Numerical Methods for Some Fractional-Order Partial Differential Equations

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Last updated on January 5, 2006

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1. The continuous problem

Initial-value problem: for $-1 < \alpha < 1$, find $u = u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} + D_t^{-\alpha} A u = f(t) \quad \text{for } t > 0, \quad \text{with } u(0) = u_0.$$

In the simplest case,

$$Au = -\frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < 1,$$

with $u(0,t) = 0 = u(1,t)$ for $t > 0$. Under *Laplace transformation*,

$$\hat{u}(z) = \mathcal{L}\{u(t)\} := \int_0^\infty e^{-zt} u(t)\, dt,$$

the fractional time derivative becomes

$$\mathcal{L}\{D^{-\alpha} v(t)\} = z^{-\alpha} \hat{v}(z).$$

Since

$$\mathcal{L}\left\{\frac{t^{\mu-1}}{\Gamma(\mu)}\right\} = z^{-\mu} \quad \text{for } \mu > 0,$$

and

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = z\hat{u}(z) - u(0),$$

we can express $D^{-\alpha}v$ in terms of a *Riemann–Liouville fractional integral*:

$$D^{-\alpha}v(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, v(s)\, ds, \qquad 0 < \alpha < 1,$$

and

$$D^{-\alpha}v(t) = D^1 D^{-(1+\alpha)}v(t) = \frac{\partial}{\partial t}\int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)}\, v(s)\, ds, \qquad -1 < \alpha < 0,$$

Laplace transformation of

$$\frac{\partial u}{\partial t} + D_t^{-\alpha} A u = f(t)$$

gives

$$\Big(z + z^{-\alpha} A\Big)\widehat{u}(z) = u_0 + \widehat{f}(z)$$

so

$$\widehat{u}(z) = \widehat{\mathcal{E}}(z)\Big(u_0 + \widehat{f}(z)\Big) \quad \text{where} \quad \widehat{\mathcal{E}}(z) = \Big(z + z^{-\alpha} A\Big)^{-1}.$$

Hence, the solution of the *homogeneous problem* ($f \equiv 0$) is

$$u(t) = \mathcal{E}(t) u_0 := \mathcal{L}^{-1}\Big\{\widehat{\mathcal{E}}(z) u_0\Big\},$$

and in the general case

$$u(t) = \mathcal{E}(t) u_0 + \int_0^t \mathcal{E}(t-s) f(s)\, ds.$$

Consider $Au = -u_{xx}$ for $0 < x < 1$ subject to homogeneous Dirichlet boundary conditions. We have orthonormal eigenfunctions and corresponding eigenvalues

$$\phi_m(x) = \sqrt{2}\sin(m\pi x) \quad \text{and} \quad \lambda_m = (m\pi)^2 \quad \text{for } m = 1, 2, 3, \ldots.$$

Taking the L_2-inner product of ϕ_m with the equation

$$\frac{\partial u}{\partial t} + D_t^{-\alpha} A u = f(t)$$

gives a sequence of scalar initial-value problems

$$\frac{du_m}{dt} + \lambda_m D^{-\alpha} u_m = f_m(t) \quad \text{for } t > 0, \quad \text{with } u_m(0) = u_{0m}.$$

Each of these problems has an explicit solution in terms of the *Mittag-Leffler function*

$$E_\mu(t) := \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(1 + \mu p)}, \quad \mu > 0.$$

In fact,

$$\mathcal{L}^{-1}\left\{\frac{1}{z+\lambda_m z^{-\alpha}}\right\} = \mathcal{L}^{-1}\sum_{p=0}^{\infty}(-\lambda_m)^p z^{-(1+\alpha)p-1}$$

$$= \sum_{p=0}^{\infty}\frac{\left(-\lambda_m t^{1+\alpha}\right)^p}{\Gamma\left(1+(1+\alpha)p\right)} = E_{1+\alpha}\left(-\lambda_m t^{1+\alpha}\right),$$

so

$$\mathcal{E}(t)u_0 = \sum_{m=1}^{\infty}\langle u_0,\phi_m\rangle E_{1+\alpha}\left(-\lambda_m t^{1+\alpha}\right)\phi_m.$$

Can show $\left|E_{1+\alpha}(-t)\right| \le 1$ for $t \ge 0$, so $\|\mathcal{E}(t)v\| \le \|v\|$ for every $v \in L_2(0,1)$ and hence

$$\|u(t)\| \le \|u_0\| + \int_0^t \|f(s)\|\, ds.$$

Thus, the initial-value problem is *well-posed*.

2. A generalised Crank–Nicolson scheme

Background:

- M. J. Sanz-Serna, *SIAM J. Numer. Anal.* 1988.
- J.-C. López-Marcos, *SIAM J. Numer. Anal.* 1990.
- W. McLean and V. Thomée, *J. Austral. Math. Soc. Ser. B (ANZIAM J.)* 1993.
- W. McLean, V. Thomée and L. Wahlbin, *J. Comput. Appl. Math.* 1996.

Introduce time levels $0 = t_0 < t_1 < t_2 < \cdots$ and put

$$t_{n-1/2} := \tfrac{1}{2}(t_{n-1} + t_n) \quad \text{and} \quad k_n := t_n - t_{n-1} \quad \text{for } n \geq 1.$$

Given a grid function V^n write $V^{n-1/2} := \frac{1}{2}(V^{n-1} + V^n)$, define a piecewise-constant function

$$\overline{V}(t) := \begin{cases} V^1 & \text{for } t_0 < t < t_1, \\ V^{n-1/2} & \text{for } t_{n-1} < t < t_n \text{ and } n \geq 2. \end{cases}$$

and a discrete fractional derivative ($-1 < \alpha < 0$) or fractional integral ($0 < \alpha < 1$),

$$(\mathrm{D}^{-\alpha} V)^{n-1/2} := \frac{1}{k_n} \int_{t_{n-1}}^{t_n} D^{-\alpha} \overline{V}(t)\, dt$$

$$= \omega_{n1} V^1 + \sum_{j=2}^{n} \omega_{nj} V^{j-1/2}.$$

When $\alpha = 0$ we have $D^0\overline{V} = \overline{V}$ so

$$(\mathrm{D}^0 V)^{n-1/2} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n} \overline{V}(t)\,dt = \begin{cases} V^1 & \text{if } n = 1, \\ V^{n-1/2} & \text{if } n \geq 2. \end{cases}$$

If $-1 < \alpha < 0$ then we find that

$$\omega_{nn} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n} \frac{(t_n - s)^\alpha}{\Gamma(1+\alpha)}\,ds = \frac{k_n^\alpha}{\Gamma(2+\alpha)} > 0$$

and, for $1 \leq j \leq n-1$,

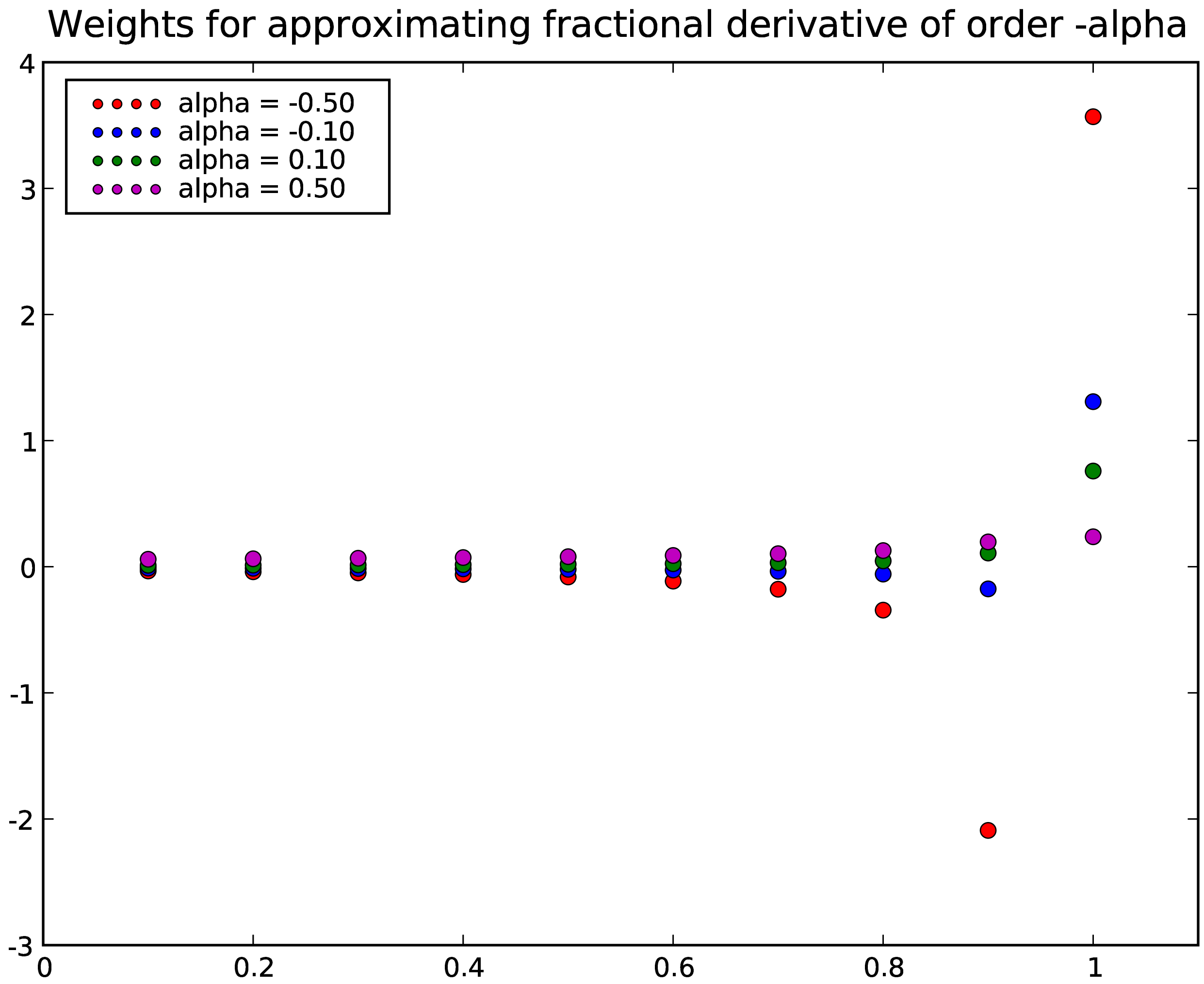
$$\omega_{nj} = \frac{1}{k_n}\int_{t_{j-1}}^{t_j} \frac{(t_n - s)^\alpha - (t_{n-1} - s)^\alpha}{\Gamma(1+\alpha)}\,ds < 0.$$

If $0 < \alpha < 1$ then

$$\omega_{nn} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\,dt = \frac{k_n^\alpha}{\Gamma(2+\alpha)} > 0$$

and, for $1 \leq j \leq n-1$,

$$\omega_{nj} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\,dt > 0.$$



Weights for approximating fractional derivative of order -alpha
alpha = -0.50
alpha = -0.10
alpha = 0.10
alpha = 0.50
4
3
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1

Recall our continuous problem

$$\frac{\partial u}{\partial t} + D_t^{-\alpha} A u = f(t) \quad \text{for } t > 0, \quad \text{with } u(0) = u_0.$$

Starting from $U^0 \approx u_0$ we generate a discrete-time solution $U^n \approx u(t_n)$ using

$$\frac{U^n - U^{n-1}}{k_n} + (\mathsf{D}^{-\alpha} A U)^{n-1/2} = f^{n-1/2} \quad \text{for } n \geq 1,$$

for a suitable approximation

$$f^{n-1/2} \approx \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(t)\, dt;$$

e.g., $f^{n-1/2} = f(t_{n-1/2})$ or $\frac{1}{2}\Big(f(t_{n-1}) + f(t_n)\Big)$.

At the first step, we have

$$\Big(I + \omega_{11} k_1 A\Big) U^1 = U^0 + f^{1/2} k_1.$$

For $n \geq 2$,

$$\Big(I + \tfrac{1}{2}\omega_{nn}k_nA\Big)U^n = \Big(I - \tfrac{1}{2}\omega_{nn}k_nA\Big)U^{n-1} + f^{n-1/2}k_n - \Bigg(\omega_{n1}AU^1 + \sum_{j=2}^{n-1}\omega_{nj}AU^{j-1/2}\Bigg)k_n.$$

Hence, at each step we must solve an elliptic problem, so the scheme is *implicit*.

Formally, the scheme is second-order accurate. However, the mth Fourier mode of the $\mathcal{E}(t)u_0$ involves a factor

$$E_{1+\alpha}\Big(-\lambda_m t^{1+\alpha}\Big) = 1 - \frac{\lambda_m t^{1+\alpha}}{\Gamma(2+\alpha)} + O\Big(t^{2(1+\alpha)}\Big) \quad \text{as } t \to 0^+,$$

so the solution of the continuous problem is not smooth at $t = 0$. To compensate, we employ a *graded mesh*, e.g.,

$$t_n = (nk)^\gamma, \qquad \text{with } \gamma \geq 1.$$

3. Convergence Analysis

Positivity of $D^{-\alpha}$: for real-valued v,

$$\int_0^\infty v(t) D^{-\alpha} v(t)\, dt = \frac{1}{\pi}\cos\left(\frac{\alpha\pi}{2}\right)\int_0^\infty \xi^{-\alpha}|\widehat{v}(i\xi)|^2\, d\xi \geq 0.$$

Take the L_2-inner product of $u(t)$ with the FPDE,

$$\left\langle u(t), \frac{\partial u}{\partial t}\right\rangle + \left\langle u(t), D^{-\alpha} A u\right\rangle = \langle u(t), f(t)\rangle$$

and observe that

$$\left\langle u(t), \frac{\partial u}{\partial t}\right\rangle = \frac{\partial}{\partial t}\frac{1}{2}\|u(t)\|^2$$

and, with $u_m(t) := \langle \phi_m, u(t)\rangle$,

$$\left\langle u(t), D^{-\alpha} A u\right\rangle = \sum_{m=1}^{\infty} \lambda_m \left\langle u_m(t), D^{-\alpha} u_m\right\rangle.$$

Thus, integration from $t = 0$ to $t = T$ gives

$$\frac{1}{2}\Big(\|u(T)\|^2 - \|u(0)\|^2\Big) + \underbrace{\int_0^T \Big\langle u(t), D^{-\alpha} Au \Big\rangle \, dt}_{\geq 0} = \int_0^T \langle u(t), f(t) \rangle \, dt.$$

and so

$$\|u(T)\|^2 \leq \|u_0\|^2 + 2 \int_0^T \langle u(t), f(t) \rangle \, dt,$$

from which

$$\|u(T)\| \leq \|u_0\| + 2 \int_0^T \|f(t)\| \, dt.$$

Can mimic this *energy argument* for our discrete-time scheme once we know

$$\sum_{n=1}^{N} \Big\langle V^{n-1/2}, (\mathsf{D}^{-\alpha} V)^{n-1/2} \Big\rangle k_n \geq 0.$$

In fact,

$$\sum_{n=1}^{N} \left\langle V^{n-1/2}, (\mathrm{D}^{-\alpha} V)^{n-1/2} \right\rangle k_n = \sum_{n=1}^{N} \left\langle V^{n-1/2}, \int_{t_{n-1}}^{t_n} D^{-\alpha} \overline{V}(t)\, dt \right\rangle$$

$$= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left\langle \overline{V}(t), D^{-\alpha} \overline{V}(t) \right\rangle dt = \int_{0}^{t_N} \left\langle \overline{V}(t), D^{-\alpha} \overline{V}(t) \right\rangle dt \geq 0,$$

and *stability* of the generalised Crank–Nicolson method follows:

$$\|U^N\| \leq \|U^0\| + 2 \sum_{n=1}^{N} \|f^{n-1/2}\|\, k_n.$$

Integrating the FPDE from $t = t_{n-1}$ to $t = t_n$ we have

$$u(t_n) - u(t_{n-1}) + \int_{t_{n-1}}^{t_n} D^{-\alpha} A u(t)\, dt = \int_{t_{n-1}}^{t_n} f(t)\, dt,$$

whereas

$$U^n - U^{n-1} + \int_{t_{n-1}}^{t_n} D^{-\alpha} A \overline{U}(t)\, dt = f^{n-1/2}\, k_n.$$

Therefore, the *error* $e^n = U^n - u(t_n)$ satisfies

$$e^n - e^{n-1} + \int_{t_{n-1}}^{t_n} D^{-\alpha} A\bar{e}(t)\,dt = \eta^{n-1/2}\,k_n$$

where $\eta^{n-1/2} = \eta_1^{n-1/2} + \eta_2^{n-1/2}$ with

$$\eta_1^{n-1/2} = f^{n-1/2} - \frac{1}{k_n}\int_{t_{n-1}}^{t_n} f(t)\,dt,$$

$$\eta_2^{n-1/2} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n} D^{-\alpha} A(u - \bar{u})\,dt.$$

Stability yields an *a priori error bound*

$$\|e^n\| \le \|U^0 - u_0\| + 2\sum_{j=1}^{n} \|\eta^{j-1/2}\|\,k_j.$$

3.1 Theorem (McLean and Mustapha, 2005). Let $0 < \alpha < 1$. If the exact solution and source term satisfy

$$t\|Au'(t)\| + t^2\|Au''(t)\| \le Mt^{\sigma-1} \quad \text{and} \quad t\|f'(t)\| + t^2\|f''(t)\| \le Mt^{\sigma-1},$$

for $t > 0$, with $\sigma > 0$, then for $0 \le t \le T$,

$$\|U^n - u(t_n)\| \le \|U^0 - u_0\| + C_{\alpha,\gamma,\sigma,T} M \times \begin{cases} k^{\gamma\sigma}, & 1 \le \gamma < 2/\sigma, \\ k^2 \log(t_n/t_1), & \gamma = 2/\sigma, \\ k^2, & \gamma > 2/\sigma. \end{cases}$$

If we discretise in space using continuous, piecewise-linear finite elements then the error bound has an additional term of order $h^2 \log(t_n/t_1)$, where h is the maximum element size.

4. Simple Numerical Examples

Consider the scalar problem

$$\frac{du}{dt} + D^{-\alpha}u = f(t) \quad \text{for } t > 0, \quad \text{with } u(0) = u_0.$$

We present results for two cases:

$$u(t) = t, \quad f(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad u_0 = 0,$$

and

$$u(t) = \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, \quad f(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}, \quad u_0 = 0.$$

In the first case, $u(t) = t$ is smooth for $t \geq 0$ and the maximum errors (over $[0, T] = [0, 1]$) using a uniform mesh ($\gamma = 1$) are as follows:

N	$\alpha = -0.6$		$\alpha = -0.2$		$\alpha = +0.2$	
20	3.07e-03		1.89e-03		1.02e-03	
40	1.24e-03	1.31	5.66e-04	1.74	2.61e-04	1.98
80	4.95e-04	1.33	1.66e-04	1.77	6.59e-05	1.98
160	1.95e-04	1.34	4.84e-05	1.78	1.66e-05	1.99
320	7.63e-05	1.35	1.40e-05	1.79	4.19e-06	1.99
640	2.96e-05	1.36	4.04e-06	1.79	1.05e-06	1.99
1280	1.15e-05	1.37	1.16e-06	1.80	2.65e-07	1.99

We observe $O(k^2)$ convergence for $0 \leq \alpha \leq 1$, but this deteriorates to $O(k^{2+\alpha})$ when $-1 \leq \alpha \leq 0$.

In the second case, when $u(t) \propto t^{1+\alpha}$ is not smooth at $t = 0$, the convergence is much worse:

N	$\alpha = -0.6$		$\alpha = -0.2$		$\alpha = +0.2$	
20	1.19e-01		1.03e-02		7.04e-04	
40	9.25e-02	0.37	5.38e-03	0.94	4.17e-04	0.76
80	7.15e-02	0.37	2.89e-03	0.89	2.12e-04	0.98
160	5.50e-02	0.38	1.59e-03	0.86	9.99e-05	1.09
320	4.22e-02	0.38	9.09e-04	0.81	4.54e-05	1.14
640	3.23e-02	0.39	5.23e-04	0.80	2.03e-05	1.16
1280	2.47e-02	0.39	3.01e-04	0.80	8.95e-06	1.18

Using a graded mesh ($\gamma = 2$) we can restore $O(k^2)$ convergence for $0 \le \alpha \le 1$, but the improvement is weaker for $-1 < \alpha \le 0$.

N	$\alpha = -0.6$		$\alpha = -0.2$		$\alpha = +0.2$	
20	3.88e-02		1.08e-03		3.96e-04	
40	2.32e-02	0.74	4.15e-04	1.38	1.05e-04	1.91
80	1.42e-02	0.71	1.54e-04	1.43	2.77e-05	1.93
160	8.50e-03	0.74	5.55e-05	1.47	7.24e-06	1.94
320	5.03e-03	0.76	1.96e-05	1.50	1.88e-06	1.95
640	2.96e-03	0.77	6.78e-06	1.53	4.85e-07	1.95
1280	1.72e-03	0.78	2.33e-06	1.54	1.25e-07	1.96

For $-1 < \alpha < 0$ we obtain better results with a different approximation to $D^{-\alpha}$. Using integration by parts we find

$$D^{-\alpha}v(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\,v(0) + \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)}\,v'(s)\,ds,$$

motivating the definition

$$\begin{aligned}(\widetilde{\mathsf{D}}^{-\alpha}V)^{n-1/2} &:= \frac{t^\alpha}{\Gamma(1+\alpha)}\,V^0 + \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)}\,\check{V}'(s)\,ds \\ &= \widetilde{\omega}_{n0}V^0 + \sum_{j=1}^{n} \widetilde{\omega}_{nj}\Big(V^j - V^{j-1}\Big),\end{aligned}$$

with all postive weights

$$\begin{aligned}\widetilde{\omega}_{n0} &= \frac{t_n^{1+\alpha} - t_{n-1}^{1+\alpha}}{k_n\Gamma(2+\alpha)}, \\ \widetilde{\omega}_{nj} &= \frac{1}{k_nk_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j} \frac{(t-s)^\alpha}{\Gamma(1+\alpha)}\,ds\,dt, \qquad 1 \le j \le n-1, \\ \widetilde{\omega}_{nn} &= \frac{1}{k_nk_j}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \frac{(t-s)^\alpha}{\Gamma(1+\alpha)}\,ds\,dt.\end{aligned}$$

The results below were obtained using $\widetilde{\mathrm{D}}^{-\alpha}$, $\gamma = 3$ and

$$f^{n-1/2} := \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(t)\, dt.$$

N	$\alpha = -0.6$		$\alpha = -0.4$		$\alpha = -0.2$	
20	2.96e-04		2.93e-04		1.81e-04	
40	7.67e-05	1.95	7.42e-05	1.98	4.55e-05	1.99
80	1.97e-05	1.96	1.87e-05	1.99	1.14e-05	2.00
160	5.04e-06	1.97	4.71e-06	1.99	2.85e-06	2.00
320	1.28e-06	1.98	1.18e-06	1.99	7.14e-07	2.00
640	3.24e-07	1.98	2.96e-07	2.00	1.79e-07	2.00
1280	8.17e-08	1.99	7.42e-08	2.00	4.47e-08	2.00

5. Laplace transformation and quadrature

Background (sinc quadrature, heat equation):

- J. McNamee, F. Stenger and E. L. Whitney, *Math. Comp.* 1971.
- D. Sheen, I. H. Sloan and V. Thomée, *Math. Comp.* 1999; *IMA J. Numer. Anal.* 2003.
- I. P. Gavrulyuk and V. L. Makarov, ????.
- W. McLean and V. Thomée, *Numer. Math.* 2005.
- Lopez-Fernandez and Palencia, *Appl. Numer. Math.* 2004.
- Weideman, ????.

The *Laplace inversion formula*

$$u(t) = \mathcal{L}^{-1}\{\widehat{u}(z)\} = \frac{1}{2\pi i}\int_\Gamma e^{zt}\widehat{u}(z)\,dz, \qquad t > 0,$$

provides an integral representation of the solution of our FPDE in terms of its Laplace transform

$$\widehat{u}(z) = \widehat{\mathcal{E}}(z)\Big(u_0 + \widehat{f}(z)\Big) = (z + z^{-\alpha}A)^{-1}\Big(u_0 + \widehat{f}(z)\Big).$$

Here, the contour Γ must pass to the right of any singularities of $\widehat{u}(z)$.

We require the spectrum of A to lie in a sector $|\arg z| < \chi$ with *spectral angle* χ and to satisfy a *resolvent estimate*

$$\Big\|(z - A)^{-1}\Big\| \le \frac{M}{1 + |z|} \quad \text{for } |\arg z| > \chi.$$

We have

$$\widehat{u}(z) = z^\alpha (z^{1+\alpha} + A)^{-1}\Big(u_0 + \widehat{f}(z)\Big)$$

and

$$\Big\|(z^{1+\alpha} + A)^{-1}\Big\| \le \frac{M}{1 + |z|^{1+\alpha}} \quad \text{for } |\arg z| < \frac{\pi}{2} + \delta_0,$$

where

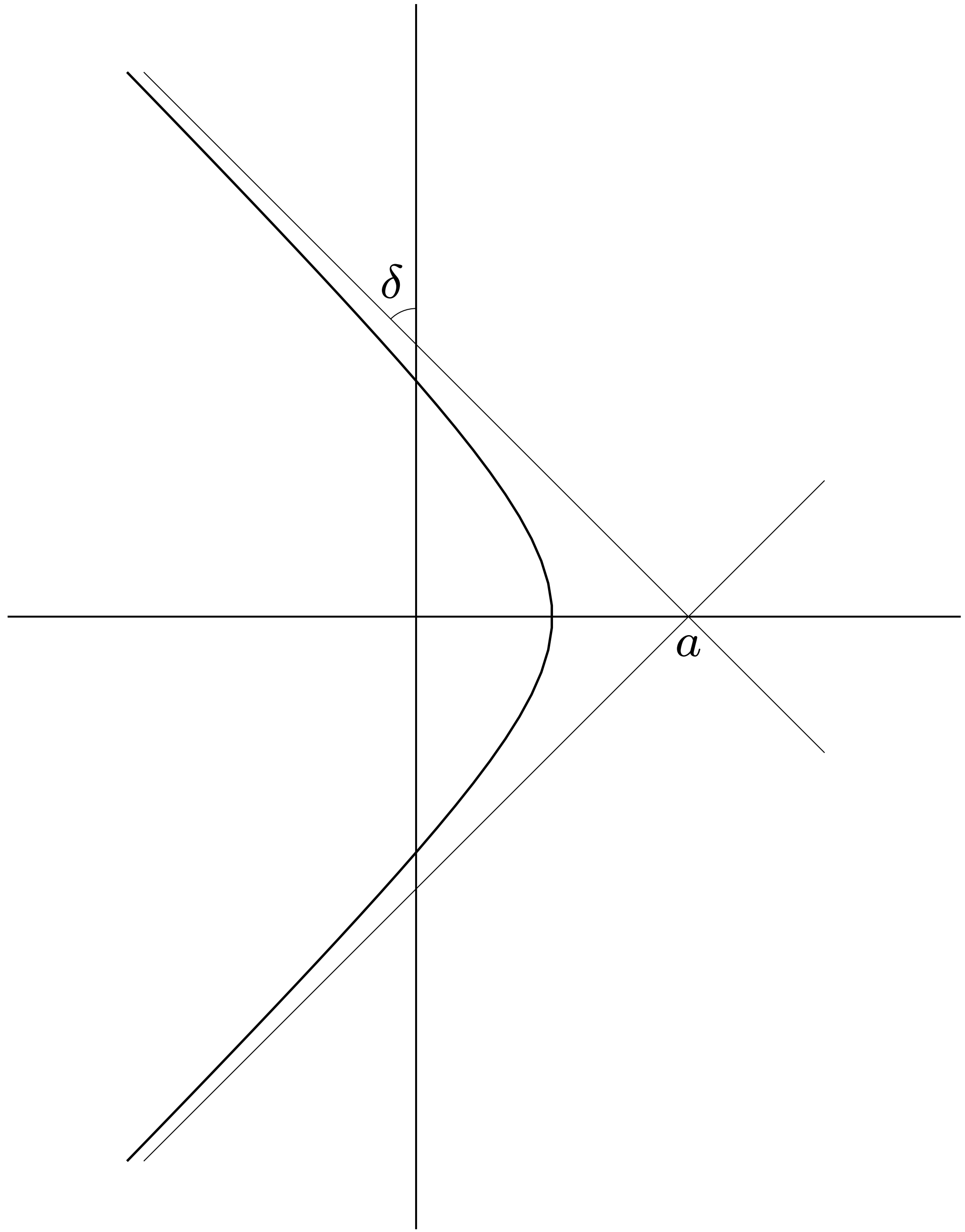
$$\delta_0 = \min\left(\frac{\pi}{2}, \frac{\pi - \chi}{1+\alpha} - \frac{\pi}{2}\right).$$

To guarantee $0 < \delta_0 \le \frac{1}{2}\pi$ we require $0 < \chi < \frac{1}{2}(1-\alpha)\pi$.

For Γ we may take the contour with parametric representation

$$\begin{aligned} z(\xi) &= a - b\sin(\delta - i\xi) \\ &= (a - b\sin\delta\cosh\xi) + ib\cos\delta\sinh\xi \end{aligned}$$

for $-\infty < \xi < \infty$.


δ
a

Putting $z = x + iy$ we find

$$\left(\frac{x(\xi) - a}{b \sin \delta}\right)^2 - \left(\frac{y(\xi)}{b \cos \delta}\right)^2 = 1,$$

implying that Γ is the left branch of the hyperbola with asymptotes $y = \pm(x - a)\cot\delta$. Thus, Γ lies in the sector $|\arg z| < \frac{1}{2}\pi + \delta_0$ provided

$$0 < \delta < \delta_0, \quad a > b \sin \delta, \quad b > 0.$$

We want to compute

$$u(t) = \int_{-\infty}^{\infty} v(\xi, t)\, d\xi \quad \text{for} \quad v(\xi, t) = \frac{1}{2\pi i} e^{z(\xi)t} \hat{u}(z) \frac{dz}{d\xi}$$

and since

$$|e^{z(\xi)t}| = \exp\Big(\mathrm{Re}\, z(\xi)t\Big) = \exp\Big((a - b \sin \delta \cosh \xi)t\Big)$$

the integrand $v(\xi, t)$ exhibits a *double exponential decay* as $|\xi| \to \infty$ for t bounded away from 0.

Take a simple equal-weight quadrature rule of the form

$$I(v) := \int_{-\infty}^{\infty} v(\xi)\, d\xi \approx Q_N(v) := k \sum_{j=-N}^{N} v(\xi_j), \quad \xi_j = jk.$$

5.1 Lemma (López-Fernández and Palencia, 2004). If $v(\zeta)$ is analytic for $|\operatorname{Im}\zeta| < r$, continuous for $|\operatorname{Im}\zeta| \le r$, and satisfies

$$|v(\xi + i\eta)| \le V e^{-\mu \cosh \xi} \quad \text{for } -\infty < \xi < \infty \text{ and } -r \le \eta \le r,$$

then

$$|Q_N(v) - I(v)| \le C L(\mu) V \Big(\exp\big(-2\pi r/k\big) + \exp\big(-\mu \cosh(Nk)\big)\Big),$$

where $L(\mu) = 1 + \log_+(1/\mu)$.

Thus, if $k = \log N/N$ then the quadrature error is of order $e^{-cN/\log N}$.

Numerical method: solve (in parallel) the $2N+1$ elliptic problems

$$(z_j + z_j^{-\alpha} A)\widehat{u}(z_j) = u_0 + \widehat{f}(z_j), \quad z_j = z(\xi_j) \in \Gamma, \quad -N \le j \le N,$$

and then put

$$U_N(t) = k \sum_{j=-N}^{N} v(\xi_j, t) = \sum_{j=-N}^{N} w_j e^{z_j t} \widehat{u}(z_j), \quad w_j = \frac{k}{2\pi i} \frac{dz}{d\xi}\bigg|_{\xi=\xi_j}.$$

The formula

$$z = T(\zeta) = a - b \sin(\delta - i\zeta)$$

defines a conformal mapping T of the strip $|\,\mathrm{Im}\, z| \le r$ onto a region

$$S_r = \{\, T(\zeta) : |\,\mathrm{Im}\, z| \le r \,\}$$

bounded by two hyperbolas. Note that $T : \mathbb{R} \to \Gamma$.

If

$$0 < r < \delta, \quad \delta + r < \delta_0, \quad a > b \sin(\delta + r),$$

then S_r lies in the sector $|\arg z| < \frac{1}{2}\pi + \delta_0$ so $v(\zeta, t)$ is analytic for $|\operatorname{Im} \zeta| < r$ and we can show that

$$|U_N(t) - u(t)| \le Ce^{at} L(t)\Big(\|u_0\| + \max_{z \in S_r} \|\hat{f}(z)\|\Big) \times \Big(\exp\big(-2\pi r/k\big) + \exp\big(-\tfrac{1}{2} at \sin(\delta - r) \cosh(Nk)\big)\Big).$$

Put

$$k = \frac{\log(N/\tau)}{N} \quad \text{with} \quad 0 < \tau < N,$$

then the error in $U_N(t) \approx u(t)$ is of order

$$\exp\big(-cN/\log(N/\tau)\big) + \exp\big(-c(t/\tau)N\big).$$