma : \mathbb{R} \to \mathbb{C}$ is any periodic arc-length parametrization of $\Gamma$, i.e., if

$$\Gamma = \{\gamma(s) : 0 \leq s \leq L\}, \quad \gamma(s + L) = \gamma(s),$$  

where $L$ is the length of $\Gamma$, then

$$|x, y| = \min\{|s - t| : s, t \in \mathbb{R} \text{ with } x = \gamma(s) \text{ and } y = \gamma(t)\}.$$  

It is not difficult to verify that the function $|\cdot, \cdot|$ is a metric on $\Gamma$; this metric induces the usual topology, because there exists a constant $c_0$ such that

$$|x - y| \leq |x, y| \leq c_0|x - y| \quad \text{for all } x, y \in \Gamma.$$
Indeed, the left hand inequality follows at once from the definition of arc-length as the supremum of the lengths of polygonal interpolants. The right hand inequality is called the chord-arc condition, and can be proved using the compactness of $\Gamma$, together with the fact that the derivative of a Lipschitz function is bounded. The inequalities (1.1) are crucial for the theory developed in the sequel.

The winding number of $\Gamma$ about any point in $\Omega_+$ is +1, and the winding number of $\Gamma$ about any point in $\Omega_-$ is 0, thus by the Cauchy Integral Theorem,

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(y)}{y-z} \, dy = \begin{cases} 
  f(z), & z \in \Omega_+ \\
  0, & z \in \Omega_- 
\end{cases},
\]

for any complex function $f$ which is holomorphic on an open set containing $\Omega_+$. In fact, it is enough to assume that $f$ is continuous on $\Omega_+$ and holomorphic on $\Omega_+$, because, by Mergelyan's Theorem [G, p.97], this implies that $f$ is the uniform limit on $\Omega_+$ of a sequence of polynomials with complex coefficients. Likewise, for the exterior region $\Omega_-$ the following result holds: if $f$ is continuous on $\Omega_-$ and holomorphic on $\Omega_-$, and if $f$ is bounded at infinity, then $f(\infty) = \lim_{z \to \infty} f(z)$ exists and

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(y)}{y-z} \, dy = \begin{cases} 
  f(\infty) - f(z), & z \in \Omega_- \\
  f(\infty), & z \in \Omega_+ 
\end{cases}.
\]

For any function $u \in L_1(\Gamma)$, the Cauchy integral of $u$ is the holomorphic function $\Phi u : C \setminus \Gamma \to C$ defined by

\[
\Phi u(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{u(y)}{y-z} \, dy, \quad z \notin \Gamma.
\]

It is obvious from (1.2) and (1.3) that, in general, $\Phi u$ is discontinuous across $\Gamma$. In the remainder of this section, we discuss the interior and exterior boundary values of the Cauchy integral.

For $x \in \Gamma$ and $0 < m < 1$, the nontangential approach regions $N_+(x,m)$ and $N_-(x,m)$ are defined by

\[
N_\pm(x,m) = \{ z \in \Omega_\pm : \text{dist}(z,\Gamma) > m|x-z| \},
\]

where $\text{dist}(z,\Gamma) = \inf\{ |y-z| : y \in \Gamma \}$ is the distance between $z$ and $\Gamma$; cf. [K, p.177]. Put

\[
m_\pm(x) = \sup\{ m : 0 < m < 1 \text{ and } x \in N_\pm(x,m) \},
\]
then it is not difficult to prove that there exists a number $m_0$ satisfying
\[ m_{\pm}(x) \geq m_0 > 0 \quad \text{for all } x \in \Gamma. \]

If $0 < m < m_{\pm}(x)$, then it makes sense to send $z \to x$ with $z \in \mathcal{N}_{\pm}(x, m)$. Thus, given a function $F : \mathbb{C} \setminus \Gamma \to \mathbb{C}$, we let
\[ F_{\pm}(x; m) = \lim_{\substack{z \to x \\text{in } \mathcal{N}_{\pm}(x, m)}} F(z), \quad x \in \Gamma, \; 0 < m < m_{\pm}(x) \]
whenever these limits exist. Notice that
\[ \mathcal{N}_{\pm}(x, m_2) \subseteq \mathcal{N}_{\pm}(x, m_1) \quad \text{for } 0 < m_1 < m_2 < 1, \]
so $F_{\pm}(x; m_1) = F_{\pm}(x, m_2)$ whenever $F_{\pm}(x; m_1)$ exists. Thus, if $F_{\pm}(x, m)$ exists for all $m$ sufficiently small, then there exist unique nontangential limits
\[ F_{\pm}(x) = F_{\pm}(x; m), \quad x \in \Gamma, \; 0 < m < m_{\pm}(x). \]

Of course, if the ordinary limits
\[ F_{\pm}(x; 0) = \lim_{\substack{z \to x \\text{in } \mathcal{N}_{\pm}}} F(z), \quad x \in \Gamma \]
ever exist, then $F_{\pm}(x) = F_{\pm}(x; 0)$, however it is necessary to allow for the possibility that in (1.5) the convergence is not uniform in $m$.

Returning to the Cauchy integral, the first problem in determining $\Phi_{\pm}u = (\Phi u)_{\pm}$ is to make sense of the integral in (1.4) when $z$ is a point lying on the contour $\Gamma$ — in this case, the singularity at $y = z$ is not integrable (except for special choices of $u$). The usual way of coping with this difficulty is to introduce the Cauchy principal value integral
\[ \int_{\Gamma} \frac{u(y)}{y - x} \, dy = \lim_{h \to 0^+} \int_{\Gamma \setminus \Gamma_h(x)} \frac{u(y)}{y - x} \, dy, \quad x \in \Gamma, \]
where $\Gamma_h(x) = \{ y \in \Gamma : |y - x| < h \}$ is the sub-arc of $\Gamma$ centred at $x$, and having length $2h$.

Let us define
\[ S_{pv}u(x) = \frac{1}{\pi i} \int_{\Gamma} \frac{u(y)}{y - x} \, dy, \quad x \in \Gamma. \]

If $1 < p < \infty$, then $S_{pv} : L_p(\Gamma) \to L_p(\Gamma)$ is a bounded linear operator. Moreover, the nontangential limits $\Phi_{\pm}u$ of the Cauchy integral
exist almost everywhere on $\Gamma$, and are given by the Plemelj-Sokhotski formulae

$$\Phi_{\pm} u = \frac{1}{2} (S_{pv} \pm I) u,$$

where $I$ denotes the identity operator. These well known facts which, for Lipschitz contours, are consequences of the Coifman-McIntosh-Meyer Theorem mentioned in the Introduction, form the foundation of the $L_p$ theory of singular integral equations. Our task now is to establish analogous results — using only elementary methods — for the Hölder theory.

2. The Operator $S$

Let $f : \Gamma \to C$, and suppose $0 < \alpha \leq 1$. If the number

$$[f]_{\alpha} = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in \Gamma \text{ and } x \neq y \right\}$$

is finite, then $f$ is said to be Hölder continuous with exponent $\alpha$; following Zygmund [Z], we denote the set of all such functions by $\Lambda^\alpha(\Gamma)$. Notice that

$$|f(x) - f(y)| \leq [f]_{\alpha}|x - y|^\alpha \quad \text{for all } x, y \in \Gamma,$$

so, in particular, $\Lambda^1(\Gamma)$ is the set of Lipschitz functions from $\Gamma$ to $C$. The norm

$$\|f\|_{(\alpha)} = \|f\|_\infty + [f]_{\alpha}, \quad 0 < \alpha \leq 1,$$

makes $\Lambda^\alpha(\Gamma)$ into a Banach space.

The Plemelj-Privalov Theorem states that if the contour $\Gamma$ is smooth, then $S_{pv}$ is a bounded linear operator on $\Lambda^\alpha(\Gamma)$ for $0 < \alpha < 1$. This is no longer true, however, if $\Gamma$ is permitted to have corners, as we will now see.

Suppose $\Gamma$ is piecewise smooth (e.g., a polygon) and let $\Delta(x)$ denote the jump in the tangent angle at the point $x \in \Gamma$. The 'interior angle' at $x$ is then $\pi - \Delta(x)$, and the 'exterior angle' is $\pi + \Delta(x)$; obviously, $\Delta(x) = 0$ if and only if a tangent to $\Gamma$ exists at $x$. Using the Cauchy Integral Theorem, it is easy to see that

$$S_{pv} 1(x) = 1 - \frac{\Delta(x)}{\pi}, \quad x \in \Gamma;$$

hence $S_{pv} 1$ is discontinuous at every corner point of $\Gamma$. 
The way out of this difficulty is to introduce a new type of ‘singular integral’, namely,

\[(2.1) \quad \frac{1}{\pi i} \int_{\Gamma} \frac{u(y)}{y-x} \, dy = u(x) + \frac{1}{\pi i} \int_{\Gamma} \frac{u(y) - u(x)}{y-x} \, dy, \quad x \in \Gamma,\]

and to define a new operator

\[(2.2) \quad Su(x) = \frac{1}{\pi i} \int_{\Gamma} \frac{u(y)}{y-x} \, dy, \quad x \in \Gamma.\]

Observe that, in contrast to $S_{pv1}$, the function $S1 = 1$ is continuous. Also, $S = S_{pv}$ when $\Gamma$ is smooth, since

\[(2.3) \quad Su(x) = \frac{\Delta(x)}{\pi} u(x) + S_{pv} u(x), \quad x \in \Gamma,\]

Notice however that (2.3) cannot serve as a definition of $S$, because $\Delta(x)$ does not make sense for a general Lipschitz contour.

We will now prove that

$S : \Lambda^{\alpha}(\Gamma) \to \Lambda^{\alpha}(\Gamma), \quad 0 < \alpha < 1,$

is a bounded linear operator. In all of the estimates in this paper, the generic constant $c$ is a positive number depending only on the contour $\Gamma$; any dependence on Hölder exponents, functions, etc., is shown explicitly. The arc-length measure is denoted by $|dy|$, so that

$$\left| \int_{\Gamma} f(y) \, dy \right| \leq \int_{\Gamma} |f(y)| \, |dy|$$

for every function $f \in L_1(\Gamma)$.

**Lemma 2.1.** For all $x \in \Gamma$ and $h > 0$,

$$\int_{\Gamma \setminus \Gamma_h(x)} |y-x|^{\alpha-1} \, |dy| \leq \frac{c h^{\alpha}}{\alpha}, \quad 0 < \alpha \leq 1$$

$$\int_{\Gamma \setminus \Gamma_h(x)} |y-x|^{\alpha-2} \, |dy| \leq \frac{c h^{\alpha-1}}{1-\alpha}, \quad 0 \leq \alpha < 1$$

$$\left| \int_{\Gamma \setminus \Gamma_h(x)} \frac{dy}{y-x} \right| \leq c.$$
PROOF: The chord-arc condition (1.1) implies
\[
\int_{\Gamma_h(x)} |y - x|^{\alpha - 1} \, dy \leq c \int_{\Gamma_h(x)} |y, x|^{\alpha - 1} \, dy \\
\leq c \int_0^h s^{\alpha - 1} \, ds = c \alpha^{-1} h^\alpha
\]
for \(0 < \alpha \leq 1\), and
\[
\int_{\Gamma \setminus \Gamma_h(x)} |y - x|^{\alpha - 2} \, dy \leq c \int_{\Gamma \setminus \Gamma_h(x)} |y, x|^{\alpha - 2} \, dy \\
\leq c \int_h^\infty s^{\alpha - 2} \, ds = c(1 - \alpha)^{-1} h^{\alpha - 1}
\]
for \(0 \leq \alpha < 1\).

If \(0 < h \leq L/2\), where \(L\) is the length of \(\Gamma\), then
\[
\left| \frac{dy}{y - x} \right| \leq c \int_h^{L/2} s^{-1} \, ds = c \log \left( \frac{L}{2h} \right),
\]
so, for the third inequality, it suffices to consider \(h\) sufficiently small. (If \(h > L/2\), then \(\Gamma \setminus \Gamma_h(x) = \emptyset\) and there is nothing to prove.) Let \(x^h_\pm\) be the two points on \(\Gamma\) satisfying \(|x^h_\pm, x| = h\), with \(x^-\) preceding \(x\), and \(x^+\) following \(x\), as \(\Gamma\) is traversed in the counterclockwise sense near \(x\).

The compactness of \(\Gamma\) implies there is a fixed \(\alpha > 0\) such that, for every \(x \in \Gamma\), there exists a Lipschitz representation \((a, \phi, \theta)\) of \(\Gamma\) at \(x\). Fix \(x\), then since
\[
\int_{\Gamma \setminus \Gamma_h(x)} \frac{dy}{y - x} = \int_{A_{x,\phi}[\Gamma \setminus \Gamma_h(x)]} \frac{dy}{y - 0},
\]
we may assume \(x = 0\) and \(\theta = 0\), and thereby simplify the notation.

For \(0 < h < a\), there is a rectifiable arc \(\Pi_h\) beginning at \(x^h_\pm\), finishing at \(x^h_\pm\), and lying wholly within \(\Omega_\pm \cap [-a, a]^2\). The point \(x = 0\) lies outside the (closed) Jordan curve \(\Pi_h \cup \Gamma \setminus \Gamma_h(x)\), therefore
\[
\int_{\Gamma \setminus \Gamma_h(x)} \frac{dy}{y - x} + \int_{\Pi_h} \frac{dy}{y - x} = 0.
\]

Choose the branch of the logarithm so that
\[-\pi/2 < \arg(y - x) < 3\pi/2 \quad \text{for all } y \in \Pi_h,\]
then

\[ \left| \int_{\Omega_h} \frac{dy}{y-x} \right| = \left| \left( \log(y-x) \right)_{y=x}^h \right| \]
\[ \leq \left| \log |x^h_+ - x| - \log |x^h_- - x| \right| \]
\[ + \left| \arg(x^h_+ - x) \right| + \left| \arg(x^h_- - x) \right| \]
\[ \leq \left| \log \frac{|x^h_+ - x|}{|x^h_- - x|} \right| + 3\pi. \]

This completes the proof, because the chord-arc condition implies that the ratio \( |x^h_+ - x|/|x^h_- - x| \) is bounded away from 0 and \( \infty \), uniformly in \( x \). \qed

**Theorem 2.2.** If \( 0 < \alpha < 1 \), then

\[ \|Su\|_{(\alpha)} \leq c \left( \frac{1}{\alpha} + \frac{1}{1-\alpha} \right) \|u\|_{(\alpha)} \]

for all \( u \in \Lambda^\alpha(\Gamma) \).

**Proof:** Define the function

\[ \psi(x) = \frac{1}{\pi i} \int_{\Gamma} \frac{u(y) - u(x)}{y-x} \, dy, \quad x \in \Gamma, \]

then \( Su = u + \psi \), so it suffices to estimate \( \|\psi\|_{(\alpha)} \). Firstly,

\[ |\psi(x)| \leq \frac{1}{\pi} \int_{\Gamma} |u\|_{\alpha} |y-x|^{\alpha-1} |dy| \leq \frac{c[u]_{\alpha}}{\alpha}, \]

therefore \( \|\psi\|_{\infty} \leq c\alpha^{-1}[u]_{\alpha} \). To estimate \( [\psi]_{\alpha} \), let \( x, z \in \Gamma \) and put \( h = |x,z| \), then

\[ \psi(x) - \psi(z) = \frac{1}{\pi i} (I_1 + I_2 + I_3), \]

where

\[ I_1 = \int_{\Gamma_{2h}(x)} \left\{ \frac{u(y) - u(x)}{y-x} - \frac{u(y) - u(z)}{y-z} \right\} \, dy \]
\[ I_2 = \int_{\Gamma \setminus \Gamma_{2h}(x)} \frac{u(z) - u(x)}{y-x} \, dy \]
\[ I_3 = \int_{\Gamma \setminus \Gamma_{2h}(x)} [u(y) - u(z)] \left\{ \frac{1}{y-x} - \frac{1}{y-z} \right\} \, dy. \]
If $y \in \Gamma_{2h}(x)$, then $|y, z| \leq |y, x| + |x, z| \leq 3h$, so $y \in \Gamma_{3h}(z)$ and hence

$$|I_1| \leq [u]_\alpha \left\{ \int_{\Gamma_{2h}(x)} |y - x|^\alpha |dy| + \int_{\Gamma_{2h}(z)} |y - z|^\alpha |dy| \right\} \leq \frac{c[u]_\alpha}{\alpha} h^\alpha.$$

Next,

$$|I_2| = |u(z) - u(x)| \int_{\Gamma \setminus \Gamma_{2h}(x)} \frac{dy}{y - x} \leq c[u]_\alpha |x - z|^\alpha.$$

If $y \in \Gamma \setminus \Gamma_{2h}(x)$, then $|y, z| \geq |y, x| - |x, z| \geq 2h - h = h$ and so $|y - x| \leq |y, x| \leq |y, z| + h \leq 2|y, z| \leq c|y - z|$. Hence, the integrand of $I_3$ can be estimated as follows:

$$\left| \frac{1}{y - x} - \frac{1}{y - z} \right| \leq \frac{|y - z|^\alpha |z - x|}{|y - x||y - z|} \leq \frac{c[u]_\alpha h}{|y - x||y - z|^{1-\alpha}} \leq \frac{c[u]_\alpha h}{|y - x|^{2-\alpha}}.$$

Therefore,

$$|I_3| \leq c[u]_\alpha h \int_{\Gamma \setminus \Gamma_{2h}(x)} |y - x|^\alpha |dy| \leq \frac{c[u]_\alpha}{1-\alpha} h^\alpha,$$

and so, combining the estimates for $I_1$, $I_2$ and $I_3$, we find

$$|\psi(x) - \psi(z)| \leq c \left( \frac{1}{\alpha} + \frac{1}{1-\alpha} \right) [u]_\alpha |x - z|^\alpha,$$

noting that the chord-arc condition implies $h \leq c|x - z|$.

3. The Plemelj-Sokhotski Formulae

Putting $f(z) = 1$ ($z \in C$) in the Cauchy integral formula (1.2), we see

$$\Phi 1(x) = \begin{cases} 1, & \text{if } z \in \Omega_+ \\ 0, & \text{if } z \in \Omega_- \end{cases},$$

therefore, if $u \in \Lambda^\alpha(\Gamma)$ and $x \in \Gamma$, then

$$\Phi u(z) = \begin{cases} u(x) + \Psi u(z, x), & z \in \Omega_+ \\ \Psi u(z, x), & z \in \Omega_- \end{cases} \quad (3.1)$$
where

$$\Psi u(z, x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{u(y) - u(x)}{y - z} \, dy, \quad z \in \Omega_+ \cup \Omega_- \cup \{x\}. $$

Hence, to find the nontangential limits $\Psi_\pm u(x)$, it suffices to determine the limiting values of $\Psi u(z, x)$ as $z \to x$ with $z \in \mathcal{N}_\pm(x, m)$. In fact,

$$\lim_{z \to x \atop z \in \mathcal{N}_\pm(x, m)} \Psi u(z, x) = \Psi u(x, x) = \frac{1}{2} [S u(x) - u(x)], \quad x \in \Gamma,$$

as the second part of the following lemma shows.

**Lemma 3.1.** Suppose $0 < \alpha < 1$ and $0 < m < 1$. If $u \in \Lambda^\alpha(\Gamma)$ and $x \in \Gamma$, then

$$|\Psi u(z, x) - \Psi u(w, x)| \leq c \frac{1}{m^2} \left(\frac{1}{\alpha} + \frac{1}{1 - \alpha}\right) [u]_\alpha |z - w|^{\alpha},$$

and

$$|\Psi u(z, x) - \Psi u(x, x)| \leq c \frac{1}{m} \left(\frac{1}{\alpha} + \frac{1}{1 - \alpha}\right) [u]_\alpha |z - x|^{\alpha},$$

for all $z, w \in \mathcal{N}_+(x, m) \cup \mathcal{N}_-(x, m)$.

**Proof:** Put $h = |z - w|$, and write

$$\Psi u(z, x) - \Psi u(w, x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{[u(y) - u(x)](z - w)}{(y - z)(y - w)} \, dy$$

$$= \frac{z - w}{2\pi i} (I_1 + I_2),$$

where

$$I_1 = \int_{\Gamma \setminus \Gamma_h(z)} \frac{u(y) - u(x)}{(y - z)(y - w)} \, dy, \quad I_2 = \int_{\Gamma \setminus \Gamma_h(z)} \frac{u(y) - u(x)}{(y - z)(y - w)} \, dy.$$ 

Assuming $z \in \mathcal{N}_+(x, m) \cup \mathcal{N}_-(x, m)$ and $y \in \Gamma$, we have

$$|y - z| \geq \text{dist}(z, \Gamma) > m|x - z|,$$

and so

$$\frac{1}{|y - z|} = \frac{1}{|y - x|} \left(1 + \frac{z - x}{y - z}\right) \leq \frac{1 + m^{-1}}{|y - x|}. $$
Similarly, the inequality
\[ \frac{1}{|y - w|} \leq \frac{1 + m^{-1}}{|y - x|} \]
holds for \( w \in \mathcal{N}_+(x, m) \cup \mathcal{N}_-(x, m) \) and \( y \in \Gamma \). Without loss of generality, we may assume \( |x - z| \geq |x - w| \), then
\[ |z - w| \leq |z - x| + |x - w| \leq 2|x - z|. \]
Hence, \( 1/|x - z| \leq 2/|z - w| \), from which it follows that
\[
|I_1| \leq \int_{\Gamma_h(z)} \frac{|u|_\alpha |y - x|^{\alpha} (1 + m^{-1})}{m |z - w|} \frac{1}{|y - x|} |dy|
\]
\[
\leq \frac{2(1 + m^{-1}) |u|_\alpha}{m |x - w|} \int_{\Gamma_h(z)} |y - x|^{-\alpha - 1} |dy|
\]
\[
\leq \frac{c |u|_\alpha}{m^2 \alpha} \frac{h^\alpha}{|z - w|}
\]
and
\[
|I_2| \leq \int_{\Gamma \setminus \Gamma_h(z)} |u|_\alpha |y - x|^{\alpha} \left( \frac{1 + m^{-1}}{|y - x|} \right)^2 |dy|
\]
\[
\leq (1 + m^{-1})^2 |u|_\alpha \int_{\Gamma \setminus \Gamma_h(z)} |y - x|^{-\alpha - 2} |dy|
\]
\[
\leq \frac{c |u|_\alpha}{m^2 (1 - \alpha)} h^{\alpha - 1}.
\]
Inserting these bounds in (3.5), we arrive at (3.3).

To prove (3.4), put \( h = |z - x| \) and write
\[
\Psi u(z, x) - \Psi u(x, x) = \frac{z - x}{2\pi i} (I_1 + I_2),
\]
where
\[
I_1 = \int_{\Gamma_h(x)} \frac{u(y) - u(x)}{(y - z)(y - x)} dy, \quad I_2 = \int_{\Gamma \setminus \Gamma_h(x)} \frac{u(y) - u(x)}{(y - z)(y - x)} dy.
\]
This time,
\[
|I_1| \leq \int_{\Gamma_h(x)} \frac{|u|_\alpha |y - x|^{\alpha - 1}}{m |x - z|} |dy| \leq \frac{|u|_\alpha}{m \alpha} \frac{h^\alpha}{|x - z|}
\]
and
\[
|I_2| \leq \int_{\Gamma \setminus \Gamma_h(x)} |u|_\alpha (1 + m^{-1}) |y - x|^{-\alpha - 2} |dy| \leq \frac{|u|_\alpha}{m(1 - \alpha)} h^{\alpha - 1},
\]
from which the result follows immediately. \( \blacksquare \)
THEOREM 3.2. Let \( u \in \Lambda^\alpha(\Gamma) \), where \( 0 < \alpha < 1 \). The Plemelj-Sokhotski formulae
\[
\Phi_{\pm} u = \frac{1}{2} (S \pm I) u
\]
hold, and the inequality
\[
(3.6) \quad |\Phi u(z) - \Phi_{\pm} u(x)| \leq c \left( \frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) \|u\|_{(\alpha)} |z - x|^{\alpha}
\]
is valid for \( z \in \Omega_\pm \) and \( x \in \Gamma \).

PROOF: The Plemelj-Sokhotski formulae follow at once from (3.1) and (3.2).

To prove (3.6), let \( z \in \Omega_\pm \) and choose \( y \in \Gamma \) such that \( |z - y| = \text{dist}(z, \Gamma) \), then
\[
\Phi u(z) - \Phi_{\pm} u(y) = \begin{cases} 
\Psi u(z, y) - \Psi u(y, y), & z \in \Omega_+
\Psi u(z, y) - \Psi u(y, y), & z \in \Omega_-
\end{cases}
\]
Since \( z \in \mathcal{N}_\pm(y, m) \) for every \( m < 1 \), it follows from (3.4) that
\[
(3.7) \quad |\Phi u(z) - \Phi_{\pm} u(y)| \leq c \left( \frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) [u]_{\alpha} |z - y|^{\alpha},
\]
while Theorem 2.2 implies
\[
(3.8) \quad |\Phi_{\pm} u(y) - \Phi_{\pm} u(x)| \leq \frac{1}{2} (|S u(y) - S u(x)| + |u(y) - u(x)|)
\leq c \left( \frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) \|u\|_{(\alpha)} |y - x|^{\alpha}.
\]
Notice \( |z - y| = \text{dist}(z, \Gamma) \leq |z - x| \) and \( |y - x| \leq |y - z| + |z - x| \leq 2|z - x| \), so (3.7) and (3.8) together imply the result.

At this point, it is convenient to extend \( \Phi_{\pm} u \) to \( \overline{\Omega}_\pm \) in the obvious way, by putting \( \Phi_{\pm} u = \Phi u \) on \( \Omega_\pm \). The final theorem for this section asserts that the function \( \Phi_{\pm} u \) is Hölder continuous on \( \overline{\Omega}_\pm \), whenever \( u \) is Hölder continuous on \( \Gamma \). This fact is, of course, well known if \( \Gamma \) is smooth, and the reader may like to compare our proof with that in Muskhelishvili [M, p.53–55]; the latter relies on the maximum modulus principle.
THEOREM 3.3. If $0 < \alpha < 1$ and $u \in \Lambda^\alpha(\Gamma)$, then

\begin{equation}
|\Phi_\pm u(z) - \Phi_\pm u(w)| \leq c \left( \frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) \|u\|_{(\alpha)} |z - w|^\alpha
\end{equation}

for all $z, w \in \Omega_\pm$.

PROOF: If at least one of the points $z$ and $w$ lies on $\Gamma$, then (3.9) follows at once from Theorems 2.2 and 3.2. Thus, we may assume that both $z$ and $w$ belong to the open set $\Omega_\pm$.

Choose $x, y \in \Gamma$ such that $|z - x| = \text{dist}(z, \Gamma)$ and $|w - y| = \text{dist}(w, \Gamma)$, then the formula (3.1) implies

$$
\Phi_\pm u(z) - \Phi_\pm u(w) = \Psi u(z, x) - \Psi u(w, x) = \Psi u(z, y) - \Psi u(w, y).
$$

Therefore, by Lemma 3.1,

$$
|\Phi_\pm u(z) - \Phi_\pm u(w)| \leq c \frac{1}{m^2} \left( \frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) |u|_{\alpha} |z - w|^\alpha
$$

if $w \in N_\pm(x, m)$ or $z \in \mathcal{N}_\pm(y, m)$.

This leaves the case when $w \in \Omega_\pm \setminus N_\pm(x, m)$ and $z \in \Omega_\pm \setminus N_\pm(y, m)$, i.e., when

\begin{equation}
|w - y| = \text{dist}(w, \Gamma) \leq m|x - w| \\
|z - x| = \text{dist}(z, \Gamma) \leq m|y - z|.
\end{equation}

By Theorems 2.2 and 3.2,

$$
|\Phi_\pm u(z) - \Phi_\pm u(w)| \leq |\Phi_\pm u(z) - \Phi_\pm u(x)| + |\Phi_\pm u(x) - \Phi_\pm u(y)| + |\Phi_\pm u(y) - \Phi_\pm u(w)|
$$

$$
\leq c \left( \frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) \|u\|_{(\alpha)} E
$$

where $E = |z - x|^\alpha + |x - y|^\alpha + |y - w|^\alpha$. Thus, to complete the proof, it suffices to show $E \leq c |z - w|^\alpha$ for $m \leq 1/4$ (with $c$ independent of $m$).

The inequalities (3.10) imply

$$
|w - y| \leq m|x - y| + m|y - w|, \quad |z - x| \leq m|y - x| + m|x - z|,
$$

therefore if $m < 1$, then

$$
|w - y| \leq \frac{m}{1 - m} |x - y|, \quad |z - x| \leq \frac{m}{1 - m} |y - x|.
$$
Next,
\[ |x - y| \leq |x - z| + |z - w| + |w - y| \leq |z - w| + \frac{2m}{1-m}|x - y|, \]
so, for \( m < 1/3 \),
\[ |x - y| \leq \frac{1-m}{1-3m}|z - w|, \]
and finally
\[ E \leq \left\{ 1 + 2 \left( \frac{m}{1-m} \right)^\alpha \right\} |x - y|^{\alpha} \]
\[ \leq \left\{ 1 + 2 \left( \frac{m}{1-m} \right)^\alpha \right\} \left( \frac{1-m}{1-3m} \right)^\alpha |z - w|^{\alpha} \leq c|z - w|^{\alpha} \]
for \( m \leq 1/4 \). \( \blacksquare \)

4. AN ALGEBRA OF SINGULAR INTEGRAL OPERATORS

A remarkable property of the operator \( S \) is that \( S^{-1} = S \), a fact we will now prove.

**Theorem 4.1.** If \( 0 < \alpha < 1 \) and \( u \in \Lambda_0(\Gamma) \), then \( S^2u = u \).

**Proof:** Let \( u \in \Lambda_0(\Gamma) \), then the Cauchy integral \( \Phi_+u \) is holomorphic on \( \Omega_+ \) and, by Theorem 3.3, is (Hölder) continuous on \( \overline{\Omega}_+ \). Hence,
\[ \Phi u(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi_+u(y)}{y - z} dy = \Phi(\Phi_+u)(z), \quad z \in \Omega_+, \]
and therefore, applying the Plemelj-Sokhotski formulae,
\[ \frac{1}{2}(S + I)u = \frac{1}{2}(S + I)\frac{1}{2}(S + I)u. \]

After some elementary algebra, this gives the result. \( \blacksquare \)

In the proof above, use was made of the strong form of the Cauchy Integral Theorem, which relies on a rather deep result from approximation theory: Mergelyan's Theorem (see §1). An alternative 'elementary' proof, based on the Poincaré-Bertrand formula, is given in Corollary A.7 of the Appendix.

Define the linear operators \( P \) and \( Q \) by
\[ Pu = \Phi_+u = \frac{1}{2}(I + S)u \]
\[ Qu = -\Phi_-u = \frac{1}{2}(I - S)u, \]

(4.1)
then obviously
\[ P + Q = I, \quad P - Q = S. \]

Moreover, it is easy to see using Theorem 4.1 that
\[ (4.2) \quad P^2 = P, \quad Q^2 = Q, \quad PQ = 0 = QP, \]

hence \( P \) and \( Q \) are complementary projections.

For any function \( a \in \Lambda^\alpha(\Gamma) \), we write
\[ (au)(x) = a(x)u(x), \quad x \in \Gamma, \]

then \( a : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma) \) is a bounded linear operator, because one can easily verify that \( \Lambda^\alpha(\Gamma) \) is a (commutative) Banach algebra, i.e.,
\[ (4.3) \quad \|au\|_{(\alpha)} \leq \|a\|_{(\alpha)}\|u\|_{(\alpha)}, \quad 0 < \alpha \leq 1. \]

Furthermore, since \( a1 = a \), the operator norm of \( a \) is the same as the norm of the function \( a \), i.e.,
\[ \|a\|_{\mathcal{L}(\Lambda^\alpha(\Gamma))} = \|a\|_{(\alpha)}, \quad 0 < \alpha \leq 1. \]

We will study the following class of operators.

**Definition 4.2.** For \( 0 < \alpha < 1 \), let \( \text{SIO}_\alpha(\Gamma) \) denote the set of all linear operators having the form
\[ A = aP + bQ + K = \frac{a + b}{2} I + \frac{a - b}{2} S + K, \]

where \( a, b \in \Lambda^\alpha(\Gamma) \), and where \( K : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma) \) is compact. The elements of \( \text{SIO}_\alpha(\Gamma) \) are called singular integral operators.

Our aim in the remainder of this section is to prove that \( \text{SIO}_\alpha(\Gamma) \) is an algebra, i.e., it is closed under composition.

Consider the commutator \([S, a] = Sa - aS\); a simple calculation shows
\[ [S, a]u(x) = \frac{1}{\pi i} \int_\Gamma \frac{a(y) - a(x)}{y - x} u(y) dy, \quad y \in \Gamma. \]

Thus, \([S, a]\) is an integral operator with a weakly singular kernel, a fact which turns out to have important consequences.
THEOREM 4.3. If $0 < \alpha \leq 1$ and $0 < \beta < 1$, then

$$
||[S, a]u||_\beta(\beta) \leq c \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{1 - \beta} \right) [a]_\beta ||u||_{(\alpha)}
$$

for all $a \in \Lambda^\beta(\Gamma)$ and $u \in \Lambda^\alpha(\Gamma)$.

PROOF: Put $f = [S, a]u$, and note first that

$$
|f(x)| \leq \frac{1}{\pi} \int_\Gamma [a]_\beta |y - x|^{\beta - 1} ||u||_\infty |dy| \leq \frac{c[a]_\beta ||u||_\infty}{\beta}
$$

for every $x \in \Gamma$. Let $x, z \in \Gamma$, and write

$$
f(x) - f(z) = \frac{1}{\pi i} (I_1 + I_2 + I_3),
$$

where

$$
I_1 = \int_{\Gamma_n(x)} \left( \frac{a(y) - a(x)}{y - x} - \frac{a(y) - a(z)}{y - z} \right) u(y) dy
$$

$$
I_2 = \int_{\Gamma \setminus \Gamma_n(x)} \frac{a(z) - a(x)}{y - x} u(y) dy
$$

$$
I_3 = \int_{\Gamma \setminus \Gamma_n(x)} [a(y) - a(z)] \left( \frac{1}{y - x} - \frac{1}{y - z} \right) u(y) dy.
$$

We choose $h = 2c_0 |x - z|$, where $c_0$ is the constant appearing in the chord-arc condition (1.1).

If $y \in \Gamma_n(x)$, then $|y, z| \leq |y, x| + |x, z| \leq h + c_0 |x - z| \leq ch$, and therefore $y \in \Gamma_{ch}(z)$. Hence, it follows that

$$
|I_1| \leq [a]_\beta ||u||_\infty \left\{ \int_{\Gamma_n(x)} |y - x|^{\beta - 1} |dy| + \int_{\Gamma_{ch}(x)} |y - z|^{\beta - 1} |dy| \right\}
$$

$$
\leq \frac{c[a]_\beta ||u||_\infty}{\beta} h^\beta.
$$

Next,

$$
I_2 = [a(z) - a(x)] \left\{ \int_{\Gamma \setminus \Gamma_n(x)} \frac{dy}{y - x} + \int_{\Gamma \setminus \Gamma_n(x)} \frac{u(y) - u(x)}{y - x} dy \right\}
$$

so

$$
|I_2| \leq [a]_\beta |z - x|^\beta \{ c||u||_\infty + c\alpha^{-1} [u]_\alpha \} \leq \frac{c[a]_\beta}{\alpha} ||u||_{(\alpha)} |x - z|^\beta.
$$
If \( y \in \Gamma \setminus \Gamma_h(x) \), then \( c_0 |y-x| \geq |y,x| \geq h = 2c_0|x-x| \), which implies 
\[ |y-x| \geq 2|x-z| \text{ and } |y-x| \geq |y-x|-|x-z| \geq |y-x|/2. \] Hence,

\[
|I_3| \leq \int_{\Gamma \setminus \Gamma_h(x)} [a]_\beta \frac{|y-z|^\beta |x-z|}{|y-x||y-z|} |u(y)||dy|
\leq c[a]_\beta \|u\|_\infty h \int_{\Gamma \setminus \Gamma_h(x)} |y-x|^{\beta-2} |dy|
\leq \frac{c[a]_\beta \|u\|_\infty}{1-\beta} h^\beta,
\]

and this, when combined with the estimates for \( I_1 \) and \( I_2 \), gives

\[
|f(x)-f(z)| \leq c \left( \frac{1}{\beta} + \frac{1}{\alpha} + \frac{1}{1-\beta} \right) [a]_\beta \|u\|_\infty |x-z|^{\beta},
\]

as required. \( \blacksquare \)

Recall that if \( 0 < \alpha < \beta \leq 1 \), then \( \Lambda^\beta(\Gamma) \subseteq \Lambda^\alpha(\Gamma) \) and the inclusion is compact; see [P, p.102].

**THEOREM 4.4.** Suppose \( 0 < \alpha \leq 1 \) and \( 0 < \beta < 1 \). If \( a \in \Lambda^\beta(\Gamma) \), then

\[
[S,a] : \Lambda^\alpha(\Gamma) \rightarrow \Lambda^\beta(\Gamma)
\]

is a compact linear operator.

**PROOF:** Let \( 0 < \epsilon < \alpha \), then the inclusion \( \Lambda^\alpha(\Gamma) \subseteq \Lambda^{\alpha-\epsilon}(\Gamma) \) is compact, and the linear operator \([S,a] : \Lambda^{\alpha-\epsilon}(\Gamma) \rightarrow \Lambda^\beta(\Gamma)\) is bounded. \( \blacksquare \)

The above result enables us to show that the composition of two singular integral operators is again a singular integral operator.

**THEOREM 4.5.** Suppose \( 0 < \alpha < 1 \). If \( A_j = a_j P + b_j Q + K_j \in \text{SIO}_\alpha(\Gamma) \) for \( j = 1 \) and \( 2 \), then \( A_1A_2 \in \text{SIO}_\alpha(\Gamma) \). In fact, there is a compact linear operator \( K : \Lambda^\alpha(\Gamma) \rightarrow \Lambda^\alpha(\Gamma) \) such that

\[
A_1A_2 = a_1 a_2 P + b_1 b_2 Q + K.
\]

**PROOF:** Since \([P,a] = \frac{1}{2}[S,a] = -[Q,a]\), it follows from Theorem 4.4 that \( P \) and \( Q \) commute modulo compact operators with every \( a \in \Lambda^\alpha(\Gamma) \). Hence, the result follows at once from the identities (4.2). In fact,

\[
K = a_1 [P,a_2]P + a_1[P,b_2]Q + b_1 [Q,a_2]P + b_1 [Q,b_2]Q \]

\[+(a_1 P + b_1 Q) K_2 + K_1 (a_2 P + b_2 Q + K_2),\]
as can be verified after an elementary but tedious calculation. ■

Thus, as mentioned earlier, $\text{SIO}_\alpha(\Gamma)$ is an algebra.

Singular integral operators usually arise in the form

$$Au(x) = h(x)u(x) + \frac{1}{\pi i} \int_{\Gamma} \frac{k(x, y)}{y - x} u(y) \, dy, \quad x \in \Gamma,$$

and in this case the coefficients are

$$a(x) = h(x) + k(x, x), \quad b(x) = h(x) - k(x, x),$$

with

$$Ku(x) = \frac{1}{\pi i} \int_{\Gamma} \frac{k(x, y) - k(x, x)}{y - x} u(y) \, dy.$$  

By adapting the proof of Theorem 4.2, it can be shown that this operator $K$ is compact on $\Lambda^\alpha(\Gamma)$ for $\alpha < \beta$ if $k \in \Lambda^{(\beta, \alpha)}(\Gamma \times \Gamma)$; see also Prößdorf [P, p.102]. (Refer to the Appendix for the definition of the space $\Lambda^{(\beta, \alpha)}(\Gamma \times \Gamma)$.)

5. REGULARIZATION

Before discussing singular integral operators, let us consider the general case when $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Denote the kernel and image of $A$ by

$$\ker A = \{ u \in \mathcal{X} : Au = 0 \},$$

$$\text{im} A = \{ f \in \mathcal{Y} : \text{there exists } u \in \mathcal{X} \text{ such that } f = Au \},$$

then the nullity $n(A)$ and defect $d(A)$ are the (possibly infinite) numbers

$$n(A) = \dim \ker A, \quad d(A) = \dim \mathcal{Y}/\overline{\text{im} A}.$$  

An operator $R \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ is said to be a (two-sided) regularizer of $A$, if

$$(5.1) \quad RA = I - K_1, \quad AR = I - K_2,$$

where $K_1 : \mathcal{X} \to \mathcal{X}$ and $K_2 : \mathcal{Y} \to \mathcal{Y}$ are compact linear operators; $R$ can be looked upon as an inverse for $A$ modulo compact operators.

If $\text{im} A$ is closed, and if $n(A)$ and $d(A)$ are both finite, then $A$ is said to be a Fredholm operator, although some authors prefer the term Noether operator. Using well known properties of Riesz-Schauder operators [R, p.103], the reader can easily verify that if $A$ possesses a regularizer, then $A$ is Fredholm. We will now prove the converse; cf. [MP, p.22].
Suppose $A$ is Fredholm, then there exist closed subspaces $\mathcal{X}_1 \subseteq \mathcal{X}$ and $\mathcal{Y}_1 \subseteq \mathcal{Y}$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \ker A$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \im A$. Let $K_1$ be the projection of $\mathcal{X}$ onto $\ker A$ parallel to $\mathcal{X}_1$, and let $K_2$ be the projection of $\mathcal{Y}$ onto $\mathcal{Y}_1$ parallel to $\im A$, then $K_1$ and $K_2$ are compact linear operators because their image spaces $\ker A$ and $\mathcal{Y}_1$ are finite dimensional. The operator $A_1 = A|_{\mathcal{X}_1} : \mathcal{X}_1 \to \im A$ is invertible, so we can define $A^{(-1)} = (A_1)^{-1}(I - K_2) \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. It is not difficult to verify that

$$A^{(-1)}A = I - K_1, \quad AA^{(-1)} = I - K_2,$$

which means $A^{(-1)}$ is a regularizer of $A$. Moreover, $A^{(-1)}$ is a generalized inverse of $A$, i.e.,

$$AA^{(-1)}A = A, \quad A^{(-1)}AA^{(-1)} = A^{(-1)}.$$

Hence, if $f \in \im A$, then a solution of the equation $Au = f$ is given by $u = A^{(-1)}f$, and every other solution differs from $A^{(-1)}f$ by an element of $\ker A$.

Denote the dual space of $\mathcal{X}$ by $\mathcal{X}'$, then for any $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the transpose $A' \in \mathcal{L}(\mathcal{Y}', \mathcal{X}')$ is defined by

$$(A'w)(u) = w(Au), \quad w \in \mathcal{Y}', u \in \mathcal{X}.$$ 

In fact, $A' \in \mathcal{L}(\mathcal{Y}', \mathcal{X}')$ and $\|A'\|_{\mathcal{L}(\mathcal{Y}', \mathcal{X}')} = \|A\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})}$. It follows at once from the definition of $A'$, that

$$f \in \im A \implies w(f) = 0 \quad \text{for all } w \in \ker A'.$$

When the converse is true, i.e., when

$$\im A = \{f \in \mathcal{Y} : w(f) = 0 \text{ for all } w \in \ker A'\},$$

we say that $A$ is normally solvable. A necessary and sufficient condition for normal solvability is that $\im A$ be closed [Y, p.205]; in particular, every Fredholm operator is normally solvable.

We return now to the special case of singular integral operators. In view of the results of §2 and §3, it shall be assumed henceforth that $0 < \alpha < 1$ — i.e., the case $\alpha = 1$ is ruled out — because $S$ is not a bounded linear operator on $\Lambda^1(\Gamma)$. Thus, we have

$$A = aP + bQ + K \in \text{SIO}_\alpha(\Gamma) \subseteq \mathcal{L}(\Lambda^\alpha(\Gamma)).$$
If the coefficients $a$ and $b$ satisfy
\[ a(x) \neq 0 \quad \text{and} \quad b(x) \neq 0 \quad \text{for all} \quad x \in \Gamma, \]
then $A$ is said to be elliptic, and we can define
\[ (5.2) \quad R = \frac{1}{a}P + \frac{1}{b}Q \in \text{SIO}_\alpha(\Gamma). \]

Theorem 4.5 implies $R$ is a regularizer of $A$, so the preceding discussion applies with $\mathcal{X} = \mathcal{Y} = \Lambda^\alpha(\Gamma)$.

The dual of $\Lambda^\alpha(\Gamma)$ is a rather complicated space: it can be identified with a certain set of finitely additive measures on $\Gamma$ (see [W]). For this reason, there may be difficulties in establishing the existence of a solution to $Au = f$ by applying normal solvability. Therefore, we are led to introduce another 'transpose' of $A$, acting on $\Lambda^\alpha(\Gamma)$ instead of on the dual space.

For brevity, write $\mathcal{X} = \Lambda^\alpha(\Gamma)$ and define the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{X} \times \mathcal{X}$ by
\[ \langle u, f \rangle = \int_{\Gamma} u(x)f(x)\,dx. \]

Note that $\langle \cdot, \cdot \rangle$ is not an inner product — e.g., $\langle u, u \rangle = 0$ whenever $u$ has an analytic extension to $\Omega_+$. However, the form $\langle \cdot, \cdot \rangle$ is non-degenerate, i.e., given any $u \in \mathcal{X}$, if $\langle u, f \rangle = 0$ for all $f \in \mathcal{X}$, then $u = 0$. (This follows from the fact that there exists a measurable function $h$ satisfying $dx = h(x)|dx|$ and $\|h\|_\infty = 1$.) Therefore, the map $J \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ defined by
\[ (Ju)(f) = \langle u, f \rangle, \quad u, f \in \mathcal{X}, \]
is one-one, and gives an imbedding of $\mathcal{X}$ in $\mathcal{X}'$.

We will say that $A^t \in \mathcal{L}(\mathcal{X})$ is a formal transpose of $A \in \mathcal{L}(\mathcal{X})$ if
\[ \langle A^tw, u \rangle = \langle w, Au \rangle, \quad \text{for all} \quad u, w \in \mathcal{X}. \]

The nondegeneracy of $\langle \cdot, \cdot \rangle$ implies that $A^t$ is unique, whenever it exists. The operators $A'$ and $A^t$ are related by
\[ (5.3) \quad JA^t = A'J : \mathcal{X} \to \mathcal{X}'. \]

As an example, consider an integral operator
\[ Ku(x) = \int_{\Gamma} k(x, y)u(y)\,dy, \quad x \in \Gamma. \]
If there exists a constant $M$ such that
\[ \int_{\Gamma} |k(x, y)| \, dy \leq M, \quad \int_{\Gamma} |k(y, x)| \, dy \leq M \]
and
\[ \int_{\Gamma} |k(x, y) - k(z, y)| \, dy \leq M|x - z|^\alpha \]
\[ \int_{\Gamma} |k(y, x) - k(y, z)| \, dy \leq M|x - z|^\alpha \]
for all $x, z \in \Gamma$, then $K \in \mathcal{L}(\mathcal{X})$ and the formal transpose $K^t \in \mathcal{L}(\mathcal{X})$ is given by
\[ K^t u(x) = \int_{\Gamma} k(y, x) u(y) \, dy, \quad x \in \Gamma. \]

The dominant part of a singular integral operator always possesses a formal transpose.

**Theorem 5.1.** If $A = aP + bQ \in \text{SIO}_\alpha(\Gamma)$, then
\[ A^t = QA + Pb = bP + aQ + [Q, a] + [P, b] \in \text{SIO}_\alpha(\Gamma). \]

**Proof:** Theorem A.1 from the Appendix implies
\[ \langle w, Su \rangle = \int_{\Gamma} w(x) \left\{ \frac{1}{\pi i} \left[ \int_{\Gamma}^* \frac{u(y)}{y - x} \, dy \right] \right\} dx \]
\[ = \int_{\Gamma} \left\{ \frac{-1}{\pi i} \int_{\Gamma}^* \frac{w(x)}{x - y} \, dx \right\} u(y) \, dy \]
\[ = \langle -Sw, u \rangle \]
for all $u, w \in \mathcal{X}$. Therefore, $S^t = -S$ and so $P^t = Q$ and $Q^t = P$.

Hence,
\[ \langle w, Au \rangle = \int_{\Gamma} [w(x)a(x)Pu(x) + w(x)b(x)Qu(x)] \, dx \]
\[ = \langle aw, Pu \rangle + \langle bw, Qu \rangle \]
\[ = \langle Q(aw) + P(bw), u \rangle \]
for all $w, u \in \mathcal{X}$. 

The final theorem of this section shows that for a large class of singular integral operators, the formal transpose can be used when applying normal solvability to establish the existence of solutions. Our proof is based on [P, p.107].
THEOREM 5.2. Let $A$ be an elliptic singular integral operator, so that there exists a regularizer $R$ of the form (5.2), and let $K_2$ be the associated compact operator satisfying (5.1). If

(i) $K_2^1 : \Lambda^a(\Gamma) \to \Lambda^a(\Gamma)$ exists and is compact;
(ii) $K_2$ can be extended to a compact linear operator on $L_2(\Gamma)$;
(iii) there exists an integer $N \geq 1$ such that $(K_2^1)^N$ and $(K_2^1)^N$ map $L_2(\Gamma) \to \Lambda^a(\Gamma)$;

then $\ker A' = J(\ker A')$, and so

$$\text{im } A = \{ f \in \Lambda^a(\Gamma) : \langle w, f \rangle = 0 \text{ for all } w \in \ker A' \}.$$

PROOF: The relation (5.3) implies $J(\ker A') = \ker A' \cap J(\mathcal{X})$, so it suffices to show that $\ker A' \subseteq J(\mathcal{X})$. Moreover, $(I - K_2^1) = (AR)' = R' A'$, hence $\ker A' \subseteq \ker (I - K_2^1)$, and it suffices to prove $\ker (I - K_2^1) \subseteq J(\mathcal{X})$.

Suppose for a contradiction that there exists $v \in \ker (I - K_2^1) \setminus J(\mathcal{X})$, then because $J(\ker (I - K_2^1)) = \ker (I - K_2^1) \cap J(\mathcal{X})$, it follows that $v \notin J(\ker (I - K_2^1))$, and so there exists a function $f \in \mathcal{X}$ satisfying

(5.4) $$\langle w, f \rangle = 0 \quad \text{for all } w \in \ker (I - K_2^1)$$

(5.5) $$v(f) = 1.$$

If $w \in L_2(\Gamma)$ and $(I - K_2^1)w = 0$, then $w = K_2^1w = (K_2^1)^Nw \in \mathcal{X}$, so the operator $I - K_2^1 : L_2(\Gamma) \to L_2(\Gamma)$ has the same kernel as the operator $I - K_2^1 : \mathcal{X} \to \mathcal{X}$. Hence, the condition (5.4) and the compactness of $K_2 : L_2(\Gamma) \to L_2(\Gamma)$ imply that there exists $u \in L_2(\Gamma)$ such that $f = (I - K_2)u$. The identity

$$I - (K_2)^N = \left[ \sum_{j=0}^{N-1} (K_2)^j \right] (I - K_2)$$

implies $u = (K_2)^Nu + \sum_{j=0}^{N-1} (K_2)^jf \in \mathcal{X}$, and so

$$v(f) = v((I - K_2)u) = [(I - K_2^1)v](u) = 0,$$

which contradicts (5.5). $\blacksquare$

6. THE INDEX FORMULA

The index of a Fredholm operator $A$ is the integer

$$\text{ind}(A) = n(A) - d(A).$$
Among the most important properties of the index are the fact that \( \text{ind}(A + K) = \text{ind}(A) \) whenever the operator \( K \) is compact or has a sufficiently small norm, and the fact that if \( A_1 : \mathcal{X} \to \mathcal{Y} \) and \( A_2 : \mathcal{Y} \to \mathcal{Z} \) are Fredholm, then so is their composition \( A_2A_1 : \mathcal{X} \to \mathcal{Z} \), and

\[
\text{ind}(A_2A_1) = \text{ind}(A_1) + \text{ind}(A_2).
\]

Furthermore, since \( n(A') = d(A) \) for any \( A \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \), and since \( d(A') = n(A) \) provided \( \text{im}(A) \) is closed, it follows that

\[
\text{ind}(A') = -\text{ind}(A).
\]

Obviously, \( \text{ind}(A) \) provides important information about the solution structure of the equation \( Au = f \). Fortunately there is a simple formula for the index of an elliptic singular integral operator. In order to derive this formula, we introduce the closed subspaces

\[
\Lambda^\alpha_+ (\Gamma) = \{ f \in \Lambda^\alpha(\Gamma) : f = Pf \},
\]

\[
\hat{\Lambda}^\alpha_+ (\Gamma) = \{ f \in \Lambda^\alpha(\Gamma) : f = Qf \};
\]

recall that we are assuming \( 0 < \alpha < 1 \). It was seen in (4.2) that \( P \) and \( Q \) are complementary projections, hence \( \Lambda^\alpha_+ (\Gamma) = \text{im} P \), \( \hat{\Lambda}^\alpha_+ (\Gamma) = \text{im} Q \), and

\[
\Lambda^\alpha(\Gamma) = \Lambda^\alpha_+ (\Gamma) \oplus \hat{\Lambda}^\alpha_+ (\Gamma).
\]

It is also useful to introduce the space

\[
\Lambda^\alpha_- (\Gamma) = \hat{\Lambda}^\alpha_+ (\Gamma) \oplus \text{span}\{1\};
\]

notice \( \Lambda^\alpha_+ (\Gamma) \cap \Lambda^\alpha_- (\Gamma) \) consists of the constant functions.

There are several interesting characterizations of \( \Lambda^\alpha_+ (\Gamma) \), cf. [MP, pp.63–65].

**Theorem 6.1.** If \( f \in \Lambda^\alpha(\Gamma) \), then the following assertions are equivalent:

(i) \( f \in \Lambda^\alpha_+ (\Gamma) \);

(ii) \( \Phi f \equiv 0 \) on \( \Omega_- \);

(iii) \( f = F_+ \) for some function \( F \in \Lambda^\alpha(\overline{\Omega}_+) \) which is holomorphic on \( \Omega_+ \);

(iv) there exists \( \mu \in \Omega_- \) such that

\[
\int_{\Gamma} \frac{f(y)}{(y - \mu)^k} \, dy = 0, \quad \text{for every integer } k \geq 1;
\]

(v) (6.1) holds for all \( \mu \in \Omega_- \);

(vi) \( \int_{\Gamma} y^k f(y) \, dy = 0 \) for every integer \( k \geq 0 \).
Moreover, the function $F$ in part (iii) is unique; in fact, $F = \Phi f$.

**Proof:** The equivalence of (i), (ii) and (iii) follows from the definition (4.1) of $P$ and $Q$, using the Cauchy integral formulae (1.2) and (1.3), together with Theorem 3.3.

If $|z - \mu| < |y - \mu|$, then

$$
\frac{1}{y - z} = \left[(y - \mu)\left(1 - \frac{z - \mu}{y - \mu}\right)\right]^{-1} = \sum_{t=0}^{\infty} \frac{(z - \mu)^t}{(y - \mu)^{t+1}},
$$

therefore, if $\mu \in \Omega_-$ and $|z - \mu| < \text{dist}(\mu, \Gamma)$, then

$$
\Phi f(z) = \sum_{t=0}^{\infty} a_t (z - \mu)^t, \quad a_t = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(y)}{(y - \mu)^{t+1}} dy
$$

is the Taylor expansion of $\Phi f$ about $\mu$. Hence, the condition (6.1) holds if and only if $\Phi f = 0$ on a neighbourhood of $\mu$; it follows that (ii), (iv) and (v) are equivalent, since $\Phi f$ is holomorphic on the connected open set $\Omega_-$.

Finally, if $|z| > |y|$, then

$$
\frac{1}{y - z} = -[z(1 - y/z)]^{-1} = - \sum_{k=0}^{\infty} y^k/z^{k+1},
$$

therefore if $|z| > \text{dist}(0, \Gamma)$, then

$$
\Phi f(z) = \sum_{k=0}^{\infty} a_k (1/z)^k, \quad a_k = \frac{-1}{2\pi i} \int_{\Gamma} y^k f(y) dy
$$

is the Laurent expansion of $\Phi f$ about $\infty$. Hence, (vi) holds if and only if $\Phi f = 0$ on a neighbourhood of $\infty$, and we conclude that (ii) is equivalent to (vi).

There are analogous characterizations of $\Lambda_0(\Gamma)$ and $\hat{\Lambda}_0(\Gamma)$; in particular, $f \in \Lambda_0(\Gamma)$ (respectively, $f \in \hat{\Lambda}_0(\Gamma)$) if and only if $f = F_-$ for some function $F \in \Lambda_0(\Omega_-)$ which is holomorphic on $\Omega_-$ and bounded at infinity (respectively, is holomorphic on $\Omega_-$ and satisfies $F(\infty) = 0$). Consequently, the following important result holds. (Recall that by (4.3), if multiplication of functions is defined pointwise, then $\Lambda_0(\Gamma)$ is a commutative Banach algebra.)
THEOREM 6.2. Each of the spaces $\Lambda_+^\alpha(\Gamma)$, $\Lambda_-^\alpha(\Gamma)$ and $\Lambda_+^\alpha(\Gamma)$ is a closed subalgebra of $\Lambda^\alpha(\Gamma)$.

COROLLARY 6.3. If $a_\pm \in \Lambda_+^\alpha(\Gamma)$, then

\[
Pa_+P = a_+P \quad Qa_+Q = a_+ \quad Qa_+P = 0 \\
Pa_-P = Pa_- \quad Qa_-Q = a_- \quad Pa_-Q = 0.
\]

PROOF: The identities $Pa_+P = a_+P$, $Qa_+Q = a_+$, $Qa_+P = 0$ and $Pa_-Q = 0$ follow at once from Theorem 6.2 together with (4.2), and then $Pa_-P = Pa_-(I - Q) = Pa_-$ and $Qa_+Q = Qa_+(I - P) = Qa_+$. \H

Suppose $a : \Gamma \to \mathbb{C}$ is a continuous function, then the set $a(\Gamma) = \{a(x) : x \in \Gamma\}$ is a closed, oriented, continuous curve in the complex plane. If $a(x) \neq 0$ for all $x \in \Gamma$, i.e., if $a(\Gamma)$ does not pass through the origin, then $a(\Gamma)$ has a winding number about 0, which we denote by

\[
W(a) = \frac{1}{2\pi i} \arg a(x)|_{x \in \Gamma}.
\]

If $b : \Gamma \to \mathbb{C}$ is another continuous function which never vanishes, then

\[
W(ab) = W(a) + W(b),
\]

and, since $W(1) = 0$, it follows that $W(1/a) = -W(a)$.

Now assume $a \in \Lambda^\alpha(\Gamma)$. The formula

\[
a(x) = a_+(x)(x - \mu)^n a_-(x), \quad x \in \Gamma,
\]

is said to be a factorization of $a$ if

(i) $n \in \mathbb{Z}$ and $\mu \in \Omega_+$;

(ii) $a_\pm \in \Lambda_+^\alpha(\Gamma)$ and $a_\pm(x) \neq 0$ for all $x \in \Gamma$;

(iii) $1/a_\pm \in \Lambda_+^\alpha(\Gamma)$.

Conditions (i) and (ii) imply $W(a_\pm) = 0$, hence

\[
W(a) = n,
\]

which means that the integer $n$ is uniquely determined by the function $a$.

Furthermore, if

\[
a(x) = \tilde{a}_+(x)(x - \tilde{\mu})^n \tilde{a}_-(x),
\]

is another factorization of $a$, then there is a constant $c$ such that

\[
\tilde{a}_+(x) = a_+(x), \quad \tilde{a}_-(x) = \frac{(x - \mu)^n}{c(x - \tilde{\mu})^n} a_-(x).
\]

This is easily proved by taking the holomorphic extensions of $a_\pm$ and $\tilde{a}_\pm$ to $\Omega_\pm$, and then applying Liouville's Theorem.
THEOREM 6.4. Let \( a \in \Lambda^\alpha(\Gamma) \). A factorization of \( a \) exists if and only if \( a(x) \neq 0 \) for all \( x \in \Gamma \).

PROOF: The necessity of the condition is obvious. To prove sufficiency, we use an argument from [GK, p.79]. Assume that \( a \) never vanishes, and let \( n = W(a) \). Choose any \( \mu \in \Omega_+ \), then there is a function \( f \in \Lambda^\alpha(\Gamma) \) such that

\[
a(x) = (x - \mu)^n e^{f(x)}, \quad x \in \Gamma;
\]

in fact, \( f(x) = \log [(x - \mu)^{-n} a(x)] \) for some branch of the logarithm. (Note that \( f \) is single-valued, because the winding number of the function \( x \mapsto (x - \mu)^{-n} a(x) \) is zero.) By Theorem 6.1, the functions

\[
a_+(x) = e^{Pf(x)}, \quad a_-(x) = e^{Qf(x)}
\]

have the required properties.

We will see shortly that a factorization can be used to reduce the problem of finding \( \text{ind}(A) \) for a general elliptic singular integral operator \( A = aP + bQ + K \) to the special case when \( a(x) = (x - \mu)^n \), \( b(x) = 1 \) and \( K = 0 \). Choose any \( \mu \in \Omega_+ \), and then define

\[
\omega(x) = (x - \mu), \quad x \in \Gamma
\]

\[
C_m = P + \omega^m Q, \quad m \in \mathbb{Z}.
\]

The following lemma will be used to find the nullity and defect of \( C_m \).

LEMMA 6.5. If \( B \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \) is a left inverse of \( A \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \), i.e., if \( BA = I \), then \( \text{im} \ A \) is closed and

\[
\mathbf{n}(A) = \mathbf{d}(B) = 0, \quad \mathbf{d}(A) = \mathbf{n}(B).
\]

PROOF: It is obvious that \( \ker A = \{0\} \), \( \text{im} \ A \) is closed, and \( \text{im} \ B = \mathcal{X} \). To prove that \( \mathbf{d}(A) = \mathbf{n}(B) \), observe

\[
(AB)^2 = AB, \quad \text{im}(AB) = \text{im} \ A, \quad \ker(AB) = \ker B.
\]

Hence, \( AB \) is a projection from \( \mathcal{Y} \) onto \( \text{im} \ A \) parallel to \( \ker B \), and \( \mathcal{Y} \) decomposes into the direct sum of closed subspaces

\[
(6.2) \quad \mathcal{Y} = \text{im} \ A \oplus \ker B.
\]

This implies that \( \mathcal{Y}/\text{im} \ A \) is isomorphic to \( \ker B \).
THEOREM 6.6. For all $m \in \mathbb{Z}$, the operator $C_m : \Lambda^a(\Gamma) \rightarrow \Lambda^a(\Gamma)$ is Fredholm. Furthermore,

(i) if $m \geq 0$, then $C_mC_{-m} = I$ and

$$n(C_m) = m, \quad d(C_m) = 0;$$

(ii) if $m \leq 0$, then $C_mC_{-m} = I$ and

$$n(C_m) = 0, \quad d(C_m) = -m.$$

PROOF: Suppose $m \geq 0$, then $\omega^{\pm m} \in \Lambda^a(\Gamma)$. By Corollary 6.3,

$$C_mC_{-m} = (P + \omega^m Q)(P + \omega^{-m} Q)$$

$$= P^2 + P\omega^{-m} Q + \omega^m QP + \omega^m Q\omega^{-m} Q$$

$$= P + 0 + 0 + Q = I,$$

and consequently $d(C_m) = 0$. To prove $n(C_m) = m$, let $u \in \ker C_m$, i.e., suppose

$$Pu + \omega^m Qu = 0. \tag{6.3}$$

The function $F(z) = -(z - \mu)^m \Phi u(z)$ is holomorphic on $\Omega_-$ and is $O(z^{m-1})$ as $z \rightarrow \infty$, hence there exists a polynomial $p_{m-1}$ of degree at most $m - 1$ such that $(p_{m-1} - F)(\infty) = 0$. Thus, $p_{m-1} - \omega^m Qu = (p_{m-1} - F)_- \in \Lambda^a(\Gamma)$, and so

$$p_{m-1} - \omega^m Qu = P(p_{m-1} - \omega^m Qu) = PF_- = 0. \tag{6.4}$$

The Taylor expansion of $p_{m-1}$ about $\mu$ is

$$p_{m-1} = \sum_{k=0}^{m-1} \eta_k \omega^k, \quad \eta_k = \frac{1}{k!} D^k p_{m-1}(\mu),$$

therefore, from (6.3) and (6.4),

$$u = Qu + Pu = Qu - \omega^m Qu = (\omega^{-m} - 1)\omega^m Qu$$

$$= (\omega^{-m} - 1)p_{m-1} = \sum_{k=0}^{m-1} \eta_k (\omega^{k+m} - \omega^k).$$

Conversely, if $u$ has this form, then (6.3) holds. Hence,

$$\ker C_m = \text{span}\{\omega^{k-m} - \omega^k : 0 \leq k \leq m - 1\} \tag{6.5}$$

and $n(C_m) = m$.

Now suppose $m \leq 0$, then $C_mC_{-m} = I$ because $-m \geq 0$. Applying Lemma 6.5 with $A = C_m$ and $B = C_{-m}$, we find $n(C_m) = 0$ and $d(C_m) = n(C_{-m}) = -m$. \[\square\]
Corollary 6.7. $\text{ind}(C_m) = m$ for all $m \in \mathbb{Z}$.

We are now ready to prove the main result for this section.

Theorem 6.8. If $A = aP + bQ + K \in \text{SIo}_a(\Gamma)$ is elliptic, then

$$\text{ind}(A) = W(b) - W(a),$$

where $A$ is considered as an operator on $\Lambda^a(\Gamma)$.

Proof: The compact operator $K$ has no effect on the index, so it suffices to consider the dominant part of $A$. By Theorem 6.4 there exists a factorization

$$(6.6) \quad a^{-1}b = r_+ \omega^m r_-,$$

where $m = W(a^{-1}b) = W(b) - W(a)$. Corollary 6.3 implies

$$aP + bQ = a(P + a^{-1}bQ) = a(P + r_+ \omega^m r_- Q)$$

$$= ar_+(r_+^{-1}P + \omega^m r_- Q)$$

$$= ar_+(P + \omega^m Q)(r_+^{-1}P + r_- Q)$$

and $[r_+^{-1}P + r_- Q]^{-1} = r_+ P + r_- Q$, therefore, since the pointwise multiplication operator $ar_+$ is obviously invertible,

$$\text{ind}(aP + bQ) = \text{ind}(P + \omega^m Q) = \text{ind}(C_m) = m,$$

as claimed. \(\blacksquare\)

To conclude, we consider in more detail the case $K = 0$, i.e., the case

$$(6.7) \quad A = aP + bQ = ar_+ C_m (r_+^{-1}P + r_- Q).$$

We define

$$A^{(-1)} = (r_+P + r_-^{-1}Q)C_m r_+^{-1}a^{-1},$$

then by Theorem 6.6,

$$AA^{(-1)} = I \quad \text{when } m = \text{ind}(A) \geq 0$$

$$A^{(-1)}A = I \quad \text{when } m = \text{ind}(A) \leq 0,$$

and therefore, for all values of $\text{ind}(A)$,

$$AA^{(-1)} = A, \quad A^{(-1)}AA^{(-1)} = A^{(-1)}.$$

Hence, $A^{(-1)}$ is a generalized inverse of $A$. (Obviously, $A^{(-1)} = A^{-1}$ if and only if $\text{ind}(A) = 0$.)
THEOREM 6.9. Suppose $A : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma)$ is an elliptic singular integral operator of the form (6.7), where $r_\pm \in \Lambda^\alpha_\pm(\Gamma)$ satisfy (6.6).

(i) If $m = \text{ind}(A) \geq 1$, then

$$\ker A = \text{span}\{ r_-^{-1}\omega^{k-m} - r_+\omega^k : 0 \leq k \leq m - 1 \}.$$  

(ii) If $m = \text{ind}(A) \leq -1$, then $\text{ind}(A^t) = -m \geq 1$ and

$$\ker A^t = \text{span}\{ a^{-1}r_+^{-1}\omega^k : 0 \leq k \leq |m| - 1 \} = \text{span}\{ b^{-1}r_-\omega^{k-|m|} : 0 \leq k \leq |m| - 1 \}.$$  

(iii) If $m = \text{ind}(A) \leq -1$ and $f \in \Lambda^\alpha(\Gamma)$, then the $|m|$ conditions

\begin{equation}
\int_\Gamma \frac{(y - \mu)^k f(y)}{a(y)r_+(y)} \, dy = 0, \quad 0 \leq k \leq |m| - 1,
\end{equation}

are necessary and sufficient for $f \in \text{im} A$.

PROOF: (i) Let $m \geq 1$, and notice that

$$A^{(-1)} A = (r_+ P + r_-^{-1} Q) C_{-m} C_m (r_-^{-1} P + r_+^{-1} Q),$$

therefore $Au = 0$ if and only if $C_m (r_+^{-1} P + r_-^{-1} Q) u = 0$. Recalling (6.5), it follows that

$$\ker A = \text{span}\{ (r_+ P + r_-^{-1} Q)(\omega^{k-m} - \omega^k) : 0 \leq k \leq m - 1 \}.$$  

This gives the result, because both $\omega^{k-m}$ and $\omega^k$ belong to $\hat{\Lambda}^\alpha_\pm(\Gamma)$ provided $0 \leq k \leq m - 1$.

(ii) Let $u \in \ker A$, then from Theorem 5.1 we see $P(bu) + Q(au) = 0$, hence $au \in \Lambda^\alpha_\pm(\Gamma)$ and $bu \in \hat{\Lambda}^\alpha_\pm(\Gamma)$. Notice that $b = a(a^{-1}b) = ar_+\omega^m r_-$, so $r_+ au = \omega^{m}|r_-^{-1} bu$ when $m \leq -1$. This leads us to introduce the function

$$F(z) = \begin{cases} 
\Phi(r_+ au)(z), & z \in \Omega_+ \\
-(z - \mu)^{ml} \Phi(r_-^{-1} bu)(z), & z \in \Omega_-.
\end{cases}$$

which satisfies $F_+ = P(r_+ au) = r_+ au$ and $F_- = \omega^{ml} Q(r_-^{-1} bu) = \omega^{ml} r_-^{-1} bu$, i.e., $F$ satisfies $F_+ = F_-$. Therefore, $F$ is an entire function, and is $O(z^{ml-1})$ as $z \to \infty$, which means $F$ is a polynomial of
degree at most $|m| - 1$. Summarizing: if $u \in \ker A^t$, then there are constants $\eta_0, \ldots, \eta_{|m|-1}$ such that

$$ r_+ au = \omega^{|m|} r_-^{-1} bu = \sum_{k=0}^{|m|-1} \eta_k \omega^k. $$

Conversely, if this relation holds, then $au \in \Lambda_+^{a}(\Gamma)$ and $bu \in \Lambda_-^{a}(\Gamma)$, implying $u \in \ker A^t$.

(iii) If one accepts that $A$ satisfies the hypotheses of Theorem 5.2, then the result follows at once using (ii). Alternatively, since $A^{(-1)} A = I$, it follows from (6.2) that every $f \in \Lambda^a(\Gamma)$ can be written as $f = Au + v$ for some $u \in \Lambda^a(\Gamma)$ and some $v \in \ker A^{(-1)}$. Making use of (6.5), one finds

$$ \ker A^{(-1)} = \operatorname{span}\{ ar_+ (\omega^{k-|m|} - \omega^k) : 0 \leq k \leq |m| - 1 \}, $$

therefore $v = \sum_{k=0}^{|m|-1} \eta_k ar_+ (\omega^{k-|m|} - \omega^k)$. Now observe that when $j$ satisfies $0 \leq j \leq |m| - 1$,

$$ \int_{\Gamma} \frac{(y - \mu)^j f(y)}{a(y)r_+(y)} \, dy = \langle a^{-1} r_-^{-1} \omega^j, Au \rangle $$

$$ + \sum_{k=0}^{|m|-1} \eta_k \int_{\Gamma} [(y - \mu)^j \omega^k - (y - \mu)^j k] \, dy $$

$$ + \frac{\eta_j}{\Gamma} \int_{\Gamma} \frac{dy}{(y - \mu)^{|m|-(j+k)}} $$

$$ = \langle A^t (a^{-1} r_-^{-1} \omega^j), u \rangle + \sum_{k=0}^{|m|-1} \eta_k \int_{\Gamma} \frac{dy}{(y - \mu)^{|m|-(j+k)}} $$

$$ = 0 + \sum_{k=0}^{|m|-1} 2\pi i \eta_k \delta_{k,|m|-j-1} $$

$$ = 2\pi i \eta_{|m|-j-1}, $$

so the conditions (6.9) hold if and only if $\eta_0 = \cdots = \eta_{|m|-1} = 0$, i.e., if and only if $f = Au$.

**APPENDIX: THE POINCARÉ-BERTRAND FORMULA**

Recall Fubini's Theorem, which asserts that if $u \in L_1(\Gamma \times \Gamma)$, then

$$ \int_{\Gamma} \left\{ \int_{\Gamma} u(x, y) \, dy \right\} \, dx = \int_{\Gamma \times \Gamma} u(x, y) \, dx \, dy $$

$$ = \int_{\Gamma} \left\{ \int_{\Gamma} u(x, y) \, dx \right\} \, dy. $$
In what follows, we shall discuss what happens if the integration with respect to \(x\) and/or \(y\) is singular. First, it is necessary to explain what is meant by Hölder continuity in several variables.

Given an \(n\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_n)\) of real numbers in the range \(0 < \alpha_j \leq 1\), we write

\[
|x|^\alpha = \sum_{j=1}^{n} |x_j|^{\alpha_j}, \quad x = (x_1, \ldots, x_n) \in \mathbb{C}^n.
\]

The number \(|x|^\alpha\) is related to \(|x| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}\), the unitary ‘length’ of \(x\), by

\[
c_1(M)|x|^{\alpha_{\text{max}}} \leq |x|^\alpha \leq c_2(M)|x|^{\alpha_{\text{min}}}, \quad |x| \leq M,
\]

where

\[
\alpha_{\text{max}} = \max\{\alpha_1, \ldots, \alpha_n\}, \quad \alpha_{\text{min}} = \min\{\alpha_1, \ldots, \alpha_n\}
\]

and

\[
c_1(M) = 1/M^{\alpha_{\text{max}} - \alpha_{\text{min}}}, \quad c_2(M) = n^{1-\alpha_{\text{min}}/2}M^{\alpha_{\text{max}} - \alpha_{\text{min}}}.
\]

This follows at once from the inequality

\[
(A.1) \quad \left(\sum_{k=1}^{N} |b_k|^\beta\right) \leq \sum_{k=1}^{N} |b_k|^\beta \leq N^{1-\beta} \left(\sum_{k=1}^{N} |b_k|^\beta\right),
\]

which is valid for \(b_k \in \mathbb{C}\) and \(0 < \beta \leq 1\). Moreover, using \((A.1)\) with \(N = 2\), \(\beta = \alpha_j\), \(b_1 = x_j\) and \(b_2 = y_j\), one finds that

\[
|x-y|^\alpha \leq |x|^\alpha + |y|^\alpha \quad \text{for all } x, y \in \mathbb{C}^n.
\]

Let \(\emptyset \neq \mathcal{G} \subseteq \mathbb{C}^n\) and \(f : \mathcal{G} \to \mathbb{C}\), then we define

\[
[f]_\alpha = \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} : x, y \in \mathcal{G} \text{ and } x \neq y \right\},
\]

so that

\[
|f(x) - f(y)| \leq [f]_\alpha |x-y|^\alpha \quad \text{for all } x, y \in \mathcal{G}.
\]

The set of all \(f\) such that \([f]_\alpha < \infty\) and \(\|f\|_\infty < \infty\) is denoted by \(\Lambda^\alpha(\mathcal{G})\), and we introduce the norm

\[
\|f\|(\alpha) = \|f\|_\infty + [f]_\alpha,
\]

then \(\Lambda^\alpha(\mathcal{G})\) is a Banach space.

For \(u \in \Lambda^\alpha(\Gamma \times \Gamma)\), we can define

\[
\frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y, x)}{y-x} \, dy = u(x, x) + \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y, x) - u(x, x)}{y-x} \, dy,
\]

which generalizes (2.1).
**Theorem A.1.** If \( u \in \Lambda^\alpha(\Gamma \times \Gamma) \), then

\[
\int_\Gamma \left\{ \frac{1}{\pi i} \int_\Gamma^* \frac{u(y,x)}{y-x} \, dy \right\} \, dx = \int_\Gamma \left\{ \frac{1}{\pi i} \int_\Gamma^* \frac{u(y,x)}{y-x} \, dx \right\} \, dy.
\]

**Proof:** Write \( u = u_1 + u_2 \), where \( u_1 \) and \( u_2 \) are the symmetric and antisymmetric parts of \( u \), respectively:

\[
u_1(y,x) = \frac{u(y,x) + u(x,y)}{2}, \quad u_2(y,x) = \frac{u(y,x) - u(x,y)}{2}.
\]

The function \( (x,y) \mapsto u_1(y,x)/(y-x) \) is antisymmetric, therefore by interchanging \( x \) and \( y \) it is easy to see that

\[
\int_\Gamma \left\{ \frac{1}{\pi i} \int_\Gamma^* \frac{u_1(y,x)}{y-x} \, dx \right\} \, dy = 0 = \int_\Gamma \left\{ \frac{1}{\pi i} \int_\Gamma^* \frac{u_1(y,x)}{y-x} \, dx \right\} \, dy.
\]

On the other hand, \(|u_2(y,x)| \leq [u]_\alpha(||y-x||^{\alpha_1} + |x-y|^{\alpha_2})/2\), so Fubini’s Theorem can be applied to the function \( (x,y) \mapsto u_2(y,x)/(y-x) \).

To investigate what happens when the inner and the outer integrals are both singular, we must first study the ‘singular integral with parameters’. Generalizing (2.2), let

\[
Su(x,t) = \frac{1}{\pi i} \int_\Gamma^* \frac{u(y,x,t)}{y-x} \, dy
\]

\[
= u(x,x,t) + \frac{1}{\pi i} \int_\Gamma \frac{u(y,x,t) - u(x,x,t)}{y-x} \, dy
\]

for \( x \in \Gamma \) and \( t \in T \subseteq C \).

**Theorem A.2.** Suppose \( 0 < \alpha_1 < \alpha_2 \leq 1 \) and \( 0 < \alpha_1 < \alpha_3 \leq 1 \). If \( u \in \Lambda^\alpha(\Gamma \times \Gamma \times T) \), then

\[
\|Su\|_{(\alpha_1, \alpha_1)} \leq c \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2 - \alpha_1} + \frac{1}{\alpha_3 - \alpha_1} \right) \|u\|_{(\alpha)}.
\]

**Proof:** We adapt the proof of Theorem 2.2, putting

\[
\psi(x,t) = \frac{1}{\pi i} \int_\Gamma \frac{u(y,x,t) - u(x,x,t)}{y-x} \, dy,
\]

so that \( Su(x,t) = u(x,x,t) + \psi(x,t) \). First observe that

\[
\|\psi\|_\infty \leq c[u]_\alpha \int_\Gamma |y-x|^{\alpha_1-1} \, |dy| \leq \frac{c[u]_\alpha}{\alpha_1},
\]
and then consider
\[ \psi(x, t) - \psi(x', t') = \frac{1}{\pi i} (I_1 + I_2 + I_3 + I_4 + I_5), \]
where, with \( h = \max\{|x, x'|, |t - t'|\} \),
\[
I_1 = \int_{\Gamma_{2h}(x)} \left\{ \frac{u(y, x, t) - u(x, x', t)}{y - x} - \frac{u(y, x', t') - u(x', x', t')}{y - x'} \right\} dy
\]
\[
I_2 = \int_{\Gamma \setminus \Gamma_{2h}(x)} \frac{u(y, x, t) - u(y, x', t)}{y - x} dy
\]
\[
I_3 = \int_{\Gamma \setminus \Gamma_{2h}(x)} \frac{u(y, x', t) - u(y, x', t')}{y - x} dy
\]
\[
I_4 = [u(x', x', t') - u(x, x, t)] \int_{\Gamma \setminus \Gamma_{2h}(x)} \frac{dy}{y - x}
\]
\[
I_5 = \int_{\Gamma \setminus \Gamma_{2h}(x)} [u(y, x', t') - u(x', x', t')] \left\{ \frac{1}{y - x} - \frac{1}{y - x'} \right\} dy.
\]
Noting that \(|x, x'| \leq h\), it follows as in the proof of Theorem 2.2, that
\[
|I_1| \leq \frac{c[u]_\alpha}{\alpha_1} h^{\alpha_1}, \quad |I_5| \leq \frac{c[u]_\alpha}{1 - \alpha_1} h^{\alpha_1},
\]
and we see from Lemma 2.1 that
\[
|I_4| \leq c[u]_\alpha (|x' - x|^{\alpha_1} + |x' - x|^{\alpha_2} + |t' - t|^{\alpha_3}).
\]
However, because
\[
(A.2) \quad \int_{\Gamma \setminus \Gamma_{2h}(x)} \frac{|dy|}{|y - x|} \leq c(1 + |\log h|),
\]
one finds that
\[
|I_2| \leq c[u]_\alpha (1 + |\log h|)(x - x')^{\alpha_2}
\]
\[
|I_3| \leq c[u]_\alpha (1 + |\log h|)|t - t'|^{\alpha_3},
\]
and hence
\[
|\psi(x, t) - \psi(x', t')| \\
\leq c\left\{ \left( \frac{1}{\alpha_1} + \frac{1}{1 - \alpha_1} \right) h^{\alpha_1} + (1 + |\log h|)(|x - x'|^{\alpha_2} + |t - t'|^{\alpha_3}) \right\}[u]_\alpha.
\]
For $0 < h < 1$, i.e., for $\log h \leq 0$, if $M = (\alpha_2 - \alpha_1)\log h$, then
\[
|\log h|^{\alpha_2} = |\log h|^{\alpha_2 - \alpha_1} h^{\alpha_1} = |\log h| e^{-(\alpha_2 - \alpha_1)\log h} h^{\alpha_1} = M e^{-M} \frac{h^{\alpha_1}}{\alpha_2 - \alpha_1}.
\]
Therefore, since $\sup_{M > 0} M e^{-M} < \infty$, we have
\[(A.3) \quad |\log h|^{\alpha_2} \leq ch^{\alpha_1}/(\alpha_2 - \alpha_1),
\]
and likewise $|\log h|^{\alpha_3} \leq ch^{\alpha_1}/(\alpha_3 - \alpha_1)$. Thus,
\[
|\psi(x, t) - \psi(x', t')| \leq c \left\{ \frac{1}{\alpha_1} + \frac{1}{\alpha_2 - \alpha_1} + \frac{1}{\alpha_3 - \alpha_1} \right\} [u]_\alpha h^{\alpha_1},
\]
which implies the result, because $h^{\alpha_1} \leq c_0 |x - x'|^{\alpha_1} + |t - t'|^{\alpha_1}$. 

Next, we generalize the Cauchy integral (1.4), by defining
\[
\Phi u_{\pm}(w, t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{u_{\pm}(y, w, t)}{y - w} \, dy, \quad w \in \Omega_{\pm}, \ t \in T.
\]
Here, $u_{\pm} : \Gamma \times \Omega_{\pm} \times T \to \mathbb{C}$ and, for $x \in \Gamma$ and $t \in T$, we have
\[(A.4) \quad \Phi u_{\pm}(w, t) = \begin{cases} u_+(x, w, t) + \Psi u_+(x, w, t), & w \in \Omega_+ \vspace{0.5cm} \\
\Psi u_-(x, w, t), & w \in \Omega_-
\end{cases},
\]
where
\[(A.5) \quad \Psi u_{\pm}(x, w, t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{u_{\pm}(y, w, t) - u_{\pm}(x, w, t)}{y - w} \, dy,
\]
cf. (3.1). Let us consider the nontangential limits as $w \to x$.

**LEMMA A.3.** For $h > 0$, $0 < m < 1$, $x \in \Gamma$ and $w \in \mathcal{N}_{\pm}(x, m)$,
\[
\left| \int_{\Gamma \setminus \Gamma_h(x)} \frac{dy}{y - x} \right| \leq c[1 + \log(m^{-1})].
\]

**PROOF:** Let $x^h_{\pm}$ be as in the proof of Lemma 2.1, then
\[(A.6) \quad \left| \int_{\Gamma \setminus \Gamma_h(x)} \frac{dy}{y - w} \right| \leq c + \left| \log \frac{|x^h_{\pm} - w|}{|x^h_{\pm} - w|} \right|,
\]
and let $c_0$ be the constant from the chord-arc condition (1.1), then
\[|x^h_+ - x| \leq c_0 |x^h_- - x|,\]
because $|x^h_-, x| = h$. Now suppose $w \in \mathcal{N}_\pm(x, m)$, then
\[|x^h_+ - w| \geq \text{dist}(w, \Gamma) > m|x - w|.
In the case $h > 2c_0|x - w|$, one has
\[2c_0|x^h_+ - w| \geq 2c_0|x^h_+ - x| - 2c_0|x - w| > 2h - h = h\]
and
\[2c_0|x^h_- - w| \leq 2c_0|x^h_- - x| + 2c_0|x - w| < 2c_0h + h = (2c_0 + 1)h,
therefore $|x^h_- - w| \leq (2c_0 + 1)|x^h_+ - w|$. On the other hand, when $h \leq 2c_0|x - w|$, one has
\[|x^h_- - w| \leq |x^h_- - x| + |x - w| \leq h + |x - w|
\[\leq (2c_0 + 1)|x - w| \leq (2c_0 + 1)m^{-1}|x^h_+ - w|.
Hence, for all $h > 0$,
\[
\frac{|x^h_- - w|}{|x^h_+ - w|} \leq cm^{-1}, \quad w \in \mathcal{N}_\pm(x, m).
Moreover, the roles of $x^h_+$ and $x^h_-$ are interchangeable in the preceding argument, so $|x^h_+ - w|/|x^h_- - w| \leq cm^{-1}$, and the result follows at once from (A.6) \]
We can now generalize Lemma 3.1.

**Lemma A.4.** Suppose $0 < \alpha_1 < \alpha_2 \leq 1$ and $0 < \alpha_1 < \alpha_3 \leq 1$. If $u \in \Lambda^\sigma(\Gamma \times \bar{\Omega}_\pm \times T)$, then
\[(A.7) \quad |\Psi u_\pm(x, w, t) - \Psi u_\pm(x, w', t')| \leq c \frac{1}{m^2} \left\{ \frac{1}{\alpha_1} + \frac{1}{\alpha_2 - \alpha_1} + \frac{1}{\alpha_3 - \alpha_1} \right\}[u_\pm]\sigma(\|w - w'|^{\alpha_1} + \|t - t'|^{\alpha_1})\]
and
\[(A.8) \quad |\Psi u_\pm(x, w, t) - \Psi u_\pm(x, x, t')| \leq c \frac{1}{m} \left\{ \frac{1}{\alpha_1} + \frac{1}{\alpha_2 - \alpha_1} + \frac{1}{\alpha_3 - \alpha_1} \right\}[u_\pm]\sigma(\|w - x|^{\alpha_1} + \|t - t'|^{\alpha_1}),\]
for all \( x \in \Gamma \), for all \( w, w' \in \mathcal{N}_\pm(x, m) \), and for all \( t, t' \in T \).

**Proof:** Put \( h = \max\{|w - w'|, |t - t'|\} \), then

\[
\Psi u_+ (x, w, t) - \Psi u_+ (x, w', t') = \frac{1}{2\pi i} (I_1 + I_2 + I_3 + I_4 + I_5)
\]

where

\[
I_1 = \int_{\Gamma \setminus \Gamma_h (x)} \frac{u_+ (y, w, t) - u_+ (x, w, t)}{y - w} \, dy
\]

\[
- \int_{\Gamma_h (x)} \frac{u_+ (y, w', t') - u_+ (x, w', t')}{y - w'} \, dy
\]

\[
I_2 = \int_{\Gamma \setminus \Gamma_h (x)} \frac{u_+ (y, w, t) - u_+ (y, w', t)}{y - w} \, dy
\]

\[
I_3 = \int_{\Gamma \setminus \Gamma_h (x)} \frac{u_+ (y, w', t') - u_+ (y, w', t')}{y - w} \, dy
\]

\[
I_4 = [u_+ (x, w', t') - u_+ (x, w, t)] \int_{\Gamma \setminus \Gamma_h (x)} \frac{dy}{y - w}
\]

\[
I_5 = \int_{\Gamma \setminus \Gamma_h (x)} [u_+ (y, w', t') - u_+ (x, w', t')] \left( \frac{1}{y - w} - \frac{1}{y - w'} \right) \, dy.
\]

As in the proof of Lemma 3.1,

\[
\frac{1}{|y - w|} \leq \frac{1 + m^{-1}}{|y - x|}, \quad \frac{1}{|y - w'|} \leq \frac{1 + m^{-1}}{|y - x|},
\]

hence

\[
|I_1| \leq \frac{c[u_+]_\alpha}{m} \int_{\Gamma_h (x)} |y - x|^{\alpha_1 - 1} |dy| \leq \frac{c[u_+]_\alpha}{m \alpha_1} h^{\alpha_1},
\]

and, recalling (A.2), one finds

\[
|I_2| \leq \frac{c[u_+]_\alpha}{m} (1 + |\log h|)|w - w'|^{\alpha_2}
\]

\[
|I_3| \leq \frac{c[u_+]_\alpha}{m} (1 + |\log h|)|t - t'|^{\alpha_3}.
\]

Lemma A.3 implies

\[
|I_4| \leq c[u_+]_\alpha [1 + \log(m^{-1})](|w' - w|^{\alpha_2} + |t' - t|^{\alpha_3}),
\]

\[
|I_5| \leq c[u_+]_\alpha [1 + \log(m^{-1})](|w' - w|^{\alpha_2} + |t' - t|^{\alpha_3}),
\]
and since $|w - w'| \leq h,$

$$
|I_6| \leq \int_{\tilde{\Gamma} \setminus \Gamma_\alpha(x)} \frac{|u_\pm|_\alpha|y - x|^\alpha |w - w'|}{|y - w||y - w'|} dy \\
\leq \frac{c|u_\pm|_\alpha}{m^2} |w - w'| \int_{\tilde{\Gamma} \setminus \Gamma_\alpha(x)} |y - x|^{\alpha - 2} dy \\
\leq \frac{c|u_\pm|_\alpha}{m^2(1 - \alpha)} |w - w'| h^{\alpha_1 - 1} \leq \frac{c|u_\pm|_\alpha}{m^2(1 - \alpha)} h^{\alpha_1}.
$$

The inequality (A.7) now follows by making use of (A.3).

The proof of the second inequality (A.8) is similar. \( \blacksquare \)

Let us write

$$
\Phi_\pm u_\pm(w, t) = \begin{cases} \\
\lim_{z \to x} \Phi u_\pm(z, t), & \text{if } w = x \in \Gamma \\
\Phi u_\pm(w, t), & \text{if } w \in \Omega_\pm,
\end{cases}
$$

then the Plemelj-Sokhotski formulae generalize as follows.

**Theorem A.5.** For $u_\pm \in \Lambda^\alpha(\Gamma \times \overline{\Omega}_\pm \times T),$

$$
\Phi_\pm u_\pm(x, t) = \frac{1}{2} \left[ \pm u_\pm(x, x, t) + S u_\pm(x, t) \right], \quad x \in \Gamma, t \in T.
$$

Furthermore, if $0 < \alpha_1 < \alpha_2 \leq 1$ and $0 < \alpha_1 < \alpha_3 \leq 1,$ then

$$
\|\Phi_\pm u_\pm\|_{(\alpha_1, \alpha_1)} \leq c \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2 - \alpha_1} + \frac{1}{\alpha_3 - \alpha_1} \right) \|u_\pm\|_{(\alpha_1)},
$$

and so $\Phi_\pm u_\pm \in \Lambda^{(\alpha_1, \alpha_1)}(\overline{\Omega}_\pm \times T).$

This theorem can be proved using the same methods as were used in the proofs of Theorems 3.2 and 3.3; the details are omitted.

We are now ready to prove the Poincaré-Bertrand formula.

**Theorem A.6.** If $u \in \Lambda^\alpha(\Gamma \times \Gamma),$ then

$$
(A.9) \quad \frac{1}{\pi} \int_{\Gamma} \int_{\Gamma} \frac{1}{x - z} \left\{ \frac{1}{\pi i} \int_{\Gamma} \frac{u(y, x)}{y - x} dy \right\} dx =
\quad
u(z, z) + \frac{1}{\pi i} \int_{\Gamma} \int_{\Gamma} \frac{1}{y - z} \left\{ \frac{1}{\pi i} \int_{\Gamma} \frac{u(y, x)}{x - z} dx - \frac{1}{\pi i} \int_{\Gamma} \frac{u(y, x)}{x - y} dx \right\} dy
$$

for all $z \in \Gamma.$

**Proof:** Define the functions

$$
\phi(x) = \frac{1}{\pi i} \int_{\Gamma} \frac{u(y, x)}{y - x} dy, \quad x \in \Gamma,
$$
and
\[ F(w) = \Phi \phi(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(x)}{x-w} \, dx, \quad w \in \mathbb{C} \setminus \Gamma, \]
then the Plemelj-Sokhotski formulae imply
\[ F_+(z) + F_-(z) = S \phi(z) = \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{1}{x-z} \left\{ \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y,x)}{y-x} \, dy \right\} \, dx \]
for all \( z \in \Gamma \). Thus, it suffices to show that \( F_+(z) + F_-(z) \) equals the right hand side of (A.9).

By Theorem A.1,
\[ F(w) = \frac{1}{2\pi i} \int_{\Gamma} \left\{ \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y,x)}{(x-w)(y-x)} \, dx \right\} \, dy, \quad w \in \mathbb{C} \setminus \Gamma, \]
and so, since
\[ \frac{1}{(x-w)(y-x)} = \frac{1}{y-w} \left\{ \frac{1}{x-w} - \frac{1}{x-y} \right\}, \]
we conclude that
\[ F(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\psi(y,w)}{y-w} \, dy, \quad w \in \mathbb{C} \setminus \Gamma, \]
where
\[ \psi(y,w) = \frac{1}{\pi i} \int_{\Gamma} \frac{u(y,x)}{x-w} \, dx - \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y,x)}{x-y} \, dx. \]

For \( z, w \in \Gamma \), let
\[ \psi_\pm(y,z) = \lim_{w \to z} \psi(y,w), \]
then
\[ \psi_\pm(y,z) = u_\pm(y,z) + \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y,x)}{x-z} \, dx - \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y,x)}{x-y} \, dx. \]

Theorem A.5 implies that \( \psi_\pm \) is Hölder continuous on \( \overline{\Omega}_\pm \times \Gamma \), and that
\[ F_\pm(z) = \frac{1}{2} \left\{ \pm \psi_\pm(z,z) + \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{\psi_\pm(y,z)}{y-z} \, dy \right\}, \quad z \in \Gamma, \]
from which follows
\[ F_+(z) + F_-(z) = \frac{\psi_+(z,z) - \psi_-(z,z)}{2} + \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{\psi_+(y,z) + \psi_-(y,z)}{2(y-z)} \, dy. \]
Substituting (A.10) into this formula, one obtains the right hand side of (A.9), as required.

Finally, here is the alternative proof of Theorem 4.1.
COROLLARY A.7. If \( v \in \Lambda^o(\Gamma) \), then \( S^2 v = v \).

PROOF: When \( u(y, x) = v(y) \) is independent of \( x \),

\[
\frac{1}{\pi i} \int_{\Gamma} \frac{u(y, x)}{x - z} \, dx - \frac{1}{\pi i} \int_{\Gamma} \frac{v(y, x)}{x - y} \, dx = v(y) - v(y) = 0,
\]

so the Poincaré-Bertrand formula implies \( S^2 v(z) = u(z, z) = v(z) \).

REFERENCES


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Department of Mathematics, University of Tasmania, Hobart 7001, Australia.