Math2130 Higher Mathematical Methods for Differential Equations

School of Maths and Stats, UNSW
Session 2, 2013

Version: July 26, 2013
Part I

Ordinary Differential Equations
First-order ODEs

Given \( f(x, t) \), seek \( x = x(t) \) satisfying

\[
\frac{dx}{dt} = f(x, t).
\]

General solution contains one arbitrary constant of integration.

Initial-value problem: given \( f(x, t) \) and \( x_0 \), find \( x(t) \) satisfying

\[
\frac{dx}{dt} = f(x, t) \quad \text{for } t > 0, \quad \text{with } x(0) = x_0.
\]

Example

Verify that \( x = \sec^2(t + \frac{1}{2}\pi) \) is the solution of

\[
\frac{dx}{dt} = 2x\sqrt{x - 1}, \quad x(0) = 1.
\]
Separable first-order ODEs

\[ \frac{dx}{dt} = f(x)g(t) \]

Solve by writing as

\[ \int \frac{dx}{f(x)} = \int g(t) \, dt. \]

Example

Solve

\[ \frac{dx}{dt} = 2t \sqrt{1 + x^2}, \quad x(0) = \frac{3}{4}. \]
Linear first-order ODEs

\[
\frac{dx}{dt} + a(t)x = f(t)
\]

Multiply both sides by the integrating factor

\[
I(t) = \exp\left(\int a(t) \, dt\right)
\]

and obtain

\[
\frac{d}{dt} \left[ I(t)a(t)x \right] = I(t)f(t).
\]

Example

Solve

\[
\frac{dx}{dt} \sin t + 2x \cos t = t.
\]
Clever substitutions

A change of variable can convert a difficult ODE into a simpler, standard form.

Example
Any ODE of the form

\[ \frac{dx}{dt} = f(x/t) \]

becomes separable if we introduce the new dependent variable \( v = x/t \).

Example
Find \( m \) such that the substitution \( x = v^m \) turns the Bernoulli equation

\[ \frac{dx}{dt} + a(t)x = f(t)x^n \]

into a linear ODE.
Homogeneous second-order linear ODEs

Given real constant coefficients \( a, b, c \) find \( u = u(x) \) satisfying

\[
au'' + bu' + cu = 0.
\]

General solution \( u_H \) now involves two arbitrary constants.

Characteristic polynomial

\[
a\lambda^2 + b\lambda + c = a(\lambda - \lambda_1)(\lambda - \lambda_2).
\]

Three cases:

- **Distinct real roots:** \( u_H = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \).
- **Equal roots:** \( u_H = C_1 e^{\lambda_1 x} + C_2 xe^{\lambda_1 x} \).
- **Complex conjugate roots:** \( u_H = e^{\alpha x} (C_1 \cos \omega x + C_2 \sin \omega x) \) where \( \lambda_1 = \alpha + i\omega \) and \( \lambda_2 = \alpha - i\omega \).
Inhomogeneous second-order linear ODEs

Given $a$, $b$, $c$ and $f(x)$ find $u = u(x)$ satisfying

$$au'' + bu' + cu = f(x).$$

General solution has the form

$$u(x) = u_H(x) + u_P(x)$$

where $u_H$ is the general solution of the homogeneous equation, and $u_P$ is any particular solution of the inhomogeneous equation.

Example

Solve

$$2u'' + 5u' + 3u = 5e^x, \quad u(0) = \frac{5}{2}, \quad u'(0) = -1.$$
First-order systems

Let $\mathbf{F} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and consider

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) \quad (1)$$

where $\mathbf{x} = \mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^T$. The vector ODE (1) is equivalent to the (coupled) system of $n$ scalar ODEs:

$$\frac{dx_1}{dt} = F_1(x_1, x_2, \ldots, x_n, t),$$
$$\frac{dx_2}{dt} = F_2(x_1, x_2, \ldots, x_n, t),$$
$$\vdots$$
$$\frac{dx_n}{dt} = F_n(x_1, x_2, \ldots, x_n, t).$$
Autonomous ODEs

We say that (1) is autonomous if $F$ is independent of $t$, so that

$$\frac{dx}{dt} = F(x). \quad (2)$$

Geometric interpretation: the curve $x(t)$ is always parallel to the vector field $F$.

By letting $y = [x, t]^T$ we can transform a non-autonomous system (1) in $\mathbb{R}^n$ to an autonomous system in $\mathbb{R}^{n+1}$:

$$\frac{dy}{dt} = \begin{bmatrix} F(y) \\ 1 \end{bmatrix}.$$
The butterfly effect

Example
The Lorenz equations

\[
\begin{align*}
\frac{dx}{dt} &= -\beta x + yz, \\
\frac{dy}{dt} &= -\sigma y + \sigma z, \\
\frac{dz}{dt} &= -xy + \rho y - z,
\end{align*}
\]

are autonomous and nonlinear. With the parameter values

\[\sigma = 10, \quad \rho = 28, \quad \beta = 8/3,\]

the system possesses a strange attractor.
Solution trajectory

Strange attractor of the Lorenz equations
Higher-order ODEs

Often, a first-order system arises as a reformulation of a single, higher-order ODE.

Example
The Van der Pol equation

\[ \frac{d^2x}{dt^2} - \mu(1 - x^2) \frac{dx}{dt} + x = 0 \]

is second-order (and nonlinear), and is equivalent to the first-order (and autonomous) system

\[ \begin{align*}
    \frac{dx}{dt} &= v, \\
    \frac{dv}{dt} &= \mu(1 - x^2)v - x.
\end{align*} \]
Phase portrait

Limit cycle of Van der Pol oscillator
Solution

Van der Pol oscillator with \( \mu = 0.75 \)
Initial-value problem (IVP)

Given a vector of initial data \( x_0 \), find \( x = x(t) \) satisfying

\[
\frac{dx}{dt} = F(x,t) \quad \text{for all } t, \quad \text{with } x(0) = x_0.
\] (4)

Theorem (Local existence and uniqueness)

Fix \( r > 0 \) and \( \tau > 0 \), and put

\[
S = \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_0| \leq r \text{ and } |t| \leq \tau \}.
\]

If \( F : S \to \mathbb{R}^n \) is continuous and

1. \( |F(x, t)| \leq M \) for \( (x, t) \in S \),
2. \( \partial F_i / \partial x_j \) exists and is bounded on \( S \) for \( i, j \in \{1, 2, \ldots, n\} \),

then the IVP (4) has a unique solution \( x(t) \) for \( |t| \leq \min(r/M, \tau) \).
Example of non-uniqueness

The initial value problem

\[ \frac{dx}{dt} = 3x^{2/3}, \quad x(0) = 0, \]

has infinitely many solutions, namely, for each \( a > 0 \),

\[ x(t) = \begin{cases} 
0 & \text{for } 0 \leq t \leq a, \\
(t-a)^3 & \text{for } t > a.
\end{cases} \]

In this case \( f(x) = 3x^{2/3} \) so \( f'(x) = 2x^{-1/3} \) and \( f'(0) \) does not exist.
Example of local but not global existence

The initial value problem

\[
\frac{dx}{dt} = 1 + x^2, \quad x(0) = 1,
\]

has a unique solution for \(0 \leq t < \pi/4\),

\[
x(t) = \tan\left(t + \frac{\pi}{4}\right),
\]

but this solution blows up at \(t = \pi/4\).
Constructing approximate solutions

Euler's method: choose a small $\Delta t > 0$ and, for a given $x_0$, compute $x_1, x_2, \ldots$ using the formula

$$x_{k+1} = x_k + (\Delta t)F(x_k), \quad k \geq 0.$$  

We expect

$$x_k \approx x(t_k) \quad \text{where} \quad t_k = k\Delta t$$

because

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \approx \frac{dx}{dt} = F(x(t)),$$

and so

$$x(t + \Delta t) \approx x(t) + (\Delta t)F(x(t)).$$

If $F$ satisfies the assumptions of the Theorem, then (see Math2301/3101) there exist positive constants $C$ and $L$ such that

$$|x(t_k) - x_k| \leq C t_k e^{L t_k} \Delta t.$$
Linear systems of ODEs

The system of ODEs (1) is linear if $F$ has the form $F(x, t) = A(t)x + b(t)$ where $A(t) \in \mathbb{R}^{n \times n}$ and $b(t) \in \mathbb{R}^n$.

Theorem (Global existence and uniqueness)

If $A(t)$ and $b(t)$ are continuous for $0 \leq t \leq T$, then the linear IVP

$$\frac{dx}{dt} = A(t)x + b(t), \quad x(0) = x_0,$$

has a unique solution $x(t)$ for $0 \leq t \leq T$. 
Basis of solutions

The linear system of ODEs (5) is homogeneous if \( b(t) \equiv 0 \), so that 
\[
\frac{dx}{dt} = A(t)x.
\]
For \( 1 \leq j \leq n \), let \( x = E_j(t) \) denote the unique solution of the homogeneous, linear ODE 
\[
\frac{dx}{dt} = A(t)x, \quad x(0) = e_j,
\]
where \( e_j \) is the \( j \)th standard basis vector in \( \mathbb{R}^n \). Then the linear combination 
\[
x(t) = \sum_{i=1}^{n} x_{0i} E_i(t) = x_{01} E_1(t) + x_{02} E_2(t) + \cdots + x_{0n} E_n(t)
\]
is the unique solution of the general linear homogeneous IVP
\[
\frac{dx}{dt} = A(t)x, \quad x(0) = x_0.
\]
Constant coefficient case

Consider a constant matrix \( A \in \mathbb{R}^{n \times n} \). If \( Av = \lambda v \), then the function \( x(t) = e^{\lambda t}v \) satisfies

\[
\frac{dx}{dt} = \lambda e^{\lambda t}v = e^{\lambda t}(\lambda v) = e^{\lambda t}Av = A(e^{\lambda t}v) = Ax.
\]

Example

Find a basis for the space of solutions of the system

\[
\frac{dx_1}{dt} = -5x_1 + 2x_2,
\]

\[
\frac{dx_2}{dt} = -6x_1 + 3x_2.
\]

Hence find the specific basis \( \{E_1(t), E_2(t)\} \).
We now switch from considering a first order ODE for a vector-valued function $x(t)$, to consider an $m$th order ODE for a scalar-valued function $u(x)$.

Given coefficients $a_0(x)$, $a_1(x)$, $\ldots$, $a_m(x)$ we define the linear differential operator $L$ of order $m$,

$$Lu(x) = \sum_{j=0}^{m} a_j(x) D^j u(x) = a_m D^m u + a_{m-1} D^{m-1} u + \cdots + a_0 u,$$  \hspace{1cm} (6)

where $D^j u = d^j u / dx^j$ (with $D^0 u = u$).

We refer to $a_m$ as the leading coefficient of $L$. For simplicity, we assume that each $a_j(x)$ is a smooth function of $x$. 

**Linear differential operators**
Any operator of the form (6) is indeed linear: for any constants $c_1$ and $c_2$ and any $(m$-times differentiable) functions $u_1$ and $u_2$,

$$L(c_1 u_1 + c_2 u_2) = c_1 Lu_1 + c_2 Lu_2.$$ 

Hence, the set of solutions to the **homogeneous equation** $Lu = 0$ forms a vector space.

**Example**

$Lu = (x - 3)u''' - (1 + \cos x)u' + 6u$ is a linear differential operator of order 3, with leading coefficient $x - 3$.

**Example**

$N(u) = u'' + u^2 u' - u$ is a **nonlinear** differential operator of order 2.
Equivalent first-order system

Consider a general third-order linear differential operator

\[ Lu = a_3 u''' + a_2 u'' + a_1 u' + a_0 u. \]

Writing \( y_1 = u, \ y_2 = u' \) and \( y_3 = u'' \) we see that \( Lu = f \) if and only if

\[
\begin{align*}
\frac{dy_1}{dx} & = y_2, \\
\frac{dy_2}{dx} & = y_3, \\
\frac{dy_3}{dx} & = \frac{1}{a_3} \left( f - a_0 y_1 - a_1 y_2 - a_2 y_3 \right).
\end{align*}
\]  \tag{7}

Thus, we can expect trouble if the leading coefficient vanishes at any \( x \) in the interval of interest; the ODE \( Lu = f \) is said to be singular in this case.
Equivalent first-order system: matrix version

We can write (7) as

\[
\frac{dy}{dx} = A(x)y + b(x),
\]

where

\[
A(x) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_0/a_3 & -a_1/a_3 & -a_2/a_3
\end{bmatrix}
\]

and

\[
b(x) = \begin{bmatrix}
0 \\
0 \\
f/a_3
\end{bmatrix}.
\]
Linear initial-value problem

Consider a general $m$th-order linear differential operator

$$Lu = \sum_{j=0}^{m} a_j(x) D^j u.$$ 

Given $f(x)$ and $m$ initial values $\nu_0, \nu_1, \ldots, \nu_{m-1}$ we seek $u = u(x)$ satisfying

$$Lu = f \quad \text{on } [a, b], \quad (8)$$

with

$$u(a) = \nu_0, \quad u'(a) = \nu_1, \quad \ldots, \quad u^{(m-1)}(a) = \nu_{m-1}. \quad (9)$$

Theorem

Assume that $L$ is not singular on $[a, b]$ and that $f$ is continuous on $[a, b]$. Then the IVP (8) and (9) has a unique solution.
Homogeneous problem

Theorem
Assume that the linear, mth-order differential operator $L$ is not singular on $[a, b]$. Then the set of all solutions to the homogeneous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension $m$.

If $\{u_1, u_2, \ldots, u_m\}$ is any basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_m u_m(x) \quad \text{for} \ a \leq x \leq b. \quad (10)$$

We refer to (10) as the general solution of the homogeneous equation $Lu = 0$ on $[a, b]$.

Example
The general solution to $u'' - u' - 2u = 0$ is $u(x) = c_1 e^{-x} + c_2 e^{2x}$. 
Inhomogeneous problem

Consider the inhomogeneous equation $Lu = f$ on $[a, b]$, and fix a particular solution $u_P$.

For any solution $u$, the difference $u - u_P$ is a solution of the homogeneous equation because

$$L(u - u_P) = Lu - Lu_P = f - f = 0 \quad \text{on } [a, b].$$

Hence, $u(x) - u_P(x) = c_1 u_1(x) + \cdots + c_m u_m(x)$ for some constants $c_1, \ldots, c_m$, and so

$$u(x) = u_P + \underbrace{c_1 u_1(x) + \cdots + c_m u_m(x)}_{u_H(x)}, \quad a \leq x \leq b,$$

is the general solution of the inhomogeneous equation $Lu = f$. 
Constant coefficient case

Suppose that \( a_j \) is constant for \( 0 \leq j \leq m \), with \( a_m \neq 0 \). We define the associated polynomial of degree \( m \),

\[
p(z) = \sum_{j=0}^{m} a_j z^j = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,
\]

so that, formally, \( L = p(D) \).

Since \( D^j e^{\lambda x} = \lambda^j e^{\lambda x} \) we have

\[
p(D) e^{\lambda x} = \left( a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0 \right) e^{\lambda x} = p(\lambda) e^{\lambda x},
\]

and so

\[
p(D) e^{\lambda x} = 0 \iff p(\lambda) = 0.
\]
Factorization

By the fundamental theorem of algebra,

\[ p(z) = a_m(z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_r)^{k_r} \]

where \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are the distinct roots of \( p \), with corresponding multiplicities \( k_1, k_2, \ldots, k_r \) satisfying

\[ k_1 + k_2 + \cdots + k_r = m. \]

Lemma
\[(D - \lambda)x^je^{\lambda x} = jx^{j-1}e^{\lambda x} \text{ for } j \geq 1.\]

Lemma
\[(D - \lambda)^{k}x^je^{\lambda x} = 0 \text{ for } j = 0, 1, \ldots, k - 1.\]
General solution

**Lemma**

If \((z - \lambda)^k\) is a factor of \(p(z)\) then the function \(u(x) = x^{k-1}e^{\lambda x}\) is a solution of \(Lu = 0\).

**Theorem**

For the constant-coefficient case, the general solution of the homogeneous equation \(Lu = 0\) is

\[
   u(x) = \sum_{j=1}^{r} \sum_{p=0}^{k_j-1} c_{jp} x^p e^{\lambda_j x},
\]

where the \(c_{jp}\) are arbitrary constants.
Distinct real roots

Example

From the factorization

\[ D^4 - 2D^3 - 11D^2 + 12D = (D + 3)D(D - 1)(D - 4) \]

we see that the general solution of

\[ u'''' - 2u'' - 11u'' + 12u' = 0 \]

is

\[ u = c_1 e^{-3x} + c_2 + c_3 e^x + c_4 e^{4x}. \]
Repeated real root

Example

From the factorization

\[ D^4 + 6D^3 + 9D^2 - 4D - 12 = (D - 1)(D + 2)^2(D + 3) \]

we see that the general solution of

\[ u''' + 6u'' + 9u' - 4u - 12u = 0 \]

is

\[ u = c_1 e^x + c_2 e^{-2x} + c_3 xe^{-2x} + c_4 e^{-3x}. \]
Example

From the factorization

\[ D^3 - 7D^2 + 17D - 15 = (D^2 - 4D + 5)(D - 3) \]
\[ = (D - 2 - i)(D - 2 + i)(D - 3) \]

we see that the general solution of

\[ u''' - 7u'' + 17u' - 15u = 0 \]

is

\[ u(x) = c_1 e^{(2+i)x} + c_2 e^{(2-i)x} + c_3 e^{3x} \]
\[ = c_4 e^{2x} \cos x + c_5 e^{2x} \sin x + c_3 e^{3x}. \]
A necessary condition for linear independence

The Wronskian of the functions \( u_1, u_2, \ldots, u_m \) is the \( m \times m \) determinant

\[
W(x) = W(x; u_1, u_2, \ldots, u_m) = \det[D^{j-1} u_i].
\]

For instance, if \( m = 2 \) then

\[
W(x) = \det \begin{bmatrix} u_1 & u'_1 \\ u_2 & u'_2 \end{bmatrix} = u_1u'_2 - u_2u'_1.
\]

Lemma

If \( u_1, \ldots, u_m \) are linearly dependent over an interval \([a, b]\) then \( W(x; u_1, \ldots, u_m) = 0 \) for \( a \leq x \leq b \).
Linear independence of solutions

Lemma
If \( u_1, u_2, \ldots, u_m \) are solutions of \( Lu = 0 \) on the interval \([a, b]\) then their Wronskian satisfies

\[
a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \quad a \leq x \leq b.
\]

Theorem
Let \( u_1, u_2, \ldots, u_m \) be solutions of a non-singular, linear, homogeneous, mth-order ODE \( Lu = 0 \) on the interval \([a, b]\). Either

\[ W(x) = 0 \text{ for } a \leq x \leq b \text{ and the } m \text{ solutions are linearly dependent,} \]

or else

\[ W(x) \neq 0 \text{ for } a \leq x \leq b \text{ and the } m \text{ solutions are linearly independent.} \]
Interlacing of zeros

Theorem (Sturm separation theorem)
Assume $a_2(x) \neq 0$ for $a \leq x \leq b$. If $u_1$ and $u_2$ are linearly independent solutions of

$$a_2(x)u'' + a_1(x)u' + a_0(x)u = 0$$

then $u_1$ has exactly one zero between any two successive zeros of $u_2$ in the interval $[a, b]$.

Example
The functions $u_1(x) = \cos x$ and $u_2(x) = \sin x$ are linearly independent solutions of $u'' + u = 0$ on any interval $[a, b]$. 
Proof

Suppose $a \leq \alpha < \beta \leq b$ with

$$u_2(\alpha) = 0 = u_2(\beta) \quad \text{and} \quad u_2(x) \neq 0 \text{ for } \alpha < x < \beta.$$ 

Case 1: $W(x) > 0$ for $a \leq x \leq b$ and $u_2(x) > 0$ for $\alpha < x < \beta$. Thus, $u_2'(\alpha) = \lim_{h \to 0^+} u_2(\alpha + h)/h \geq 0$ and $u_2'(\beta) = \lim_{h \to 0^-} u_2(\beta + h)/h \leq 0$. In fact, $u_2'(\alpha) > 0$ because otherwise $u_2$ must equal the unique solution $u \equiv 0$ of the IVP

$$u(\alpha) = u'(\alpha) = 0 \quad \text{and} \quad Lu = 0.$$ 

Similarly, $u(\beta) < 0$. Since $W = u_1u_2' - u_1'u_2$, we see

$$0 < W(\alpha) = u_1(\alpha)u_2'(\alpha) \quad \text{and} \quad 0 < W(\beta) = u_1(\beta)u_2'(\beta),$$

so $u_1(\alpha) > 0$ and $u_1(\beta) < 0$. 
Proof (continued)

Case 2: \( W(x) > 0 \) for \( a \leq x \leq b \) and \( u_2(x) < 0 \) for \( \alpha < x < \beta \). Similar argument shows \( u_1(\alpha) < 0 \) and \( u_1(\beta) > 0 \).

Case 3: \( W(x) < 0 \) for \( a \leq x \leq b \) and \( u_2(x) > 0 \) for \( \alpha < x < \beta \). Similar argument shows \( u_1(\alpha) < 0 \) and \( u_1(\beta) > 0 \).

Case 4: \( W(x) < 0 \) for \( a \leq x \leq b \) and \( u_2(x) < 0 \) for \( \alpha < x < \beta \). Similar argument shows \( u_1(\alpha) > 0 \) and \( u_1(\beta) < 0 \).

In all cases, \( u_1(\alpha) \) and \( u_1(\beta) \) have opposite signs, and so by the Intermediate Value Theorem \( u_1(\xi) = 0 \) for at least one \( \xi \in (\alpha, \beta) \).

Finally, suppose for a contradiction that \( u_1 \) has more than one zero in \( (\alpha, \beta) \), say \( u_1(\xi) = 0 = u_1(\eta) \) with \( \alpha < \xi < \eta < \beta \). Then, by interchanging the roles of \( u_1 \) and \( u_2 \), it follows that \( u_2 \) must have a zero in \( (\xi, \eta) \), contradicting the assumption that \( u_2(x) > 0 \) for \( \alpha < x < \beta \).
Constructing a series solution

Consider the initial-value problem

\[ Lu = (1 - x^2)u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2. \]

Look for a solution in the form of a power series

\[ u(x) = \sum_{k=0}^{\infty} A_k x^k = A_0 + A_1 x + A_2 x^2 + \cdots. \]

Formal calculations show that

\[ Lu = \sum_{k=0}^{\infty} (k + 2)[(k + 1)A_{k+2} - (k + 2)A_k]x^k, \]

and the initial conditions imply \( A_0 = 1 \) and \( A_1 = 2 \).
Convergence?

Since $Lu$ is identically zero iff the coefficient of $x^k$ vanishes for every $k$, we obtain the recurrence relation

$$A_{k+2} = \frac{k + 2}{k + 1} A_k \quad \text{for } k = 0, 1, 2, \ldots.$$

Thus,

$$A_0 = 1, \quad A_1 = 2, \quad A_2 = 2, \quad A_3 = 3, \quad \ldots.$$

Since

$$\lim_{k \to \infty} \frac{A_{k+2}x^{k+2}}{A_kx^k} = \lim_{k \to \infty} \frac{k + 2}{k + 1} x^2 = x^2,$$

the ratio test shows that $\sum_{j=0}^{\infty} A_{2j}x^{2j}$ and $\sum_{j=0}^{\infty} A_{2j+1}x^{2j+1}$ converge for $x^2 < 1$ but diverge for $x^2 > 1$. 
General case

Consider a general second-order, linear, homogeneous ODE

\[ Lu \overset{\text{def}}{=} a_2(x)u'' + a_1(x)u' + a_0(x)u = 0. \]

Equivalently,

\[ u'' + p(x)u' + q(x)u = 0, \]

where

\[ p(x) = \frac{a_1(x)}{a_2(x)} \quad \text{and} \quad q(x) = \frac{a_0(x)}{a_2(x)}. \]

Assume that \( a_j \) is analytic at 0 for \( 0 \leq j \leq 2 \), and that \( a_2(0) \neq 0 \). Then \( p \) and \( q \) are analytic at 0, that is, they admit power series expansions

\[ p(z) = \sum_{k=0}^{\infty} p_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{\infty} q_k z^k \quad \text{for } |z| < \rho, \]

for some \( \rho > 0 \).
Formal expansions

If

\[ u(z) = \sum_{k=0}^{\infty} A_k z^k \]

then we find that

\[ Lu(z) = (2A_2 + p_0 A_1 + q_0 A_0) + (6A_3 + 2p_0 A_2 + p_1 A_1 + q_0 A_1 + q_1 A_0)z + \cdots, \]

where, on the RHS, the coefficient of \( z^{n-1} \) for a general \( n \geq 1 \) is

\[ (n + 1)nA_{n+1} + \sum_{j=0}^{n-1} [(n - j)p_j A_{n-j} + q_j A_{n-1-j}]. \]
Convergence theorem

Given \( u(0) \) and \( u'(0) \), we put

\[
A_0 = u(0) \quad \text{and} \quad A_1 = u'(0),
\]

and compute recursively

\[
A_{n+1} = \frac{-1}{n(n + 1)} \sum_{j=0}^{n-1} [(n - j)p_j A_{n-j} + q_j A_{n-1-j}], \quad n \geq 1.
\]

Theorem

If the coefficients \( p(z) \) and \( q(z) \) are analytic for \( |z| < \rho \), then the formal power series for the solution \( u(z) \), constructed above, is also analytic for \( |z| < \rho \).
Earlier we considered
\[ Lu \overset{\text{def}}{=} (1 - x^2) u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2. \]

In this case,
\[ p(z) = \frac{-5z}{1 - z^2} = -5 \sum_{k=0}^{\infty} z^{2k+1} \]
and
\[ q(z) = \frac{-4}{1 - z^2} = -4 \sum_{k=0}^{\infty} z^{2k} \]
are analytic for \(|z| < 1\), so the theorem guarantees that \( u(z) \), given by the formal power series, is also analytic for \(|z| < 1\).
A simple generalization

Suppose that we want to expand about \( z = z_0 \) where \( z_0 \neq 0 \). Just make a simple change of variable

\[
\zeta = z - z_0,
\]

and put \( u(z) = U(\zeta) \) so that

\[
U'' + p(\zeta + z_0)U' + q(\zeta + z_0)U = 0.
\]

If \( p(z) \) and \( q(z) \) are analytic at \( z = z_0 \), then \( p(\zeta + z_0) \) and \( q(\zeta + z_0) \) are analytic at \( \zeta = 0 \) so we can construct

\[
U(\zeta) = \sum_{k=0}^{\infty} A_k \zeta^k, \quad |\zeta| < \rho,
\]

and then

\[
u(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k, \quad |z - z_0| < \rho.
\]
Reduction of order

If we know (somehow) one solution \( u = u_1(x) \) to a second-order, linear, homogeneous ODE

\[
u'' + p(x)u' + q(x)u = 0,
\]

then, to find a second (linearly independent) solution, substitute \( u = \nu(x)u_1(x) \) into the ODE and rearrange to obtain

\[
\begin{aligned}
(u''_1 + pu'_1 + qu_1)\nu + u_1\nu'' + (2u'_1 + pu_1)\nu' &= 0.
\end{aligned}
\]

This is just a first-order, linear ODE for the derivative of the unknown factor \( \nu(x) \): put \( w = \nu' \) then

\[
u_1 w' + (2u'_1 + pu_1)w = 0.
\]
Reduction of order (continued)

Writing the ODE for \( w \) in the standard form

\[
w' + (2u_1'u_1^{-1} + p)w = 0,
\]

we seek an integrating factor

\[
I(x) = \exp\left(\int (2u_1'u_1^{-1} + p) \, dx\right) = u_1^2 \exp\left(\int p \, dx\right),
\]

so that

\[
\frac{d}{dx}(Iw) = lw' + l'w = I(w' + (2u_1'u_1^{-1} + p)w) = 0.
\]

Then \( lw = C \) for some constant \( C \), and so

\[
v = \int \frac{C}{I(x)} \, dx.
\]

Example

For the ODE \( u'' - 6u' + 9u = 0 \), take \( u_1 = e^{3x} \) and find \( v \).
Variation of parameters

Consider a linear, second-order, *inhomogeneous* ODE with leading coefficient 1:

\[ Lu \equiv u''(x) + p(x)u'(x) + q(x)u(x) = f(x). \quad (11) \]

Let \( u_1(x) \) and \( u_2(x) \) be linearly independent solutions to the homogeneous equation and let \( W(x) = W(x; u_1, u_2) \) denote their Wronskian. Thus,

\[ Lu_1 = 0, \quad Lu_2 = 0, \quad W \neq 0. \]

We seek \( v_1 \) and \( v_2 \) such that

\[ u(x) = v_1(x)u_1(x) + v_2(x)u_2(x) \]

is a solution to \( Lu = f \).
Variation of parameters (continued)

To simplify the expression

\[ u' = v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2' \]

we impose the condition \( v_1' u_1 + v_2' u_2 = 0 \), then (as if \( v_1 \) and \( v_2 \) were constants)

\[ u' = v_1 u_1' + v_2 u_2'. \]

A short calculation now shows

\[ Lu = v_1 Lu_1 + v_2 Lu_2 + v_1 u_1' + v_2 u_2' = v_1 u_1' + v_2 u_2', \]

since by assumption \( Lu_1 = 0 = Lu_2 \).

Conclusion: \( u = v_1 u_1 + v_2 u_2 \) satisfies \( Lu = f \) if

\[ v_1' u_1 + v_2' u_2 = 0, \]

\[ v_1' u_1' + v_2' u_2' = f. \]
Variation of parameters (continued)

Thus, we have a pair of equations for the unknown \( v_1' \) and \( v_2' \). In matrix form

\[
\begin{bmatrix}
  u_1(x) & u_2(x) \\
  u_1'(x) & u_2'(x)
\end{bmatrix}
\begin{bmatrix}
  v_1'(x) \\
  v_2'(x)
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  f(x)
\end{bmatrix},
\]

so

\[
\begin{bmatrix}
  v_1'(x) \\
  v_2'(x)
\end{bmatrix}
= \frac{1}{W(x)} \begin{bmatrix}
  u_2'(x) & -u_2(x) \\
  -u_1'(x) & u_1(x)
\end{bmatrix}
\begin{bmatrix}
  0 \\
  f(x)
\end{bmatrix},
\]

or in other words,

\[
v_1'(x) = \frac{-u_2(x)f(x)}{W(x)}, \quad v_2'(x) = \frac{u_1(x)f(x)}{W(x)}.
\]

Example

Find the general solution to

\[3u'' - 6u' + 30u = e^x \tan 3x.\]
A singular ODE

A second-order Cauchy–Euler ODE has the form

\[ Lu \overset{\text{def}}{=} ax^2 u'' + bxu' + cu = f, \]

where \( a, b \) and \( c \) are constants, with \( a \neq 0 \). This ODE is singular at \( x = 0 \).

Noticing that

\[ Lx^r = \left[ ar(r - 1) + br + c \right] x^r, \]

we see that \( u = x^r \) is a solution of the homogeneous equation \( (f = 0) \) iff

\[ ar(r - 1) + br + c = 0. \]
Suppose

\[ ar(r - 1) + br + c = a(r - r_1)(r - r_2). \]

If \( r_1 \neq r_2 \) then the general solution of the homogeneous equation \( Lu = 0 \) is

\[ u(x) = C_1 x^{r_1} + C_2 x^{r_2}, \quad x > 0. \]

**Lemma**

*If* \( r_1 = r_2 \) *then the general solution of the homogeneous Cauchy–Euler equation* \( Lu = 0 \) *is*

\[ u(x) = C_1 x^{r_1} + C_2 x^{r_1} \log x, \quad x > 0. \]

**Example**

Solve \( x^2 u'' - xu' + u = 0 \).
More general singular ODEs

A number of important applications lead to ODEs that can be written in the Frobenious normal form

\[ z^2 u'' + zP(z)u' + Q(z)u = 0, \]

where \( P(z) \) and \( Q(z) \) are analytic at \( z = 0 \):

\[
\begin{align*}
P(z) &= \sum_{k=0}^{\infty} P_k z^k \quad \text{and} \quad Q(z) = \sum_{k=0}^{\infty} Q_k z^k, \\
|z| &< \rho.
\end{align*}
\]

Notice that \( u'' + p(z)u' + q(z)u = 0 \) but \( p(z) = z^{-1}P(z) \) and \( q(z) = z^{-2}Q(z) \) are not analytic at \( z = 0 \) unless \( P(0) = 0 \) and \( Q(0) = Q'(0) = 0 \).

So in general we cannot expect a solution \( u(z) \) to be analytic at \( z = 0 \).
A clue

We can think of an ODE in Frobenius normal form as a Cauchy–Euler ODE with variable coefficients.

For $z$ near 0 we have $P(z) \approx P_0$ and $Q(z) \approx Q_0$ so $u(z)$ should behave like a solution of

$$z^2 u'' + P_0 zu' + Q_0 u = 0.$$ 

We therefore consider the indicial polynomial

$$I(r) = r(r - 1) + P_0 r + Q_0 = (r - r_1)(r - r_2).$$

If $r_1 \neq r_2$ then the approximating Cauchy–Euler ODE has the general solution $c_1 z^{r_1} + c_2 z^{r_2}$. 
Generalized series expansion

We therefore seek to construct a solution in the form

\[ u(z) = z^r \sum_{k=0}^{\infty} A_k z^k = \sum_{k=0}^{\infty} A_k z^{k+r}, \quad |z| < \rho. \]

Formal manipulations show that \( Lu(z) \) equals

\[
I(r)A_0z^r + \sum_{k=1}^{\infty} \left( I(k + r)A_k + \sum_{j=0}^{k-1} [(j + r)P_{k-j} + Q_{k-j}]A_j \right) z^{k+r},
\]

so we define \( A_0(r) = 1 \) and

\[
A_k(r) = \frac{-1}{I(k + r)} \sum_{j=0}^{k-1} [(j + r)P_{k-j} + Q_{k-j}]A_j(r), \quad k \geq 1,
\]

provided \( I(k + r) \neq 0 \) for all \( k \geq 1 \).
Choice of exponent

The preceding calculations show that the series

\[ F(z; r) = \sum_{k=0}^{\infty} A_k(r)z^{k+r} \]

satisfies

\[ z^2 F'' + zP(z)F' + Q(z)F = I(r)z^r. \]

Assume, with no loss of generality, that \( \text{Re} \ r_1 \geq \text{Re} \ r_2 \). It follows that \( I(k + r_1) \neq 0 \) for all integers \( k \geq 1 \), and therefore \( u_1(z) = F(z; r_1) \) is (formally) a solution.

If \( r_1 - r_2 \) is not a whole number, then a second, linearly independent solution is \( u_2(z) = F(z; r_2) \).

Example

Solve \( z^2(z + 2)u'' - zu' + (z + 1)u = 0 \).
Equal roots of the indicial equation

Suppose that \( r_1 = r_2 \). In this case, \( I(r) = (r - r_1)^2 \) and so

\[
z^2 F'' + zP(z)F' + Q(z)F = (r - r_1)^2 z^r.
\]

The function \( v = \frac{\partial F}{\partial r} \) satisfies

\[
z^2 v'' + zP(z)v' + Q(z)v = 2(r - r_1)z^r + (r - r_1)^2 z^r \log z,
\]

and the RHS is zero if \( r = r_1 \), so a second, linearly independent solution is

\[
u_2(z) = \frac{\partial F}{\partial r}(z; r_1) = \sum_{k=0}^{\infty} A'_k(r_1)z^{k+r_1} + \sum_{k=0}^{\infty} A_k(r_1)z^{k+r_1} \log z.
\]

Example

Construct two linearly independent solutions to

\[
z^2 u'' - zu' + (1 - z)u = 0.
\]
Roots differing by an integer

Suppose \( r_2 = r_1 + n \) for an integer \( n \geq 1 \), and thus \( I(k + r_2) = (k - n)k = 0 \) when \( k = n \). We put \( A_0(r) = r - r_2 \) (instead of \( A_0(r) = 1 \)), so that in the recursion

\[
A_k(r) = \frac{-1}{I(k + r)} \sum_{j=0}^{k-1} [(j + r)P_{k-j} + Q_{k-j}]A_j(r), \quad k \geq 1,
\]

a factor \( r - r_2 \) in each \( A_j(r) \) cancels the one in \( I(n + r) = (r - r_2)(n + r - r_2) \).
We find that $F(z; r) = \sum_{k=0}^{\infty} A_k z^{k+r}$ now satisfies

$$z^2 F'' + zP(z)F' + Q(z)F = I(r)A_0(r)z^r = (r - r_1)(r - r_2)^2 z^r,$$

so for $v = \partial F / \partial r$ we have

$$z^2 v'' + zP(z)v' + Q(z)v = (r - r_2)^2 z^r$$

$$+ 2(r - r_1)(r - r_2)z^r + (r - r_1)(r - r_2)^2 z^r \log z.$$

Thus, a second, linearly independent solution is given by

$$u_2(z) = \frac{\partial F}{\partial r} (z; r_2) = \sum_{k=0}^{\infty} A'_k(r_2) z^{k+r_2} + \sum_{k=0}^{\infty} A_k(r_2) z^{k+r_2} \log z$$
The Bessel equation with parameter $\nu$ is

$$z^2 u'' + zu' + (z^2 - \nu^2)u = 0.$$  

This ODE is in Frobenius normal form, with indicial polynomial

$$I(r) = (r + \nu)(r - \nu),$$

and we seek a series solution

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

We assume $\text{Re}\, \nu \geq 0$, so $r_1 = \nu$ and $r_2 = -\nu$. 

Recurrence relation

We find that if

\[(r + 1 + \nu)(r + 1 - \nu)A_1 = 0,\]
\[(k + r + \nu)(k + r - \nu)A_k + A_{k-2} = 0, \quad k \geq 2.\]

then

\[z^2 u'' + zu' + (z^2 - \nu^2)u = (r + \nu)(r - \nu)A_0 z^r.\]

Taking \(r = \nu\) gives

\[A_k = \frac{-A_{k-2}}{k(k + 2\nu)} \quad \text{for } k \geq 2,
\]

so with \(A_0\) arbitrary and \(A_1 = 0\) we obtain

\[u(z) = A_0 z^\nu \left[ 1 - \frac{(z/2)^2}{1 + \nu} + \frac{(z/2)^4}{2(2 + \nu)(1 + \nu)} - \frac{(z/2)^6}{3!(3 + \nu)(2 + \nu)(1 + \nu)} + \cdots \right].\]
Bessel function

With the normalisation

\[ A_0 = \frac{1}{2^\nu \Gamma(1 + \nu)} \]

the series solution is called the Bessel function of order \( \nu \) and is denoted

\[ J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1 + \nu)} \left[ 1 - \frac{(z/2)^2}{1 + \nu} + \frac{(z/2)^4}{2!(1 + \nu)(2 + \nu)} - \cdots \right]. \]

From the functional equation \( \Gamma(1 + z) = z\Gamma(z) \) we see that

\[ J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1 + \nu)} - \frac{(z/2)^{\nu+2}}{\Gamma(2 + \nu)} + \frac{(z/2)^{\nu+4}}{2!\Gamma(3 + \nu)} - \frac{(z/2)^{\nu+6}}{3!\Gamma(4 + \nu)} + \cdots \]

and so

\[ J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{2k+\nu}}{k!\Gamma(k + 1 + \nu)}. \]
Bessel function of negative order

If $\nu$ is not an integer, then a second, linearly independent, solution is

$$J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k-\nu}}{k! \Gamma(k + 1 - \nu)}.$$ 

For an integer $\nu = n \in \mathbb{Z}$, since $\Gamma(n + 1) = n!$ we have

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k + n)!}.$$ 

Also, since $1/\Gamma(z) = 0$ for $z = 0, -1, -2, \ldots$, we find that $J_n$ and $J_{-n}$ are linearly dependent; in fact,

$$J_{-n}(z) = (-1)^n J_n(z).$$
Some plots

Bessel functions of integer order

$n = 0$
$n = 1$
$n = 2$
$n = 3$
$n = 4$
The Neumann function (or Bessel function of the second kind) is

\[ Y_\nu(z) = \frac{J_\nu(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi}, \quad \text{if } \nu \not\in \mathbb{Z}. \]

For \( n \in \mathbb{Z} \), L’Hospital’s rule shows that if \( \nu \to n \) then \( Y_\nu(z) \) tends to a finite limit

\[ Y_n(z) \overset{\text{def}}{=} \lim_{\nu \to n} Y_\nu(z). \]

The functions \( J_\nu \) and \( Y_\nu \) are linearly independent solutions of Bessel’s equation for all complex \( \nu \).

As \( z \to 0 \) with \( \nu \) fixed,

\[ J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu + 1)}, \quad \nu \not\in \{-1, -2, -3, \ldots\}, \]

\[ Y_0(z) \sim \frac{2}{\pi} \log z, \quad Y_\nu(z) \sim \frac{-\Gamma(\nu)}{\pi(z/2)^\nu}, \quad \text{Re} \nu > 0. \]
Some more plots

Neumann functions of integer order

- $n = 0$
- $n = 1$
- $n = 2$
- $n = 3$
- $n = 4$
Interlacing of zeros
Legendre equation

The Legendre equation with parameter $\nu$ is

$$(1 - z^2)u'' - 2zu' + \nu(\nu + 1)u = 0.$$ 

This ODE is not singular at $z = 0$ so the solution has an ordinary Taylor series expansion

$$u = \sum_{k=0}^{\infty} A_k z^k.$$ 

The $A_k$ must satisfy

$$(k + 1)(k + 2)A_{k+2} - [k(k + 1) - \nu(\nu + 1)]A_k = 0$$

for $k \geq 0$, and since

$$k(k + 1) - \nu(\nu + 1) = (k - \nu)(k + \nu + 1),$$

the recurrence relation is

$$A_{k+2} = \frac{(k - \nu)(k + \nu + 1)}{(k + 1)(k + 2)} A_k \quad \text{for } k \geq 0.$$
General solution

We have

\[ u(z) = A_0 u_0(z) + A_1 u_1(z) \]

where

\[ u_0(z) = 1 - \frac{\nu(\nu + 1)}{2!} z^2 + \frac{(\nu - 2)\nu(\nu + 1)(\nu + 3)}{4!} z^4 - \cdots \]

and

\[ u_1(z) = z - \frac{(\nu - 1)(\nu + 2)}{3!} z^3 \]

\[ + \frac{(\nu - 3)(\nu - 1)(\nu + 2)(\nu + 4)}{5!} z^5 - \cdots . \]

Suppose now that \( \nu = n \) is a non-negative integer. If \( n \) is even then the series for \( u_0(z) \) terminates, whereas if \( n \) is odd then the series for \( u_1(z) \) terminates.
The terminating solution is called the Legendre polynomial of degree $n$ and is denoted by $P_n(z)$ with the normalization

$$P_n(1) = 1.$$ 

The first few Legendre polynomials are

$$P_0(z) = 1,$$
$$P_1(z) = z,$$
$$P_2(z) = \frac{1}{2}(3z^2 - 1),$$
$$P_3(z) = \frac{1}{2}(5z^3 - 3z),$$
$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3),$$
$$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z).$$

Notice that $P_n$ is an even or odd function according to whether $n$ is even or odd.
Plots

Legendre polynomials

\[ P_n(x) \]

- Blue, \( n = 1 \)
- Green, \( n = 2 \)
- Red, \( n = 3 \)
- Cyan, \( n = 4 \)