Discontinuous Galerkin methods for fractional diffusion problems

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Outline

- Sub-diffusion Equation
- Various Numerical Methods
- Discontinuous Galerkin Methods

Part I

Sub-diffusion Equation

Classical diffusion

Density of particles $u = u(x, t)$ satisfies

$$u_t - \nabla \cdot (K \nabla u) = f(x, t) \quad \text{for } x \in \Omega \text{ and } t > 0,$$

for a diffusivity $K > 0$ and source density f.

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Zero-flux boundary condition $\partial u / \partial n = 0$ for $x \in \partial\Omega$.

Absorbing boundary condition $u = 0$ for $x \in \partial\Omega$.

Density satisfies

$$u_t - \nabla \cdot (\omega_\nu * K \nabla u)_t = f(x,t) \quad \text{for } x \in \Omega \text{ and } t > 0,$$

where $0 < \nu < 1$ and

$$\omega_\nu(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}.$$

Anomalous sub-diffusion

Density satisfies

$$u_t - \nabla \cdot (\omega_\nu * K\nabla u)_t = f(x,t) \quad \text{for } x \in \Omega \text{ and } t > 0,$$

where $0 < \nu < 1$ and

$$\omega_\nu(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}.$$

Riemann–Liouville fractional integral of order ν:

$$\omega_\nu * v(t) = \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}\, v(s)\, ds.$$

Thus, $(\omega_\nu * v)_t =$ “$\partial_t^{1-\nu} v$”.

Interpretation of ν

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$$(\omega_\nu * K\nabla u)_t \to K\nabla u.$$

Hilbert space setting

Write

$$Au = -\nabla \cdot (K \nabla u) \quad \text{and} \quad \mathcal{B}_\nu v(t) = (\omega_\nu * v)_t.$$

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$$\langle u, v \rangle = \int_\Omega uv\, dx \quad \text{and} \quad \|u\| = \sqrt{\langle u, u \rangle}.$$

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$$\mathrm{A}(u, v) = \langle Au, v\rangle = \int_\Omega K\nabla u\cdot\nabla v\,dx.$$

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Weak formulation:

$$\langle u_t, v \rangle + \mathrm{A}(\mathcal{B}_\nu u, v) = \langle f, v \rangle \quad \text{for all } v \in H^1(\Omega).$$

Separation of variables

Complete orthonormal eigensystem:

$$\begin{gathered} A\phi_m = \lambda_m \phi_m \quad \text{for } m = 0,\, 1,\, 2,\, \ldots \\ \langle \phi_m, \phi_n \rangle = \delta_{mn}, \\ 0 \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots , \\ \lambda_m \to \infty \text{ as } m \to \infty. \end{gathered}$$

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$$\lambda_m \to \infty \text{ as } m \to \infty.$$

Fourier modes $u_m(t) = \langle u(t), \phi_m \rangle$ satisfy scalar IVP:

$$u_m' + \lambda_m \mathcal{B}_\nu u_m = f_m(t), \quad u_m(0) = u_{0m},$$

where

$$f_m(t) = \langle f(t), \phi_m \rangle, \qquad u_{0m} = \langle u_0, \phi_m \rangle.$$

Laplace transformation

Notation:

$$\hat{v}(z) = \mathcal{L}\{v(t)\} = \int_0^\infty e^{-zt} v(t)\, dt.$$

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to obtain

$$z\hat{u}_m(z) - u_{0m} + \lambda_m z^{1-\nu} \hat{u}_m(z) = \hat{f}_m(z).$$

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Thus,

$$\hat{u}_m(z) = \frac{u_{0m} + \hat{f}_m(z)}{z + \lambda_m z^{1-\nu}}.$$

Explicit solution

Mittag–Leffler function satisfies

$$E_\nu(-\lambda t^\nu) = \mathcal{L}^{-1}\left\{\frac{1}{z+\lambda z^{1-\nu}}\right\} \quad \to \quad e^{-\lambda t} = \mathcal{L}^{-1}\left\{\frac{1}{z+\lambda}\right\},$$

so

$$u_m(t) = E_\nu(-\lambda_m t^\nu)u_{0m} + \int_0^t E_\nu\bigl(-\lambda_m(t-s)^\nu\bigr)\, f_m(s)\, ds.$$

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Hence,

$$u(t) = \mathcal{E}(t)u_0 + \int_0^t \mathcal{E}(t-s)f(s)\, ds$$

where

$$\mathcal{E}(t)v = \sum_{m=0}^{\infty} E_\nu(-\lambda_m t^\nu)\langle v, \phi_m\rangle \phi_m.$$

Positivity

Plancherel Theorem implies

$$\int_0^\infty \mathcal{B}_\nu v(t)v(t)\,dt = \frac{\sin(\frac{1}{2}\pi\nu)}{\pi}\int_0^\infty y^{1-\nu}|\hat{v}(iy)|^2\,dy \geq 0.$$

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So

$$\int_0^T \mathrm{A}(\mathcal{B}_\nu u, u)\,dt = \sum_{m=0}^\infty \lambda_m \int_0^T \mathcal{B}_\nu u_m(t)u_m(t)\,dt \geq 0.$$

Stability via energy argument

Take $v = u$ in the weak formulation:

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Since

$$\langle u', u\rangle = \frac{1}{2}\,\frac{d}{dt}\|u\|^2,$$

integrating from $t = 0$ to $t = T$ gives

$$\tfrac{1}{2}\|u(T)\|^2 - \tfrac{1}{2}\|u_0\|^2 \le \int_0^T \|f\|\|u\|\,dt.$$

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$$\tfrac{1}{2}\|u(T)\|^2 - \tfrac{1}{2}\|u_0\|^2 \le \int_0^T \|f\|\|u\|\,dt.$$

Existence of a unique mild solution follows, with

$$\|u(t)\| \le \|u_0\| + 2\int_0^t \|f(s)\|\,ds.$$

Regularity for homogeneous problem

If $f(t) \equiv 0$ then

$$u(t) = \mathcal{E}(t)u_0 = \sum_{m=0}^{\infty} E_\nu(-\lambda_m t^\nu)\langle u_0, \phi_m\rangle \phi_m.$$

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implying the smoothing property

$$t^q \|A^\mu u^{(q)}(t)\| \le C t^{-\mu\nu} \|u_0\|, \quad 0 \le \mu \le 1, q = 0, 1, 2, \ldots.$$

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For classical diffusion, $E_1(-\lambda_m t) = e^{-\lambda_m t}$ leads to stronger smoothing:

$$t^q \|A^\mu u^{(q)}(t)\| \le Ct^{-\mu}\|u_0\|, \quad 0 \le \mu < \infty.$$

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For instance, $\sigma = \nu/2$ valid if

$$\|A^{1/2} u_0\| < \infty \quad \text{and} \quad t^j \|f^{(j)}(t)\| \le Ct^{\nu/2-1}, \quad 0 \le j \le q+1.$$

Part II

Various Numerical Methods

Implicit finite difference scheme

Langlands and Henry, *J. Comput. Phys.* 205:719–737, 2005.

Consider

$$u_t - (\omega_\nu * u_{xx})_t = 0 \quad \text{for } 0 \le x \le L \text{ and } 0 \le t \le T,$$

with

$$u(x,0) = u_0(x), \qquad u_x(0,t) = 0 = u_x(L,t).$$

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For step sizes $\Delta x = L/P$ and $\Delta t = T/N$, seek

$$U_p^n \approx u(x_p, t_n) \quad \text{for } (x_p, t_n) = (p\Delta x, n\Delta t),$$

where $0 \le p \le P$ and $0 \le n \le N$.

Approximations

A formula of Oldham and Spanier gives

$$(\omega_\nu * v)_t \approx \frac{\Delta t^{\nu-1}}{\Gamma(1+\nu)}\left(\frac{\nu V_p^0}{n^{1-\nu}} + \sum_{j=1}^{n}(V_p^j - V_p^{j-1})\left[(n-j+1)^\nu - (n-j)^\nu\right]\right)$$

at $(x, t) = (x_p, t_n)$.

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at $(x,t) = (x_p, t_n)$.

Combine with

$$u_t \approx \frac{U_p^n - U_p^{n-1}}{\Delta t}$$

$$u_x \approx \frac{U_{p+1}^n - U_{p-1}^n}{2\Delta x},$$

$$u_{xx} \approx \frac{U_{p+1}^n - 2U_p^n + U_{p-1}^n}{\Delta x^2},$$

Final scheme

Write

$$w_j = (j+1)^\nu - j^\nu \qquad \text{and} \qquad \delta^2 U_p^n = \frac{U_{p+1}^n - 2U_p^n + U_{p-1}^n}{\Delta x^2},$$

then

$$\frac{U_p^n - U_p^{n-1}}{\Delta t} = \frac{\Delta t^{\nu-1}}{\Gamma(1+\nu)}\left(\nu n^{\nu-1}\delta^2 U_p^0 + \sum_{j=1}^{n} w_{n-j}(\delta^2 U_p^j - \delta^2 U_p^{j-1})\right)$$

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$$\frac{U_p^n - U_p^{n-1}}{\Delta t} = \frac{\Delta t^{\nu-1}}{\Gamma(1+\nu)} \left(\nu n^{\nu-1} \delta^2 U_p^0 + \sum_{j=1}^{n} w_{n-j} (\delta^2 U_p^j - \delta^2 U_p^{j-1}) \right)$$

for $1 \le n \le N$ and $0 \le p \le P$,

with

$$U_{-1}^n = U_1^n \quad \text{and} \quad U_{P+1}^n = U_{P-1}^n.$$

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Scheme is implicit: put $a = w_0 \Delta t^\nu / \Gamma(1+\nu)$ then

$$U_p^n - a\delta^2 U_p^n = U_p^{n-1} + (\text{terms in } \delta^2 U_p^0, \ldots, \delta^2 U_p^{n-1}).$$

Convergence behaviour

Scheme is unconditionally stable, and computational experiments indicate that

$$U_p^n = u(x_p, t_n) + O(\Delta t^\nu + \Delta x^2).$$

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$$U_p^n = u(x_p, t_n) + O(\Delta t^\nu + \Delta x^2).$$

As $\nu \to 1$ we have $w_j = (j+1)^\nu - j^\nu \to 1$ so

$$\frac{\Delta t^{\nu-1}}{\Gamma(1+\nu)} \left(\nu n^{\nu-1} \delta^2 U_p^0 + \sum_{j=1}^{n} w_{n-j} (\delta^2 U_p^j - \delta^2 U_p^{j-1}) \right) \to \delta^2 U_p^n,$$

and we recover the implicit Euler scheme for classical diffusion:

$$\frac{U_p^n - U_p^{n-1}}{\Delta t} = \delta^2 U_p^n.$$

Convolution quadrature

Lubich, *Numer. Math.* 52: 129–145, 1988.

Seek weights ϕ_j such that

$$(f * g)(t_n) \approx \sum_{j=0}^{n} \phi_{n-j} g(t_j).$$

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Laplace inversion formula gives

$$(f * g)(t) = \int_0^t \underbrace{\frac{1}{2\pi i} \int_\Gamma e^{z(t-s)} \hat{f}(z)\, dz}_{f(t-s)}\, g(s)\, ds$$

$$= \frac{1}{2\pi i} \int_\Gamma \hat{f}(z) \int_0^t e^{z(t-s)} g(s)\, ds\, dz.$$

Auxiliary ODE

Put

$$y(t;z) = \int_0^t e^{z(t-s)} g(s)\, ds$$

so that

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Since $y' = zy + g$ and $y(0) = 0$, we have $y(t_n; z) \approx Y^n(z)$ where

$$\frac{Y^n - Y^{n-1}}{\Delta t} = zY^n + g(t_n), \quad Y^0 = 0,$$

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so

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Generating functions

Write

$$\tilde{Y}(\zeta) = \sum_{n=0}^{\infty} Y^n \zeta^n,$$

then

$$\frac{\delta(\zeta)}{\Delta t}\,\tilde{Y}(\zeta) = z\tilde{Y}(\zeta) + \tilde{g}(\zeta), \quad \delta(\zeta) = 1 - \zeta,$$

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so

$$\tilde{Y}(\zeta) = \big(\delta(\zeta)\,\Delta t^{-1} - z\big)^{-1} \tilde{g}(\zeta),$$

and by Cauchy's theorem,

$$\widetilde{f * g}(\zeta) \approx \frac{1}{2\pi i} \int_\Gamma \hat{f}(z)\tilde{Y}(\zeta)\, dz = \hat{f}\big(\delta(\zeta)\,\Delta t^{-1}\big)\tilde{g}(\zeta).$$

Definition of weights

Compute ϕ_0, ϕ_1, ... so that

$$\hat{f}\big(\delta(\zeta)/\Delta t\big) = \sum_{n=0}^{\infty} \phi_n \zeta^n = \tilde{\phi}(\zeta)$$

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then

$$\widetilde{f * g}(\zeta) \approx \tilde{\phi}(\zeta)\tilde{g}(\zeta)$$

and so

$$(f * g)(t_n) \approx \sum_{j=0}^{n} \phi_{n-j} g(t_j).$$

Explicit finite difference scheme

Yuste and Acedo, *SIAM J. Numer. Anal.* 42: 1862–1874, 2005.

Consider

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Since $\mathcal{L}\{(\omega_\nu * v)_t\} = z^{1-\nu}\hat{v}(z)$ we have

$$(\omega_\nu * v)_t \approx \Delta t^{\nu-1} \sum_{j=0}^{n} \omega_{n-j} V_p^j \quad \text{at } (x,t) = (x_p, t_n),$$

where

$$\sum_{j=0}^{\infty} \omega_j \zeta^j = (1-\zeta)^{1-\nu}.$$

Forward difference in time

Finite difference approximations

$$u_t \approx \frac{U_p^{n+1} - U_p^n}{\Delta t} \quad \text{and} \quad u_{xx} \approx \delta^2 U_p^n = \frac{U_{p+1}^n - 2U_p^n + U_{p-1}^n}{\Delta x^2}$$

lead to the scheme

$$\frac{U_p^{n+1} - U_p^n}{\Delta t} = K \Delta t^{\nu-1} \sum_{j=0}^{n} \omega_{n-j} \delta^2 U_p^j.$$

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If $\nu \to 1$ then $\omega_j \to 0$ for $j \geq 1$ and we recover the explicit Euler scheme for classical diffusion:

$$\frac{U_p^{n+1} - U_p^n}{\Delta t} = K \delta^2 U_p^n.$$

Convergence behaviour

Von-Neumann stability condition:

$$K\,\frac{\Delta t^{\nu}}{\Delta x^2} \le \frac{1}{2^{2-\nu}}.$$

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Formally, error is $O(\Delta t + \Delta x^2)$, but in practice . . . .

Implicit convolution quadrature scheme

Cuesta, Lubich and Palencia, *Math. Comp.*, 75: 673–696, 2006.

Work with integrated equation

$$u + \omega_\nu * Au = u_0 + F(t),$$

where $F(t) = \int_0^t f(s)\,ds$.

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Compute weights ϕ_n satisfying

$$\sum_{n=0}^{\infty} \phi_n \zeta^n = \hat{\omega}_\nu\big(\delta(\zeta)/\Delta t\big) = \left(\frac{\delta(\zeta)}{\Delta t}\right)^{-\nu},$$

so that

$$\omega_\nu * v \approx \sum_{j=0}^{n} \phi_{n-j} V^j.$$

Convergence behaviour

Rearrange equation as

$$(I + \omega_\nu * A)(u - u_0) = -\omega_{1+\nu}(t)Au_0 + F(t),$$

and discretize:

$$U^n - u_0 + \sum_{j=0}^{n} \phi_{n-j} A(U^j - u_0) = -\omega_{1+\nu}(t_n)Au_0 + F(t_n).$$

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Error analysis shows

$$\begin{aligned}\|U^n - u(t_n)\| &\le C\Delta t^{1+\nu}\,\frac{\|A^\rho u_0\|}{t_n^{1+\nu(1-\rho)}} \\ &\quad + C\Delta t^2\left(\frac{\|A^\sigma f(0)\|}{t_n^{1-\nu\sigma}} + \|f'(0)\| + \int_0^{t_n}\|f''(s)\|\,ds\right)\end{aligned}$$

for $0 \le \rho \le 2$ and $0 \le \sigma \le 1$.

McLean and Thomée, *J. Integral Equations Appl.*, 22: 57–94, 2010.

We consider

$$u_t + (\omega_\nu * Au)_t = f(t).$$

Laplace transforms and quadrature

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$$z\hat{u}(z) - u_0 + z^{1-\nu} A\hat{u}(z) = \hat{f}(z),$$

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$$(zI + z^{1-\nu}A)\hat{u}(z) = u_0 + \hat{f}(z).$$

Thus, we can find $\hat{u}(z)$ by solving an elliptic boundary value problem with complex coefficients.

Resolvent estimate

Spectrum of A lies in $[0,\infty)$ so for each $\varphi > 0$,

$$\|(\lambda I - A)^{-1}\| \le \frac{C_\varphi}{|\lambda|} \quad \text{for} \quad |\arg\lambda| \ge \varphi.$$

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We have

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which is solvable with

$$\|\hat{u}(z)\| \le C_\varphi \frac{\|u_0 + \hat{f}(z)\|}{|z|} \quad \text{for} \quad |\arg z| \le \frac{\pi - \varphi}{\nu}.$$

Integral representation

Laplace inversion formula:

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{zt} \hat{u}(z) \, dz.$$

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$$z(\xi) = a + b\bigl(1 - \sin(\delta - i\xi)\bigr), \qquad -\infty < \xi < \infty,$$

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which is the left branch of a hyperbola with asymptotes $y = \pm(x - a - b)\cot\delta$.

Must assume $\hat{f}(z)$ is analytic to the right of Γ.

Quadrature

We have

$$u(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{z(\xi)t} \hat{u}\big(z(\xi)\big) z'(\xi)\, d\xi.$$

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Choose a quadrature step $k > 0$, then

$$u(t) \approx U_N(t) = \frac{k}{2\pi i} \sum_{j=-N}^{N} e^{z_j t} \hat{u}(z_j) z_j',$$

where

$$z_j = z(\xi_j), \qquad z_j' = z'(\xi_j), \qquad \xi_j = jk.$$

Exponential convergence

Given N and a time scale T, can choose a, b, δ so that,

$$\|U_N(t) - u(t)\| \le Ce^{-\mu N}\big(\|u_0\| + \max\|\hat{f}(z)\|\big) \quad \text{for } T \le t \le 2T,$$

where $\mu > 0$ and the max is over z in a certain set including Γ.

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The elliptic problems can be solved in parallel.

A variation on the method avoids using $\hat{f}(z)$ and achieves accuracy $O(e^{-\mu\sqrt{N}})$. The elliptic problems take the form

$$(z^\nu I + A)w(z, t) = z^{\nu-1}\left(e^{zt}u_0 + \int_0^t e^{z(t-s)}f(s)\,ds\right).$$

Part III

Discontinuous Galerkin Methods

Trial space

Nonuniform grid: $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ with $k_n = t_n - t_{n-1}$ and $k = \max_{1 \le n \le N} k_n$.

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For each half-open interval $I_n = (t_{n-1}, t_n]$ we choose a subspace $S_n \subseteq D(A^{1/2}) \subseteq H^1(\Omega)$.

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For each half-open interval $I_n = (t_{n-1}, t_n]$ we choose a subspace $S_n \subseteq D(A^{1/2}) \subseteq H^1(\Omega)$.

Let $\mathcal{W}_r$ denote the space of piecewise polynomial functions V of order r:

$$V(t) = \sum_{p=1}^{r} a_p t^{p-1} \quad \text{for } t \in I_n, \text{ with } a_p \in S_n.$$

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Since V may be discontinuous at t_n, write

$$V^n = V(t_n) = V(t_n^-), \qquad V_+^n = V(t_n^+), \qquad [V]^n = V_+^n - V^n.$$

Fractional derivative of a discontinuous function

Integration by parts shows that for $t \in I_n$,

$$\mathcal{B}_\nu V(t) = \omega_\nu(t) V_+^0 + \sum_{j=1}^{n-1} \omega_\nu(t - t_j)[V]^j + \text{continuous terms.}$$

Thus, $\mathcal{B}_\nu V(t)$ is left-continuous at $t = t_n$ but behaves like $(t - t_{n-1})^{\nu-1}$ as $t \to t_{n-1}^+$.

Weak formulation

Exact solution satisfies

$$\int_{I_n} \big[\langle u', v\rangle + \mathsf{A}(\mathcal{B}_\nu u, v)\big]\,dt = \int_{I_n} \langle f, v\rangle\,dt$$

for any continuous test function $v : [t_{n-1}, t_n] \to D(A^{1/2})$.

Exact solution satisfies

$$\int_{I_n} \left[\langle u', v \rangle + \mathrm{A}(\mathcal{B}_\nu u, v) \right] dt = \int_{I_n} \langle f, v \rangle \, dt$$

for any continuous test function $v : [t_{n-1}, t_n] \to D(A^{1/2})$.

Discontinuous Galerkin (DG) solution $U \in \mathcal{W}_r$ defined by

$$\langle U_+^{n-1}, V_+^{n-1} \rangle + \int_{I_n} \left[\langle U', V \rangle + \mathrm{A}(\mathcal{B}_\nu U, V) \right] dt$$
$$= \langle U^{n-1}, V_+^{n-1} \rangle + \int_{I_n} \langle f, V \rangle \, dt$$

for every test function $V \in \mathcal{W}_r$, with $U^0 \approx u_0$.

Piecewise-constant case (implicit Euler)

When $r = 1$, we have $U(t) = U^n$ for $t \in I_n$, and the DG method reduces to

$$\langle U^n, \chi\rangle + k_n \mathrm{A}(\bar{\mathcal{B}}_\nu^n U, \chi) = \langle U^{n-1}, \chi\rangle + k_n\langle \bar{f}^n, \chi\rangle$$

for all $\chi \in S_n$, where

$$\bar{\mathcal{B}}_\nu^n = \frac{1}{k_n}\int_{I_n} \mathcal{B}_\nu U\,dt \quad \text{and} \quad \bar{f}^n = \frac{1}{k_n}\int_{I_n} f\,dt.$$

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In other words,

$$\left\langle \frac{U^n - U^{n-1}}{k_n}, \chi \right\rangle + \mathrm{A}(\bar{\mathcal{B}}^n_\nu U, \chi) = \langle \bar{f}^n, \chi\rangle.$$

Discrete fractional derivative

We find

$$\bar{\mathcal{B}}_\nu^n V = k_n^{-1}\left(\beta_{nn} V^n - \sum_{j=1}^{n-1} \beta_{nj} V^j\right)$$

where

$$\beta_{nn} = \int_{I_n} \omega_\nu(t_n - s)\, ds = \omega_{1+\nu}(k_n) = k_n^\nu / \Gamma(1+\nu) > 0,$$

$$\beta_{nj} = \int_{I_j} \big[\omega_\nu(t_{n-1} - s) - \omega_\nu(t_n - s)\big]\, ds > 0, \quad 1 \le j \le n-1.$$

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Thus,

$$\langle U^n, \chi\rangle + \beta_{nn} \mathrm{A}(U^n, \chi) = \langle U^{n-1} + k_n \bar{f}^n, \chi\rangle + \sum_{j=1}^{n-1} \beta_{nj} \mathrm{A}(U^j, \chi).$$

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If $\nu \to 1$ then $\beta_{nn} \to k_n$ and $\beta_{nj} \to 0$.

Stability via energy argument

Taking $V = U$ and using $\langle U', U\rangle = \frac{1}{2}(d/dt)\|U\|^2$ we find

$$\tfrac{1}{2}\bigl(\|U^j\|^2 + \|U_+^{j-1}\|^2\bigr) + \int_{I_j} \mathrm{A}(\mathcal{B}_\nu U, U)\,dt = \langle U^{j-1}, U_+^{j-1}\rangle + \int_{I_j}\langle f, U\rangle\,dt.$$

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Summing over j and using positivity we find

$$\|U^n\|^2 + \|U_+^0\|^2 + \sum_{j=1}^{n-1}\|[U]^j\|^2 \le 2\bigg(\langle U^0, U_+^0\rangle + \int_0^{t_n}\langle f, U\rangle\,dt\bigg),$$

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and then

$$\|U^n\| \le C_r\Big(\|U^0\| + \int_0^{t_n} \|f\|\,dt\Big).$$

Projection operator

Define Πu on I_j to be the unique polynomial of order r satisfying

$$\Pi u(t_j) = u(t_j)$$

and

$$\int_{I_j} [u(t) - \Pi u(t)] t^{p-1} \, dt = 0 \quad \text{for } p = 1, 2, \ldots, r-1.$$

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If $r = 2$ then

$$\Pi u(t) = u(t_n) + \frac{u(t_n) - \bar{u}^n}{k_n/2}\,(t - t_n) \quad \text{for } t \in I_n,$$

where $\bar{u}^n = k_n^{-1} \int_{I_n} u(t)\, dt$.

Error from the time discretization

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Split the DG error into

$$U - u = \theta + \eta \quad \text{where} \quad \theta = U - \Pi u \in \mathcal{W}_r \quad \text{and} \quad \eta = \Pi u - u.$$

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Can show

$$\|\theta\|_{J_n}^2 + \|\theta^n\|^2 + \sum_{j=1}^{n-1} \|[\theta]^j\|^2 \le C_r \left| \int_0^{t_n} \langle \mathcal{B}_\nu A\eta, \theta\rangle\, dt \right|.$$

Piecewise-constant case

If $r = 1$ then

$$\int_0^{t_n} \langle \mathcal{B}_\nu A\eta, \theta \rangle \, dt = \sum_{j=1}^{n} \left\langle \int_{I_j} \mathcal{B}_\nu A\eta \, dt, \theta^j \right\rangle = \sum_{j=1}^{n} k_j \langle \bar{\mathcal{B}}_\nu^j A\eta, \theta^j \rangle$$

and

$$k_n \bar{\mathcal{B}}_\nu^n A\eta = \int_0^{t_n} \delta_n(t) A u''(t) \, dt.$$

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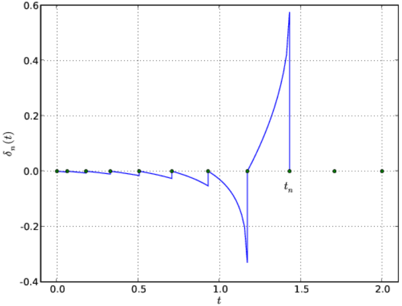

Error analysis ($r = 1$)

Easy to estimate $\sum_{j=1}^{n} |\delta_j(t)|$ and hence show

$$\|U^n - u(t_n)\| \le \|U^0 - u_0\| + \frac{4}{\Gamma(1+\nu)} \sum_{j=1}^{n} \int_{I_j} (t - t_{j-1})^{\nu} \|Au'(t)\| \, dt.$$

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Tricky argument involving

$$\Delta_j(t) = -\int_0^t \delta_j(s)\, ds \quad \text{for } t_1 \le t \le t_j,\ 2 \le j \le n,$$

shows

$$\|U^n - u(t_n)\| \le \|U^0 - u_0\| + \frac{1}{\Gamma(1+\nu)} \biggl(4 \int_{I_1} t^{\nu} \|Au'(t)\|\, dt + 2k_n\, t_n^{\nu} \|Au'(t_n)\| + 6 \sum_{j=2}^{n} k_j \int_{I_j} t^{\nu} \|Au''(t)\|\, dt \biggr).$$

Convergence behaviour ($r = 1$)

Numer. Algorithms, 52: 69–88, 2009.

Graded mesh: $t_n = (n/N)^\gamma T$ for $0 \le n \le N$, with $\gamma \ge 1$.

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Spatial discretization with piecewise-linear finite elements leads to additional error of order $h^2 \log(t_1/t_n)$.

Convergence behaviour for piecewise-linears ($r = 2$).

Numer. Algorithms, to appear.

Can show

$$\|U - u\|_{J_n} \le \|U^0 - u_0\| + \|u - \Pi u\|_{J_n} + C \int_{I_1} t^\nu \|Au'(t)\|\,dt + C \sum_{j=2}^{n} k_j^{1+\nu} \int_{I_j} \|Au''(t)\|\,dt.$$

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New paper proves error of order k^ρ for $\rho = \min(2, \frac{3}{2} + \nu)$.