Discontinuous Galerkin Method for a Non-Local Evolution Equation

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Integrodifferential Equation

Initial boundary-value problem (fractional wave equation):

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\begin{aligned}
u_t(x,t) - \int_0^t \beta(t,s)\nabla^2 u(x,s)\,ds = f(x,t) &\qquad \text{for } x \in \Omega \\
&\qquad \text{and } 0 < t < T, \\
u(x,0) = u_0(x) &\qquad \text{for } x \in \Omega, \\
u(x,t) = 0 &\qquad \text{for } x \in \partial\Omega.
\end{aligned}
$$

Standard kernel:

$$\beta(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\mu(t-s)}, \quad 0 < \alpha < 1, \quad \mu \geq 0.$$

Abstract version:

$$\frac{du}{dt} + \mathcal{B}Au = f(t) \quad \text{for } 0 < t < T, \quad \text{with } u(0) = u_0.$$

Problem is well-posed if $\mathcal{B}$ is positive semi-definite:

$$\int_0^T v(t)\mathcal{B}v(t)\,dt \geq 0 \quad \text{for all } v \in C^\infty_{\text{comp}}(0, T).$$

Laplace transformation gives

$$\mathcal{L}\{t^{\alpha-1}e^{-\mu t}/\Gamma(\alpha)\} = (z+\mu)^{-\alpha}$$

so, for the standard kernel, the Parseval–Plancherel theorem gives

$$\int_0^\infty v(t)\mathcal{B}v(t)\,dt$$
$$= \frac{1}{2\pi}\int_0^\infty (\mu^2+y^2)^{-\alpha/2}\cos\bigl(\alpha\arg(\mu+iy)\bigr)\,|\hat{v}(iy)|^2\,dy$$
$$\geq 0.$$

Related work:

Finite differences Sanz-Serna 1988, López Marcos 1990, McLean and Thomée 1993, Mustapha and McLean 2007.

Convolution quadrature Lubich, Sloan and Thomée 1996.

Laplace transforms López-Fernández and Palencia 2004, McLean and Thomée 2004.

Fast summation Schädle, López-Fernández and Lubich 2006.

Adaptive error control using piecewise-constant DG Adolfsson, Enelund and Larsson 2003.

Galerkin Approximation

Arbitrarily-spaced time levels

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T \quad \text{with} \quad k_n = t_n - t_{n-1}.$$

Associate with each time interval $I_n = (t_{n-1}, t_n]$ a conforming finite element space $S_n \subseteq H_0^1(\Omega)$.
Finite dimensional trial space $\mathcal{W}$ consists of functions U having the form

$$U(x,t) = U_+^{n-1}(x)\frac{t_n - t}{k_n} + U^n(x)\frac{t - t_{n-1}}{k_n} \quad \text{for } t_{n-1} < t \le t_n,$$

with U_+^{n-1}, $U^n \in S_n$.
Thus, $U(x,t)$ is continuous in x but discontinuous in t.

Exact solution satisfies

$$\int_{I_n} \big[\langle u_t, v\rangle + \langle \mathcal{B}\nabla u, \nabla v\rangle\big]\, dt = \int_{I_n} \langle f, v\rangle\, dt$$

for all $v \in C\big([0, T], H_0^1(\Omega)\big)$.
Discontinuous Galerkin method: given $U^0 \approx u_0$ find $U \in \mathcal{W}$ such that

$$\langle U_+^{n-1}, X_+^{n-1}\rangle + \int_{I_n} \big[\langle U_t, X\rangle + \langle \mathcal{B}\nabla U, \nabla X\rangle\big]\, dt \\ = \langle U^{n-1}, X_+^{n-1}\rangle + \int_{I_n} \langle f, X\rangle\, dt$$

for $n = 1, 2, \ldots, N$ and for all $X \in \mathcal{W}$.

Practical implementation takes the form

$$
\begin{aligned}
(\tfrac{1}{2} + \omega_{nn}^{11} A) U_+^{n-1} + (\tfrac{1}{2} + \omega_{nn}^{12}) U^n &= U^{n-1} + f^{n1} \\
&\quad - \sum_{j=1}^{n-1} \big(\omega_{nj}^{11} A U_+^{j-1} + \omega_{nj}^{12} A U^j \big), \\
(-\tfrac{1}{2} + \omega_{nn}^{21} A) U_+^{n-1} + (\tfrac{1}{2} + \omega_{nn}^{22}) U^n &= f^{n2} \\
&\quad - \sum_{j=1}^{n-1} \big(\omega_{nj}^{21} A U_+^{j-1} + \omega_{nj}^{22} A U^j \big),
\end{aligned}
$$

so the scheme is *implicit*: at each step we must solve a 2×2 elliptic system.

Notation:

$$\|\cdot\| = \|\cdot\|_{L_2(\Omega)}, \quad \|U\|_J = \sup_{t\in J} \|U\|, \quad J_n = (0, t_n] = \bigcup_{j=1}^{n} I_j.$$

Energy arguments show stability of the continuous problem,

$$\|u\|_{J_n} \le \|u_0\| + 2\int_0^{t_n} \|f(t)\|\,dt,$$

and of DG,

$$\|U\|_{J_n} \le 12\Big(\|U^0\| + \int_0^{t_n} \|f(t)\|\,dt\Big) \quad \text{for } 0 \le t_n \le T.$$

In addition, jumps $[U]^n = U^n_+ - U^n$ satisfy

$$\sum_{j=1}^{n-1} \|[U]^j\|^2 \le 24\Big(\|U^0\| + \int_0^{t_n} \|f\|\,dt\Big)^2.$$

A Priori Error Bounds

Singular behaviour of exact solution typically

$$t\|u_t\|_{H_0^2(\Omega)} + t^2\|u_{tt}\|_{H_0^2(\Omega)}\| \le Ct^{\sigma-1}$$

and

$$\|u_t\| + t\|u_{tt}\| \le Ct^{\sigma-1},$$

with $0 < \sigma \le 1$.
Introduce a projector Π defined by

$$\Pi u \in \mathcal{W}, \quad \Pi u(t_n) = u(t_n), \quad \int_{I_n} [u - \Pi u]\, dt = 0,$$

for $n = 1, 2, \ldots N$. Find

$$\|u - \Pi u\|_{I_n} \le Ck_n^{r-1} \int_{I_n} \|\partial_t^r u\|\, dt, \quad r = 1 \text{ or } 2.$$

For simplicity, consider semi-discrete method $S_n = H_0^1(\Omega)$ with no spatial error.
Global error bound

$$\|U - \Pi u\|_{J_n} \leq C\|U^0 - u_0\| + Ct_n^{\alpha} \int_{I_1} t\|u_t\|_{H_0^2(\Omega)}\,dt + Ct_n^{\alpha} \sum_{j=2}^{n} k_j^2 \int_{I_n} \|u_{tt}\|_{H_0^2(\Omega)}\,dt.$$

Putting

$$t_n = \left(\frac{n}{N}\right)^{\gamma} T \quad \text{and} \quad k = \max_{1\leq n\leq N} k_n,$$

we find

$$\|U - u\|_{J_n} \leq C\|U^0 - u_0\| + C \times \begin{cases} k^{\gamma\sigma}, & 1 \leq \gamma < 2/\sigma, \\ k^2 \log(t_n/t_1), & \gamma = 2/\sigma, \\ k^2, & \gamma > 2/\sigma. \end{cases}$$

Also, the DG solution is superconvergent at the break points,

$$\begin{aligned}\|U^n - u(\cdot, t_n)\| &\le C\biggl(\|U^0 - u_0\| + \epsilon_{n1}\int_{I_1} t\|u_t\|_{H_0^2(\Omega)}\,dt \\ &\qquad + \sum_{j=2}^{N} \epsilon_{nj} k_j^2 \int_{I_j} \|u_{tt}\|_{H_0^2(\Omega)}\,dt\biggr) \\ &\le C\|U^0 - u_0\| + C \times \begin{cases} k^{\gamma(\sigma+\alpha)}, & 1 \le \gamma < 3/(\sigma+\alpha), \\ k^3, & \gamma \ge 3/(\sigma+\alpha), \end{cases}\end{aligned}$$

where

$$\epsilon_{nj} = \sup_{t \in I_j}\biggl(\int_t^{t_j} |\beta(s,t)|\,ds + \int_{t_j}^{t_n} |\beta(s,t) - \beta(s,t_j)|\,ds\biggr).$$